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ROTATIVE MAPPINGS IN HILBERT SPACE

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ABSTRACT. The aim of this paper is to give some conditions providing existence of fixed points for lipschitzian mappings in a Hilbert space which are n-rotative with $n \geq 3$.

1. PRELIMINARIES.

Let C be a nonempty subset of a Banach space X. A mapping $T : C \to C$ is said to be (a, n)-*rotative* if for an integer $n \geq 2$ and $0 \leq a < n$,

$$\|x - T^n x\| \leq a \|x - Tx\| \quad \text{for any } x \in C. \tag{1}$$

We will simply say that the mapping is n-*rotative* if it is (a, n)-rotative with some $a < n$, and *rotative* if it is n-rotative for some $n \geq 2$.

Recall that $T : C \to C$ is called k-lipschitzian if for all $x, y \in C$,

$$\|Tx - Ty\| \leq k \|x - y\| .$$

If $k = 1$ such a mapping is said to be nonexpansive.

It is known that any nonexpansive and rotative selfmapping of a closed and convex subset of a Banach space has fixed points (see, e.g., [1] , [2] , [4]). Moreover, if we consider k-lipschitzian mappings with $k > 1$, the condition of rotativeness (1) assures the existence of fixed points provided k is not to large. Namely, we have the following

Theorem 1. [3] *If C is a nonempty, closed and convex subset of a Banach space X, then for any $n \geq 2$ and $a < n$ there exists $\gamma > 1$ such that any (a, n)-rotative and k-lipschitzian mapping $T : C \to C$ has a fixed point provided $k < \gamma$.*

Clearly, γ which appears in the above theorem depends on a, n and the space in which the set C is contained. Thus it is convenient to define the function $\gamma_n^X(a)$ as follows

$$\gamma_n^X(a) = \inf\{k : \text{there is a closed and convex set } C \subset X \text{ and a fixed point free } k\text{-lipschitzian } (a,n)\text{-rotative selfmapping of } C\}.$$

Now we can reformulate Theorem 1 in following way:

For any Banach space X, $n \geq 2$ and $a < n$, we have $\gamma_n^X(a) > 1$.

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In general, precise values of $\gamma_n^X(a)$ are not known. If n is arbitrary, we have only an estimate from below of the function γ_n^X at $a = 0$ (see [6]):

$$\gamma_n^X(0) \geq \begin{cases} 2 & \text{for } n = 2, \\ \sqrt[n-1]{\frac{1}{n-2}\left(-1+\sqrt{n(n-1)-\frac{1}{n-1}}\right)} & \text{for } n \geq 3. \end{cases} \tag{2}$$

Moreover, in an arbitrary Banach space

$$\gamma_2^X(a) \geq \max\left\{\frac{1}{2}\left(2-a+\sqrt{(2-a)^2+a^2}\right), \frac{1}{8}\left(a^2+4+\sqrt{\left(a^2+4\right)^2-64a+64}\right)\right\} \tag{3}$$

(see [1]). There exist also some evaluations of γ_2^H, where H is a Hilbert space (see [1], [8]). In this case, the best known estimate is due to T. Komorowski [7]:

Theorem 2.

$$\gamma_2^H(a) \geq \sqrt{\frac{5}{a^2+1}}. \tag{4}$$

Proof. Suppose that for $\varepsilon \in (0,1)$ and $x \in C$, $\left\|T^2x - Tx\right\| \geq (1-\varepsilon)\|Tx - x\|$. Then if we put $u = \frac{1}{2}\left(T^2x + Tx\right)$, we get

$$\begin{aligned} \|u - Tu\|^2 &= \left\|\frac{1}{2}\left(T^2x - Tu\right) + \frac{1}{2}(Tx - Tu)\right\|^2 \\ &= \frac{1}{2}\left\|T^2x - Tu\right\|^2 + \frac{1}{2}\|Tx - Tu\|^2 - \frac{1}{4}\left\|T^2x - Tx\right\|^2 \\ &\leq \frac{k^2}{2}\|Tx - u\|^2 + \frac{k^2}{2}\|x - u\|^2 - \frac{1}{4}\left\|T^2x - Tx\right\|^2 \\ &= \frac{k^2}{8}\left\|T^2x - Tx\right\|^2 - \frac{1}{4}\left\|T^2x - Tx\right\|^2 \\ &\quad + \frac{k^2}{2}\left(\frac{1}{2}\left\|T^2x - x\right\|^2 + \frac{1}{2}\|Tx - x\|^2 - \frac{1}{4}\left\|T^2x - Tx\right\|^2\right) \\ &\leq \left[\frac{k^2\left(a^2+1\right)}{4} - \frac{1}{4}(1-\varepsilon)\right]\|Tx - x\|^2. \end{aligned}$$

Let $x_1 = x$ and for $n \in \mathbb{N}$ set

$$\begin{aligned} x_{n+1} &= Tx_n && \text{if } \left\|T^2x - Tx\right\| < (1-\varepsilon)\|Tx - x\|, \\ x_{n+1} &= \frac{T^2x_n + Tx_n}{2} && \text{if } \left\|T^2x - Tx\right\| \geq (1-\varepsilon)\|Tx - x\|. \end{aligned}$$

Then $\{x_n\}$ converges to a fixed point of T provided $\frac{1}{4}\left(k^2\left(a^2+1\right) - 1 + \varepsilon\right) < 1$. Since ε was arbitrarily chosen, this gives (4). $\square$

Although the estimate (4) is better than (3) and better than that obtained in [8] for a Hilbert space, it is still not sharp. Namely, one can prove (see [8]) that $\gamma_2^H(0) \geq \sqrt{\pi^2 - 3} \approx 2.62$, while it follows from (4) that $\gamma_2^H(0) \geq \sqrt{5} \approx 2.24$.

In [5] J. Górnicki gives an evaluation of $\gamma_3^H(a)$ in a Hilbert space. Unfortunately, there are some miscalculations in his paper.

2. Evaluation of $\gamma_n^H(a)$ for $n \geq 3$.

We will start with two lemmas.

Lemma 1. *Let* $a_1, a_2, ...a_n \in H,\ \alpha_1, \alpha_2, ...\alpha_n \in (0,1),\ \sum_{i=1}^n \alpha_i = 1$. *Then*

$$\left\| \sum_{i=1}^n \alpha_i a_i \right\|^2 = \sum_{i=1}^n \alpha_i \|a_i\|^2 - \sum_{1 \leq i < j \leq n} \alpha_i \alpha_j \|a_i - a_j\|^2 . \tag{5}$$

Proof. It is known that for $u, v \in H,\ \alpha \in (0,1)$,

$$\|(1-\alpha)u + \alpha v\|^2 = (1-\alpha)\|u\|^2 + \alpha\|v\|^2 - \alpha(1-\alpha)\|u-v\|^2 .$$

Suppose now that (5) holds for some $n \in \mathbb{N}$. Let $a_i \in H,\ i = 1, ...n+1$ and $\alpha_i \in (0,1),\ \sum_{i=1}^{n+1} \alpha_i = 1$. Setting $\alpha = \sum_{i=1}^n \alpha_i$ we have $\alpha_{n+1} = 1 - \alpha$ and

$$\begin{aligned}
\left\| \sum_{i=1}^{n+1} \alpha_i a_i \right\|^2 &= \left\| \alpha \sum_{i=1}^n \frac{\alpha_i}{\alpha} a_i + (1-\alpha) a_{n+1} \right\|^2 \\
&= \alpha \left\| \sum_{i=1}^n \frac{\alpha_i}{\alpha} a_i \right\|^2 + (1-\alpha)\|a_{n+1}\|^2 - \alpha(1-\alpha) \left\| \sum_{i=1}^n \frac{\alpha_i}{\alpha} (a_i - a_{n+1}) \right\|^2 \\
&= \alpha \left(\sum_{i=1}^n \frac{\alpha_i}{\alpha} \|a_i\|^2 - \sum_{1 \leq i < j \leq n} \frac{\alpha_i \alpha_j}{\alpha^2} \|a_i - a_j\|^2 \right) + (1-\alpha)\|a_{n+1}\|^2 \\
&\quad - \alpha(1-\alpha) \left(\sum_{i=1}^n \frac{\alpha_i}{\alpha} \|a_i - a_{n+1}\|^2 - \sum_{1 \leq i < j \leq n} \frac{\alpha_i \alpha_j}{\alpha^2} \|a_i - a_j\|^2 \right) \\
&= \sum_{i=1}^n \alpha_i \|a_i\|^2 + (1-\alpha)\|a_{n+1}\|^2 - (1-\alpha) \sum_{i=1}^n \alpha_i \|a_i - a_{n+1}\|^2 \\
&\quad - \sum_{1 \leq i < j \leq n} \alpha_i \alpha_j \|a_i - a_j\|^2 \\
&= \sum_{i=1}^{n+1} \alpha_i \|a_i\|^2 - \sum_{1 \leq i < j \leq n+1} \alpha_i \alpha_j \|a_i - a_j\|^2 ,
\end{aligned}$$

which, by induction argument, ends the proof. □

In particular, if $\alpha_i = \frac{1}{n},\ i = 1, ...n$, then

$$\left\| \frac{1}{n} \sum_{i=1}^n a_i \right\|^2 = \frac{1}{n} \sum_{i=1}^n \|a_i\|^2 - \frac{1}{n^2} \sum_{1 \leq i < j \leq n} \|a_i - a_j\|^2 . \tag{5'}$$

Lemma 2. *Let C be a convex subset of a Hilbert space H and let $T : C \to C$ be k-lipschitzian and (a,n)-rotative with $n \geq 3$. For $x \in C$ put*

$$z = \frac{1}{n}\left(Tx + T^2x + \cdots + T^nx\right).$$

Then

$$\|z - Tz\|^2 \leq \frac{k^2a^2 + k^{2n-2}}{n^2}\|x - Tx\|^2 + \frac{1}{n^2}\sum_{j=2}^{n-1} k^{2(n-j)}\left\|x - T^jx\right\|^2 - \frac{1}{n^2}\sum_{j=1}^{n-1}\left\|T^nx - T^jx\right\|. \tag{6}$$

Proof. Using (5′) we get

$$\begin{aligned}
\|z - Tz\|^2 &= \frac{1}{n}\sum_{i=1}^{n}\left\|T^ix - Tz\right\|^2 - \frac{1}{n^2}\sum_{1\leq i<j\leq n}\left\|T^ix - T^jx\right\|^2 \\
&\leq \frac{k^2}{n}\sum_{i=1}^{n}\left\|T^{i-1}x - \frac{1}{n}(Tx + \cdots + T^nx)\right\|^2 - \frac{1}{n^2}\sum_{1\leq i<j\leq n}\left\|T^ix - T^jx\right\|^2 \\
&= \frac{k^2}{n}\sum_{i=1}^{n}\left(\frac{1}{n}\sum_{j=1}^{n}\left\|T^{i-1}x - T^jx\right\|^2 - \frac{1}{n^2}\sum_{1\leq i<j\leq n}\left\|T^ix - T^jx\right\|^2\right) \\
&\quad - \frac{1}{n^2}\sum_{1\leq i<j\leq n}\left\|T^ix - T^jx\right\|^2 \\
&= \frac{k^2}{n^2}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left\|T^{i-1}x - T^jx\right\|^2\right) - \frac{k^2+1}{n^2}\sum_{1\leq i<j\leq n}\left\|T^ix - T^jx\right\|^2 \\
&= \frac{k^2}{n^2}\sum_{j=1}^{n}\left\|x - T^jx\right\|^2 \\
&\quad + \frac{k^2}{n^2}\left(\sum_{j=1}^{n}\left\|Tx - T^jx\right\|^2 + \cdots + \sum_{j=1}^{n}\left\|T^{n-1}x - T^jx\right\|^2\right) \\
&\quad - \frac{k^2+1}{n^2}\left(\sum_{j=2}^{n}\left\|Tx - T^jx\right\|^2 + \sum_{j=3}^{n}\left\|T^2x - T^jx\right\|^2 + \cdots \right. \\
&\qquad\qquad \left. + \left\|T^{n-1}x - T^nx\right\|^2\right) \\
&= \frac{k^2}{n^2}\sum_{j=1}^{n}\left\|x - T^jx\right\|^2 - \frac{1}{n^2}\left(\sum_{j=2}^{n}\left\|Tx - T^jx\right\|^2 + \sum_{j=3}^{n}\left\|T^2x - T^jx\right\|^2\right. \\
&\quad \left. + \cdots + \left\|T^{n-1}x - T^nx\right\|^2\right) + \frac{k^2}{n^2}\left[\left\|T^2x - Tx\right\|^2\right.
\end{aligned}$$

$$+\left(\left\|T^3x-Tx\right\|^2+\left\|T^3x-T^2x\right\|^2\right)+\cdots+\sum_{j=1}^{n-2}\left\|T^{n-1}x-T^jx\right\|^2\Bigg]$$

$$=\frac{k^2}{n^2}\sum_{j=1}^{n}\left\|x-T^jx\right\|^2+\frac{k^2-1}{n^2}\left(\sum_{j=2}^{n-1}\left\|Tx-T^jx\right\|^2+\sum_{j=3}^{n-1}\left\|T^2x-T^jx\right\|^2\right.$$

$$\left.+\cdots+\left\|T^{n-2}x-T^{n-1}x\right\|^2\right)-\frac{1}{n^2}\sum_{j=1}^{n-1}\left\|T^nx-T^jx\right\|^2$$

Since T is (a,n)-rotative, we have

$$(*)\qquad \|z-Tz\|^2\le\frac{k^2a^2}{n^2}\|x-Tx\|^2+\frac{k^2}{n^2}\sum_{j=1}^{n-1}\left\|x-T^jx\right\|^2$$

$$+\frac{k^2-1}{n^2}\sum_{1\le i<j\le n-1}\left\|T^ix-T^jx\right\|^2-\frac{1}{n^2}\sum_{j=1}^{n-1}\left\|T^nx-T^jx\right\|^2$$

Observe that

$$\frac{k^2}{n^2}\sum_{j=1}^{n-1}\left\|x-T^jx\right\|^2+\frac{k^2-1}{n^2}\sum_{1\le i<j\le n-1}\left\|T^ix-T^jx\right\|^2$$

$$=\frac{k^2}{n^2}\left(\|x-Tx\|^2+\left\|x-T^2x\right\|^2+\cdots\left\|x-T^{n-1}x\right\|^2\right)$$

$$+\frac{k^2-1}{n^2}\left(\left\|Tx-T^2x\right\|^2+\left\|Tx-T^3x\right\|^2+\cdots+\left\|Tx-T^{n-1}x\right\|^2\right.$$

$$+\left\|T^2x-T^3x\right\|^2+\cdots+\left\|T^2x-T^{n-1}x\right\|^2$$

$$\vdots$$

$$\left.+\left\|T^{n-2}x-T^{n-1}x\right\|^2\right)$$

$$\le\frac{k^2}{n^2}\left(\|x-Tx\|^2+\left\|x-T^2x\right\|^2+\cdots\left\|x-T^{n-1}x\right\|^2\right)$$

$$+\frac{k^2-1}{n^2}\left(k^2\|x-Tx\|^2+k^2\left\|x-T^2x\right\|^2+\cdots+k^2\left\|x-T^{n-2}x\right\|^2\right.$$

$$+k^4\|x-Tx\|^2+\cdots+\left\|x-T^{n-3}x\right\|^2$$

$$\vdots$$

$$\left.+k^{2(n-2)}\|x-Tx\|^2\right)$$

$$=\sum_{j=1}^{n-1}\left(\frac{k^2}{n^2}+\frac{k^2-1}{n^2}\cdot k^2\frac{k^{2(n-j-1)}-1}{k^2-1}\right)\left\|x-T^jx\right\|^2,$$

which together with $(*)$ gives desired inequality (6). □

Using only the triangle inequality and the fact that T is k-lipschitzian we have

$$\left\|x - T^j x\right\| \leq \left(1 + k + ... + k^j\right) \|x - Tx\| = \frac{k^{j+1} - 1}{k - 1} \|x - Tx\|, \qquad j = 1, ...n - 1. \tag{7}$$

On the other hand, using the condition of (a, n)-rotativeness of T we can also evaluate the expression $\left\|x - T^{n-1}x\right\|$ in a different manner. Namely,

$$\left\|x - T^{n-1}x\right\| \leq \|x - T^n x\| + \left\|T^n x - T^{n-1}x\right\| \leq \left(a + k^{n-1}\right) \|x - Tx\|. \tag{8}$$

Now by (6) and (7) we obtain

$$\|z - Tz\|^2 \leq \frac{1}{n^2} \left[k^2 a^2 + \sum_{j=1}^{n-1} k^{2j} \left(\frac{k^{n-j} - 1}{k - 1}\right)^2\right] \|x - Tx\|^2 - \frac{1}{n^2} \sum_{j=1}^{n-1} \left\|T^n x - T^j x\right\|^2, \tag{9}$$

while by (6), (7) and (8) we get

$$\|z - Tz\|^2 \leq \frac{1}{n^2} \left[2k^2 a^2 + 2ak^{n+1} + k^{2n} + \sum_{j=2}^{n-1} k^{2j} \left(\frac{k^{n-j} - 1}{k - 1}\right)^2\right] \|x - Tx\|^2 - \frac{1}{n^2} \sum_{j=1}^{n-1} \left\|T^n x - T^j x\right\|^2. \tag{10}$$

We are now ready to formulate the main theorem of our paper, which is a generalization of Theorem 2.

Theorem 3. *Given an integer $n \geq 2$, let $\gamma_n^1(a)$ be the solution of the equation*

$$k^2 a^2 + \sum_{j=1}^{n-1} k^{2j} \left(\frac{k^{n-j} - 1}{k - 1}\right)^2 = n^2 + 1, \tag{11}$$

and additionally for $n \geq 3$ let $\gamma_n^2(a)$ be the solution of the equation

$$2k^2 a^2 + 2ak^{n+1} + k^{2n} + \sum_{j=2}^{n-1} k^{2j} \left(\frac{k^{n-j} - 1}{k - 1}\right)^2 = n^2 + 1. \tag{12}$$

If $T : C \to C$, $C \subset H$, is a k-lipschitzian and (a, n)-rotative mapping such that $k < \max\left(\gamma_n^1(a), \gamma_n^2(a)\right)$, then T has fixed points. In other words,

$$\gamma_n^H(a) \geq \max\left(\gamma_n^1(a), \gamma_n^2(a), 1\right). \tag{**}$$

Proof. For $n = 2$ our claim follows from Theorem 2.

Given an integer $n \geq 3$, suppose that for $\varepsilon \in (0, 1)$ and $x \in C$,

$$\sum_{j=1}^{n-1} \left\|T^n x - T^j x\right\|^2 \geq (1 - \varepsilon) \|Tx - x\|.$$

It then follows from (9) and (10) that

$$\|Tz-z\|^2 \le \frac{1}{n^2}\left[k^2a^2+\sum_{j=1}^{n-1}k^{2j}\left(\frac{k^{n-j}-1}{k-1}\right)^2-1+\varepsilon\right]\|x-Tx\|^2$$

and

$$\|Tz-z\|^2 \le \frac{1}{n^2}\left[2k^2a^2+2ak^{n+1}+k^{2n}+\sum_{j=2}^{n-1}k^{2j}\left(\frac{k^{n-j}-1}{k-1}\right)^2-1+\varepsilon\right]\|x-Tx\|^2,$$

where $z=\frac{1}{n}\left(Tx+T^2x+\cdots+T^nx\right)$. Consider now the sequence defined as follows: $x_1=x$ and for $n\in\mathbb{N}$,

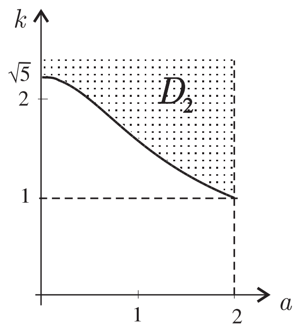
$$x_{n+1}=T^{n-1}x_n \quad \text{if } \sum_{j=1}^{n-1}\left\|T^nx-T^jx\right\|^2<(1-\varepsilon)\|Tx-x\|,$$
$$x_{n+1}=\frac{1}{n}\sum_{j=1}^{n}T^jx \quad \text{if } \sum_{j=1}^{n-1}\left\|T^nx-T^jx\right\|^2\ge(1-\varepsilon)\|Tx-x\|.$$

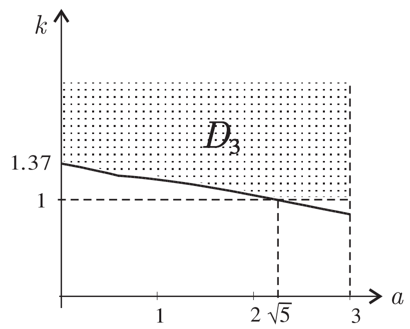
Then the sequence $\{x_n\}$ converges to a fixed point of T provided
$\frac{1}{n^2}\left[k^2a^2+\sum_{j=1}^{n-1}k^{2j}\left(\frac{k^{n-j}-1}{k-1}\right)^2-1+\varepsilon\right]<1$
or
$\frac{1}{n^2}\left[2k^2a^2+2ak^{n+1}+k^{2n}+\sum_{j=2}^{n-1}k^{2j}\left(\frac{k^{n-j}-1}{k-1}\right)^2-1+\varepsilon\right]<1.$
Since ε was arbitrarily chosen, the proof is complete. □

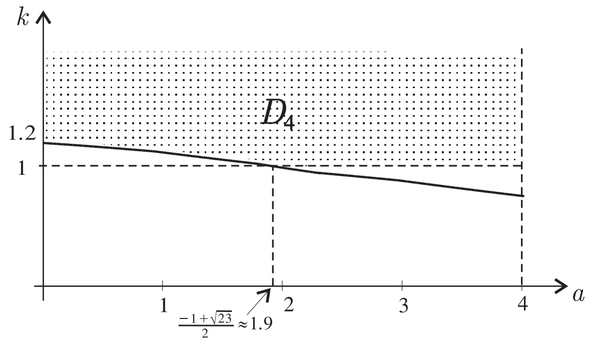
Unfortunately, inequality $(**)$ does not give any estimate of $\gamma_n^H(a)$ for $n>6$. Indeed, one can check that only for $n\le 6$ there exists $b_n\in\langle 0,n\rangle$ such that $\max\left(\gamma_n^1(a),\gamma_n^2(a)\right)>1$ for $a\in\langle 0,b_n)$ $(b_2=2$ and for $n\ge 3,\ b_n<n)$.

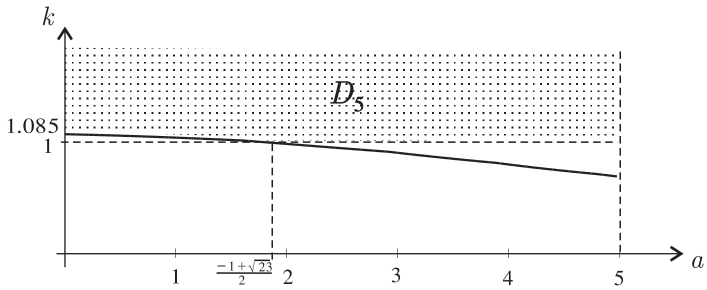
It follows from Theorem 3 that $\gamma_3^H(0)\ge 1.3666,\ \gamma_4^H(0)\ge 1.1962,\ \gamma_5^H(0)\ge 1.0849$ and $\gamma_6^H(0)\ge 1.0228$. All these evaluations are slightly better than those obtained by W. A. Kirk [6] in the general case of Banach space X. Indeed, it follows from (2) that $\gamma_3^X(0)\ge 1.3452,\ \gamma_4^X(0)\ge 1.065,\ \gamma_5^X(0)\ge 1.0351$ and $\gamma_6^X(0)\ge 1.022$.

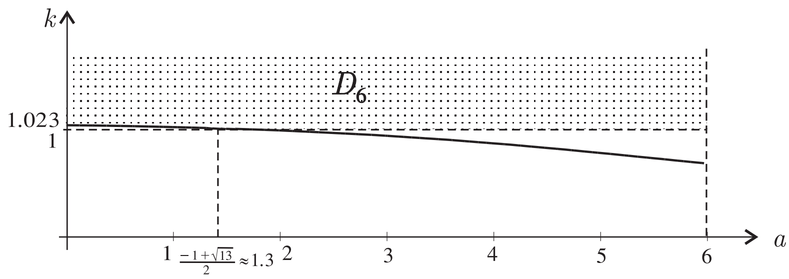
Theorem 3 allows us also to evaluate $\gamma_n^H(a)$ for some a slightly greater than 0. Using computer techniques one can sketch the lower boundaries of the sets $D_n\subset\langle 0,n)\times(1,\infty)$ such that $\left(a,\gamma_n^H(a)\right)\in D_n$, $n=2,...6$. In the following figures the thicker lines are the graphs of $k=\max\left(\gamma_n^1(a),\gamma_n^2(a)\right)$.











3. Uniformly lipschitzian rotative mappings.

Recall, that a mapping $T : C \to C$ is called uniformly k-lipschitzian if for all $m \in \mathbb{N}$ and $x, y \in C$,

$$\|T^m x - T^m y\| \leq k \|x - y\| .$$

If such a mapping is $(a, 2)$-rotative, we have exactly the same situation as in the general case of lipschitzian mappings. However, if we consider mappings which are uniformly k-lipschitzian and (a, n)-rotative with $n \geq 3$, then, instead of the inequality in the last part of the proof of Lemma 2, we get

$$\frac{k^2}{n^2} \sum_{j=1}^{n-1} \left\|x - T^j x\right\|^2 + \frac{k^2 - 1}{n^2} \sum_{1 \leq i < j \leq n-1} \left\|T^i x - T^j x\right\|^2$$
$$\leq \sum_{j=1}^{n-1} \left(\frac{k^2}{n^2} + \frac{k^2 - 1}{n^2} \cdot k^2 (n - j - 1) \right) \left\|x - T^j x\right\|^2 .$$

This evaluation and inequality $(*)$ lead to the following

Lemma 3. *Let C be a convex subset of a Hilbert space H and let $T : C \to C$ be uniformly k-lipschitzian and (a,n)-rotative, $n \geq 3$. For $x \in C$ put*

$$z = \frac{1}{n}\left(Tx + T^2x + \cdots + T^n x\right).$$

Then

$$\begin{aligned} \|z - Tz\|^2 \leq {} & \frac{k^2}{n^2}\left(a^2 + (n-2)k^2 - (n-3)\right)\|x - Tx\|^2 \\ & + \frac{k^2}{n^2}\sum_{j=2}^{n-1}\left((n-j-1)k^2 - (n-j-2)\right)\left\|x - T^j x\right\|^2 \\ & - \frac{1}{n^2}\sum_{j=1}^{n-1}\left\|T^n x - T^j x\right\|^2. \end{aligned} \tag{6'}$$

Since T is uniformly k-lipschitzian, we see that

$$\left\|x - T^j x\right\| \leq (1 + (j-1)k)\|x - Tx\|, \quad j = 1, ...n-1. \tag{7'}$$

Using additionally the condition of (a,n)-rotativeness of T we get also

$$\left\|x - T^{n-1}x\right\| \leq (a+k)\|x - Tx\|. \tag{8'}$$

It follows from (6′) and (7′) that

$$\begin{aligned} \|z - Tz\|^2 \leq {} & \frac{k^2}{n^2}\left[a^2 + (n-2)k^2 - (n-3)\right]\|x - Tx\|^2 \\ & + \frac{k^2}{n^2}\sum_{j=2}^{n-1}\left((n-j-1)k^2 - (n-j-2)\right)(1 + (j-1)k)^2\|x - Tx\|^2 \\ & - \frac{1}{n^2}\sum_{j=1}^{n-1}\left\|T^n x - T^j x\right\|^2, \end{aligned} \tag{9'}$$

while using additionally (8′) we get

$$\begin{aligned} \|z - Tz\|^2 \leq {} & \frac{k^2}{n^2}\left[a^2 + (n-2)k^2 - (n-3) + (a+k)^2\right]\|x - Tx\|^2 \\ & + \frac{k^2}{n^2}\sum_{j=2}^{n-2}\left((n-j-1)k^2 - (n-j-2)\right)(1 + (j-1)k)^2\|x - Tx\|^2 \\ & - \frac{1}{n^2}\sum_{j=1}^{n-1}\left\|T^n x - T^j x\right\|^2. \end{aligned} \tag{10'}$$

Consequently, we obtain an analogue of Theorem 3 for uniformly lipschitzian mappings in a Hilbert space.

Theorem 4. *Given an integer $n \geq 3$, let $\tilde{\gamma}_n^1(a)$ be the solution of the equation*

$$\begin{aligned} & k^2\left(a^2 + (n-2)k^2 - (n-3)\right) \\ & + k^2\sum_{j=2}^{n-1}\left((n-j-1)k^2 - (n-j-2)\right)(1 + (j-1)k)^2 = n^2 + 1, \end{aligned} \tag{11'}$$

and let $\tilde{\gamma}_n^2(a)$ *be the solution of the equation*

$$(12') \qquad \begin{aligned} &k^2\left(a^2+(n-2)k^2-(n-3)+(a+k)^2\right)\\ &+k^2\sum_{j=2}^{n-2}\left((n-j-1)k^2-(n-j-2)\right)(1+(j-1)k)^2=n^2+1. \end{aligned}$$

If $T: C \to C$, $C \subset H$, *is a uniformly* k*-lipschitzian and* (a,n)*-rotative mapping such that* $k < \max\left(\tilde{\gamma}_n^1(a), \tilde{\gamma}_n^2(a)\right)$, *then* T *has fixed points.*

If we define

$$\tilde{\gamma}_n^H(a) = \inf\{k: \text{ there is a closed and convex set } C \subset H \text{ and a fixed point free uniformly } k\text{-lipschitzian } (a,n)\text{-rotative selfmapping of } C\},$$

then, obviously, $\tilde{\gamma}_n^H(a) \geq \gamma_n^H(a)$. Nevertheless, it turns out that Theorem 4, similarly to Theorem 3, gives us an evaluation of $\tilde{\gamma}_n^H(a)$ only for $n = 3, ...6$. It is also surprising that $\max\left(\tilde{\gamma}_n^1(a), \tilde{\gamma}_n^2(a)\right) > 1$ if and only if $\max\left(\gamma_n^1(a), \gamma_n^2(a)\right) > 1$, which means that the lower boundaries of the sets $\tilde{D}_n \subset \langle 0, n) \times (1, \infty)$ such that $\left(a, \tilde{\gamma}_n^H(a)\right) \in \tilde{D}_n$ lie above the line $k = 1$ for exactly the same intervals as the lower boundaries of the sets D_n do (i.e. for $a \in \langle 0, \sqrt{5})$ when $n = 3$, for $a \in \left\langle 0, \frac{\sqrt{23}-1}{2}\right)$ when $n = 4, 5$ and for $a \in \left\langle 0, \frac{\sqrt{13}-1}{2}\right)$ when $n = 6$).

However, it follows from Theorem 4 that $\tilde{\gamma}_3^H(0) \geq 1.5447$, $\tilde{\gamma}_4^H(0) \geq 1.2418$, $\tilde{\gamma}_5^H(0) \geq 1.1429$ and $\tilde{\gamma}_6^H(0) \geq 1.0277$; and these evaluations are better than those obtained for $\gamma_n^H(0)$, $n = 3, 4, 5, 6$, from Theorem 3.

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