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CONJUGATE POINTS FOR A CONSTRAINED NONLINEAR PROGRAMMING PROBLEM

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Abstract. Conjugate point is a global concept in calculus of variations, and is a key factor of optimality conditions. In variational problems, the variable is not a vector x in R^n but a function $x(t)$. So a simple and natural question arises. Is it possible to establish a conjugate points theory for a minimization problem: minimize $f(x)$ on $x \in R^n$? In [4], the author positively answered this question. He introduced the Jacobi equation and conjugate points for it, and described optimality conditions in terms of conjugate points. In this paper, we extend it to a constrained nonlinear programming problem.

1. Introduction

In this section, we first review the classical conjugate points theory for the simplest problem in calculus of variations in brief:

$$(SP) \quad \text{Minimize} \quad \int_0^T f(t, x(t), \dot{x}(t))dt$$
$$\text{subject to} \quad x(0) = A,\ x(T) = B.$$

If $\bar{x}$ is a weak minimum for (SP), then it satisfies the Euler equation

$$\frac{d}{dt} f_{\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t)) = f_x(t, \bar{x}(t), \dot{\bar{x}}(t))$$

and the Legendre condition $f_{\dot{x}\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t)) \geq 0$. Legendre attempted to prove its inverse, that is, he expected that if a feasible solution $\bar{x}(t)$ satisfies the Euler equation and the strengthened Legendre condition: $f_{\dot{x}\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t)) > 0$, then $\bar{x}(t)$ would be a weak minimum. However, his conjecture was false. Jacobi solved this problem by introducing "conjugate points" in 1837. For a feasible solution $\bar{x}(t)$ for (SP), conjugate points are defined via the Jacobi equation:

$$\frac{d}{dt}\{\bar{f}_{\dot{x}x}(t)y(t) + \bar{f}_{\dot{x}\dot{x}}(t)\dot{y}(t)\} = \bar{f}_{xx}(t)y(t) + \bar{f}_{x\dot{x}}(t)\dot{y}(t). \tag{1.1}$$

where $\bar{f}_{\dot{x}\dot{x}}(t) := f_{\dot{x}\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t))$, etc. A point $c \in (0, T]$ is said to be *conjugate to* $t = 0$ if there exists a non-trivial solution $y(t)$ of the Jacobi equation (1.1) on $[0, c]$ and $y(0) = y(c) = 0$. Then Jacobi proved the following result:

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THEOREM **1.1.** *(Jacobi) If $\bar{x}(t)$ is a weak minimum for the simplest problem (SP) and it satisfies the strengthened Legendre condition, then there is no point conjugate to $t = 0$ on $[0, T)$. Conversely, if $\bar{x}(t)$ satisfies the Euler equation and the strengthened Legendre condition, and if there is no point conjugate to $t = 0$ on $[0, T]$, then $\bar{x}(t)$ is a weak minimum.*

Recently, conjugate point has been extended to complex extremal problems such as optimal control problems and variational problems with state constraints, see e.g. Kawasaki-Zeidan [5], Loewen-Zheng [6], Warga [8], Zeidan [9], and Zeidan-Zezza [10]-[12]. The present paper is outside of this trend. We deal with a minimization problem in a finite dimensional space:

$$
\begin{array}{llll}
(P) & \text{Minimize} & f(x) & \\
 & \text{subject to} & g_i(x) \leq 0 & i \in I := \{1, \ldots, \ell\}, \\
 & & g_i(x) = 0 & i \in J := \{\ell + 1, \ldots, m\}, \\
 & & x \in R^n, &
\end{array}
$$

where $f : R^n \to R$ and $g_i : R^n \to R$ are assumed to be twice continuously differentiable. It seems to the author that one reason why researchers have not paid much attention to conjugate points for (P) even in the unconstrained case lies in the following elementary results on the unconstrained problem:

THEOREM **1.2.** *If $\bar{x}$ is a minimum for (P), then it satisfies $f'(\bar{x}) = 0$ and $f''(\bar{x}) \geq 0$, Conversely, if $\bar{x}$ satisfies $f'(\bar{x}) = 0$ and $f''(\bar{x}) > 0$, then it is a minimum for (P), where "$\geq$" and "$>$" stand for non-negative definite and positive definite, respectively.*

Theorem 1.2 seems to assert that there is no room for conjugate points to play a role in (P). However, the author succeeded to establish a conjugate points theory based on an insight of Gelfand and Fomin [2] for an unconstrained nonlinear programming problem in [4]. The purpose of this paper is to extend it to the constrained case.

In Section 2, we review the outline of [4]. Section 3 is the main part of this paper. We define conjugate points for (P), and describe optimality conditions in terms of conjugate points. In Section 4, we give an example which is a finite-dimensional version of the shortest path problem on the unit sphere.

2. UNCONSTRAINED PROBLEM

In this section, we briefly review the conjugate points theory presented in [4], where we dealt with

$$(P_0) \qquad \text{Minimize} \quad f(x), \quad x \in R^n.$$

According to Sylvester's criterion, an $n \times n$-symmetric matrix $A = (a_{ij})$ is positive-definite if and only if its descending principal minors $|A_k|$, $(k = 1, \ldots, n)$ are positive, where

$$A_k := \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix}.$$

The following lemma shows that the determinant of any square matrix is expanded w.r.t. the descending principal minors.

LEMMA **2.1.** *([4]) For any $n \times n$-matrix $A = (a_{ij})$, its determinant is expanded as follows:*

$$|A| = \sum_{k=0}^{n-1} \sum_{\rho} \varepsilon(\rho) a_{k+1\rho(k+1)} a_{k+2\rho(k+2)} \cdots a_{n\rho(n)} |A_k| \tag{2.1}$$

where $|A_0| := 1$, $\varepsilon(\rho)$ denotes the sign of ρ,and the summation is taken over all permutations ρ on $\{k+1, \ldots, n\}$ satisfying that there is no $\ell > k$ such that ρ is closed on $\{\ell+1, \ldots, n\}$.

EXAMPLE **2.1.** *When A is a tridiagonal matrix:*

$$\begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ & & b_{n-1} & a_n \end{pmatrix}, \tag{2.2}$$

the expansion (2.1) reduces to

$$|A_k| = a_k |A_{k-1}| - b_{k-1}^2 |A_{k-2}|, \tag{2.3}$$

which coincides with (81) in [2, p.127].

DEFINITION **2.1.** *([4]) For any $n \times n$-matrix $A = (a_{ij})$, we call the recursion relation on $y_0, \ldots, y_n$*

$$y_k = \sum_{i=0}^{k-1} \sum_{\rho} \varepsilon(\rho) a_{i+1\rho(i+1)} a_{i+2\rho(i+2)} \cdots a_{k\rho(k)} y_i, \qquad k = 1, \ldots, n \tag{2.4}$$

the Jacobi equation for A. We say that k is conjugate to 1 if a solution $\{y_i\}$ of the Jacobi equation with $y_0 > 0$ changes the sign from positive to non-positive at k. Namely,

$$y_0 > 0,\ y_1 > 0,\ \ldots, y_{k-1} > 0,\ \text{and } y_k \leq 0. \tag{2.5}$$

Readers may refer to [4] regarding the reason why we call the recursion relation (2.4) the Jacobi equation.

THEOREM **2.1.** *([4]) For any $n \times n$-symmetric matrix A, $A > 0$ if and only if there is no point conjugate to 1.*

THEOREM **2.2.** *([4]) A sufficient condition for an extremal $\bar{x}$ to be a minimum for (P_0) is that there is no point conjugate to 1 concerning the Hesse matrix $f''(\bar{x})$.*

Next, we consider a necessary optimality condition for (P_0). Since the descending principal minors $|A_1|, \ldots, |A_n|$ are not enough to characterize $A \geq 0$, the situation is slightly different from the sufficiency case.

DEFINITION **2.2.** *([4]) Let $A = (a_{ij})$ be an $n \times n$ symmetric matrix, and let $1 \leq i,\ j \leq n$ be two distinct integers. Then we say that j is strictly conjugate to i if there exist a permutation σ and $1 < k \leq n$ such that $\sigma(1) = i$, $\sigma(k) = j$, and if a solution $\{y_i\}$ of the Jacobi equation (2.4) for A^σ with $y_0 > 0$ changes the sign from nonnegative to negative at k, that is,*

$$y_0 > 0, y_1 \geq 0, \ldots, y_{k-1} \geq 0, \text{ and } y_k < 0, \tag{2.6}$$

where A^σ denotes the matrix whose (i, j)-component is $(\sigma(i), \sigma(j))$-component of A.

THEOREM **2.3.** *([4]) Let A be a symmetric matrix. Then $A \geq 0$ if and only if there is no pair $1 \leq i,\ j \leq n$ of integers such that j is strictly conjugate to i.*

THEOREM **2.4.** *([4]) A necessary condition for an extremal $\bar{x}$ to be a minimum for (P_0) is that there is no pair $1 \leq i,\ j \leq n$ of integers such that j is strictly conjugate to i concerning the Hesse matrix $f''(\bar{x})$.*

3. CONSTRAINED PROBLEM

In this section, we extend the results in the previous section to the constrained case.

Let $\bar{x}$ be a feasible solution for (P), and $I(\bar{x}) := \{i \in I : g_i(\bar{x}) = 0\}$. We assume that

$$\{g_i'(\bar{x}) : i \in I(\bar{x}) \cup J\} \quad \text{are linearly independent.} \tag{3.1}$$

Then, if $\bar{x}$ is a minimum, there exists $\lambda = (\lambda_1, \ldots, \lambda_m)^T \in R^m$ such that

$$L'(\bar{x}) = 0 \tag{3.2}$$

and

$$\lambda_i \geq 0, \quad \lambda_i g_i(\bar{x}) = 0 \quad \forall i \in I, \tag{3.3}$$

where $L(x) := f(x) + \sum_{i=1}^m \lambda_i g_i(x)$. Furthermore, it holds that

$$\xi^T L''(\bar{x})\xi \geq 0 \quad \forall \xi \in R^n \quad \text{s.t.} \quad \begin{cases} g_i'(\bar{x})\xi \leq 0 & \forall i \in I(\bar{x}) \cap \{i : \lambda_i = 0\} \\ g_i'(\bar{x})\xi = 0 & \forall i \in (I(\bar{x}) \cap \{i : \lambda_i > 0\}) \cup J \end{cases} \tag{3.4}$$

Under the assumption of the strict complementarity:

$$\lambda_i > 0 \quad \forall i \in I(\bar{x}), \tag{3.5}$$

the second-order condition (3.4) reduces to

$$\xi^T L''(\bar{x})\xi \geq 0 \quad \forall \xi \in R^n \quad \text{satisfying} \quad (g_i'(\bar{x})\xi = 0 \quad \forall i \in I(\bar{x}) \cup J). \tag{3.6}$$

Conversely, if there exists $\lambda \in R^m$ such that (3.2), (3.5), and

$$\xi^T L''(\bar{x})\xi > 0 \quad \forall \xi \neq 0 \quad \text{satisfying} \quad (g_i'(\bar{x})\xi = 0 \quad \forall i \in I(\bar{x}) \cup J), \tag{3.7}$$

then $\bar{x}$ is a minimum, see Fiacco and McCormick[1].

Now, let $k := |I(\bar{x}) \cup J|$ and G' denote the $k \times n$-matrix whose row vectors are $\{g_i'(\bar{x}) : i \in I(\bar{x}) \cup J\}$. Then it follows from (3.1) that rank$G' = k$, so that G' can be divided as $G' = (B, N)$, where B is a $k \times k$-nonsingular matrix and N a

$k \times (n-k)$-matrix. Similarly, we divide $\xi \in R^n$ as $\xi^T = (y^T, z^T) \in R^k \times R^{n-k}$. Then $G'\xi = 0$ is equivalent to $y = -B^{-1}Nz$, so that

$$\xi^T L''(\bar{x})\xi = z^T(-N^T B^{-T}, I)L''(\bar{x}) \begin{pmatrix} -B^{-1}N \\ I \end{pmatrix} z. \tag{3.8}$$

Hence the following matrix is a key to describe optimality conditions in terms of conjugate points.

$$M := (-N^T B^{-T}, I)L''(\bar{x}) \begin{pmatrix} -B^{-1}N \\ I \end{pmatrix} \tag{3.9}$$

Indeed, conditions (3.6) and (3.7) are equivalent to $M \geq 0$ and $M > 0$, respectively. Therefore we get the following theorem.

THEOREM **3.1.** *A sufficient condition for a feasible solution of* (P) *be a minimum is that there exists* $\lambda \in R^m$ *such that (3.2), (3.5), and there is no point conjugate to 1 concerning* M *defined by (3.9). Conversely, if* $\bar{x}$ *is a minimum for* (P)*, then there exists* $\lambda \in R^m$ *such that (3.2), (3.3), and there is no pair* $1 \leq i,\ j \leq n$ *of integers such that* j *is strictly conjugate to* i *concerning* M*.*

4. EXAMPLE

In this section, we present an example that can be regarded as a finite-dimensional analogue to the classical shortest path problem on the unit sphere S. We compute conjugate points for the former one.

EXAMPLE **4.1.** *The original variational problem is finding a shortest path on* S *joining* $A = (1,0,0)$ *and* $B = (\cos T, \sin T, 0)$*, where* $0 < T < 2\pi$ *is given. Its finite-dimensional analogue is obtained by the following procedure:*

1. For $k = 1, \ldots, n+1$*, let* R_k *be the ring defined by* $\{(\cos \frac{kT}{n+1},\ y,\ z) \in S\}$*.*
2: Choose one point, say X_k*, on each* R_k *for* $k = 1, 2, \ldots n$*.*
3: Minimize the length of the polygonal curve joining $A, X_1, \ldots, X_n$*, and* B*.*

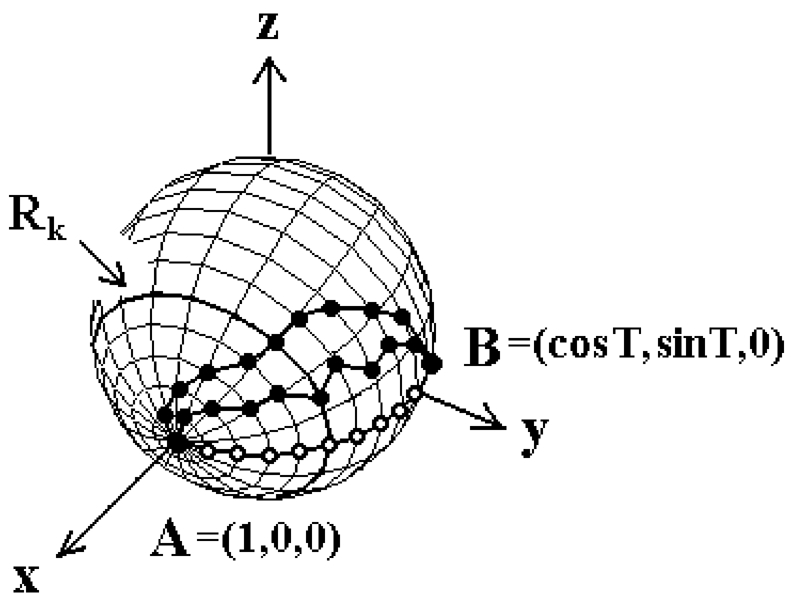

Figure 4.1

Then it is formulated as follows:

$$\begin{aligned}
\textit{Min} \quad & f(y_1,\ldots,y_n,z_1,\ldots,z_n) \\
& := \sum_{k=0}^{n} \sqrt{(y_{k+1}-y_k)^2+(z_{k+1}-z_k)^2+(\cos(k+1)\Delta t-\cos k\Delta t)^2} \\
\textit{s.t.} \quad & (y_0,z_0)=(0,0), \quad (y_{n+1},z_{n+1})=(\sin T,0) \\
& g_k(y_1,\ldots,y_n,z_1,\ldots,z_n) := y_k^2+z_k^2-\sin^2 k\Delta t=0, \quad k=1,\ldots,n,
\end{aligned}$$

where $\Delta t := T/(n+1)$*. Since the equatorial arc corresponds to*

$$(\bar{y},\bar{z}) := (\bar{y}_1,\ldots,\bar{y}_n,\bar{z}_1,\ldots,\bar{z}_n) = (\sin\Delta t,\ldots,\sin n\Delta t,0,\ldots,0),$$

it can be easily seen that

$$f'(\bar{y},\bar{z}) = 2\sin\frac{\Delta t}{2}(\sin\Delta t,\ldots,\sin n\Delta t,0,\ldots,0) \in R^{2n} \tag{4.1}$$

and

$$g'(\bar{y},\bar{z}) = 2\begin{pmatrix} \sin\Delta t & & & & 0 & \cdots & 0 \\ & \sin 2\Delta t & & & 0 & \cdots & 0 \\ & & \ddots & & \vdots & & \vdots \\ & & & \sin n\Delta t & 0 & \cdots & 0 \end{pmatrix} \tag{4.2}$$

Hence $\lambda_k = -\sin\frac{\Delta t}{2}$, *so that* $L(y,z) = f(y,z) - \sin\frac{\Delta t}{2}\sum_{k=1}^{n}(y_k^2 + z_k^2 - \sin^2 k\Delta t)$. *Furthermore, we may choose as* B *and* N *in (3.9)*

$$B = 2\begin{pmatrix} \sin\Delta t & & & \\ & \sin 2\Delta t & & \\ & & \ddots & \\ & & & \sin n\Delta t \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix},$$

respectively. Thus

$$\begin{aligned} M &= (-N^T B^{-T}, I)\begin{pmatrix} L_{yy} & L_{yz} \\ L_{xx} & L_{zz} \end{pmatrix}\begin{pmatrix} -B^{-1}N \\ I \end{pmatrix} \\ &= L_{zz} \\ &= \frac{1}{2\sin\frac{\Delta t}{2}}\begin{pmatrix} 2c & -1 & & \\ -1 & 2c & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2c \end{pmatrix} \end{aligned} \tag{4.3}$$

where $c := \cos\Delta t$. *On the other hand, it is easily seen that*

$$k\left\{\begin{vmatrix} 2c & -1 & & \\ -1 & 2c & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2c \end{vmatrix}\right. = \frac{\sin(k+1)\Delta t}{\sin\Delta t}.$$

Since $\Delta t = T/(n+1)$, *we conclude that*

(a) when $T < \pi$, *there is no point conjugate to* 1,

(b) when $T \geq \pi$, *the first number* k *satisfying* $(k+1)/(n+1) \geq \pi/T$ *is conjugate to* 1,

which matches the classical result. Additionally speaking, we gave in [4] *another finite-dimensional analogue to the same classical shortest problem. We treated it as an unconstrained extremal problem by using the spherical coordinate. The key matrix was*

$$\frac{1}{\sqrt{2(1-c)}}\begin{pmatrix} 2c & -1 & & \\ -1 & 2c & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2c \end{pmatrix}, \tag{4.4}$$

which is same with (4.3) up to constant. So we had the same conclusion.

References

[1] A.V. Fiacco and G.P. McCormick, *Nonlinear Programming: Sequential Unconstrained Minimization Techniques*, John Wiley and Sons, New York, 1968.

[2] I. M. Gelfand and S. V. Fomin, *Calculus of variations*, Prentice Hall, 1963.

[3] H. Kawasaki, Conjugate points for a minimization problem in a finite dimensional space, Kyushu University Preprint Series in Mathematics, No. 23 (1999), pp. 1-12.

[4] H. Kawasaki, A conjugate points for a minimization problem in a finite dimensional space, revised.

[5] H. Kawasaki and V. Zeidan, Conjugate points for variational problems with equality and inequality state constraints, to appear in SIAM J. Control Optim.

[6] P. D. Loewen and H. Zheng, Generalized conjugate points for optimal control problems, Nonlinear Anal., 22 (1994), pp. 771–791.

[7] G. Strang, *Linear Algebra and its Applications*, Academic Press, New York, 1976.

[8] J. Warga, A second-order Lagrangian condition for restricted control problems, J. Optim. Theory Appl., 24 (1978), pp. 475-483.

[9] V. Zeidan, The Riccati equation for optimal control problems with mixed state-control constraints: necessity and sufficiency, SIAM J. Control and Optim., 32 (1994), pp. 1297-1321.

[10] V. Zeidan and P. Zezza, The conjugate point condition for smooth control sets, J. Math. Anal. Appl., 132 (1988), pp. 572-589.

[11] V. Zeidan and P. Zezza, Coupled points in the calculus of variations and applications to periodic problems, Trans. Amer. Math. Soc., 315 (1989), pp. 323-335.

[12] V. Zeidan and P. Zezza, Coupled points in optimal control theory, IEEE Trans. Automat. Control, 36 (1991), pp. 1276-1281.

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