Journal of Nonlinear and Convex Analysis
Volume 1, Number 3, 2000, 271–286

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

ON THE TOPOLOGICAL STRUCTURE OF FIXED POINT SETS FOR ABSTRACT VOLTERRA OPERATORS ON FRÉCHET SPACES

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Abstract. In this paper we show that the solution set for certain abstract Volterra equations is an R_δ set. Application of our results to nonlinear integral equations are also presented.

1. Introduction

In this paper we study the topological structure of the solution set for the equation

$$y = F(y). \tag{1.1}$$

Here $F : E \to E$ is a nonlinear abstract Volterra operator with $E = C([0,\infty),\mathbf{R}^n)$ or $E = L^p_{loc}([0,\infty),\mathbf{R}^n)$ ($1 < p < \infty$ and $n \in \mathbf{N}$). Recall that an operator $F : C([0,\infty),\mathbf{R}^n) \to C([0,\infty),\mathbf{R}^n)$ is an abstract Volterra operator if

$$\begin{cases} \forall \epsilon > 0,\ \forall x, y \in C([0,\infty),\mathbf{R}^n), \text{ if } x(t) = y(t)\ \forall t \in [0,\epsilon] \\ \text{then } F(x)(t) = F(y)(t)\ \forall t \in [0,\epsilon]. \end{cases}$$

(When considering $E = L^p_{loc}([0,\infty),\mathbf{R}^n)$, the relevant equalities are replaced by equalities almost everywhere.)

The solution set of the given equation (1.1) coincides with the set of all fixed points of F, i.e.

$$Fix(F) = \{x \in E : x = F(x)\}.$$

We also recall that a set S in E is called an R_δ set if S is homeomorphic with a decreasing sequence of compact absolute retracts. We note that some general results relating to the structure of $Fix(F)$ can be found in [1, 3, 4, 6]. In this paper by combining some of the ideas in [1, 3] together with a trick involving the Urysohn function [2] we are able to present new results which guarantee that $Fix(F)$ is an R_δ set; in particular we are able to remove the strong compactness type assumption in [3].

2. $Fix(F)$ when $E = C([0,\infty),\mathbf{R}^n)$

Recall $C([0,\infty),\mathbf{R}^n)$ is a Fréchet space with the topology given by the complete family of seminorms $\{p_m\}_{m\geq 1}$ (here $p_m(y) = \sup_{t\in[0,m]} |y(t)|$), or, equivalently, by the distance d defined by

$$d(x,y) = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{p_m(x-y)}{1+p_m(x-y)},$$

2000 *Mathematics Subject Classification.* 45N05, 47H15.

Key words and phrases. Abstract Volterra equations, topological structure of the solution set, Urysohn function, strong compactness.

for $x, y \in C([0,\infty), \mathbf{R}^n)$.

In this section we consider an operator

$$F : C([0,\infty), \mathbf{R}^n) \to C([0,\infty), \mathbf{R}^n).$$

Define the sequence of operators $\{F_n\}_n$,

$$F_n : C([0,\infty), \mathbf{R}^n) \to C([0,\infty), \mathbf{R}^n),$$

as follows:

$$F_n(x)(t) = F(x)(r_n(t)), \text{ for } x \in C([0,\infty), \mathbf{R}^n) \text{ and } t \geq 0, \tag{2.1}$$

where

$$r_n(t) = \begin{cases} 0, \text{ if } t \in [0, 1/n]; \\ t - \frac{1}{n}, \text{ if } t > 1/n. \end{cases} \tag{2.2}$$

The main results in Sections 2 and 3 rely on the following well known result from the literature [3].

Theorem 2.1. *Let X be a closed set in a Fréchet space (E, d), and $F : X \to E$ a continuous compact operator. Assume there exists a sequence $\{U_n\}_n$ of closed convex sets in X such that*

$$\forall n \in \mathbf{N},\ 0 \in U_n, \tag{2.3}$$

$$\lim_{n\to\infty} diam(U_n) = 0, \tag{2.4}$$

and there exists a sequence $\{F_n\}_n$ of operators $F_n : X \to E$, with

$$\forall n \in \mathbf{N},\ \forall x \in X,\ F(x) - F_n(x) \in U_n, \tag{2.5}$$

and

$$I - F_n \text{ is a homeomorphism of the set } (I - F_n)^{-1}(U_n) \text{ onto } U_n \tag{2.6}$$

holding. Then $\{x \in X :\ x = F\,x\}$ is an R_δ set.

Our first result concerns the case when $F : C([0,\infty), \mathbf{R}^n) \to C([0,\infty), \mathbf{R}^n)$ is a continuous, compact operator. We use Theorem 2.1 (with $X = E = C([0,\infty), \mathbf{R}^n)$) to show that $Fix(F)$ is an R_δ set. The proof is based on ideas presented in [1, 3] (for completeness we provide the details).

Theorem 2.2. *Let $F : C([0,\infty), \mathbf{R}^n) \to C([0,\infty), \mathbf{R}^n)$ be a continuous, compact map. Also assume that the following conditions hold:*

(i) $\exists u_0 \in \mathbf{R}^n$ with $F(x)(0) = u_0$, for all $x \in C([0,\infty), \mathbf{R}^n)$;

(ii) $\forall \epsilon > 0$, $\forall x, y \in C([0,\infty), \mathbf{R}^n)$, if $x(t) = y(t)$, $\forall t \in [0, \epsilon]$, then $F(x)(t) = F(y)(t)$, $\forall t \in [0, \epsilon]$ (i.e. F is an abstract Volterra operator);

Then $Fix(F)$ is an R_δ set.

PROOF: Let $E = X = C([0,\infty), \mathbf{R}^n)$. Consider the sequence $\{F_n\}_n$ defined by (2.1)-(2.2). We show that there exists a sequence $\{U_n\}_n$ of closed convex sets in E and there exists a subsequence $\{G_n\}_n$ of $\{F_n\}_n$ such that $\{U_n\}_n$ and $\{G_n\}_n$ satisfy conditions (2.3) – (2.6) in Theorem 2.1.

First, we show that $\forall n$, $I - F_n$ is a homeomorphism from E onto E.

The continuity of $I - F_n$ follows immediately from the continuity of F and the fact that

$$\begin{aligned} p_m(F_n(x) - F_n(y)) &= \sup\{|F_n(x)(t) - F_n(y)(t)|, t \in [0, m]\} \\ &= \sup\{|F(x)(t) - F(y)(t)|, t \in [0, m - 1/n]\} \\ &\leq \sup\{|F(x)(t) - F(y)(t)|, t \in [0, m]\} \\ &= p_m(F(x) - F(y)). \end{aligned}$$

To see that $I - F_n$ is one-to-one, let $x, y \in E$ be such that $(I - F_n)(x) = (I - F_n)(y)$. Then, for $t \in [0, 1/n]$,

$$x(t) - y(t) = F(x)(r_n(t)) - F(y)(r_n(t)) = F(x)(0) - F(y)(0) = u_0 - u_0 = 0.$$

If $t \in [1/n, 2/n]$, then $t - \frac{1}{n} \in [0, 1/n]$ and

$$x(t) - y(t) = F(x)\left(t - \frac{1}{n}\right) - F(y)\left(t - \frac{1}{n}\right) = F(x)(s) - F(y)(s),$$

with $s \in [0, 1/n]$, therefore the difference is 0, because (ii) holds. Proceed by induction. Assume we know that $x(t) - y(t) = 0$ for $t \in [0, (k-1)/n]$, for some positive integer k. Since (ii) holds, this means that $F(x)(t) - F(y)(t) = 0$ for $t \in [0, (k-1)/n]$. Then, for $t \in [(k-1)/n, k/n]$,

$$x(t) - y(t) = F(x)\left(t - \frac{1}{n}\right) - F(y)\left(t - \frac{1}{n}\right) = F(x)(s) - F(y)(s),$$

with $s \in [0, (k-1)/n]$, so the difference is 0, and $I - F_n$ is one-to-one.

To see that $(I - F_n)^{-1}$ is continuous, let $\{x_j\}_j$ be a sequence in E. Let $x \in E$ be such that

$$\lim_{j\to\infty} ((x_j - F_n(x_j)) - (x - F_n(x))) = 0 \text{ in } E, \tag{2.7}$$

which implies that

$$\lim_{j\to\infty} ((x_j(t) - F_n(x_j)(t)) - (x(t) - F_n(x)(t))) = 0, \tag{2.8}$$

uniformly for t in every compact in $[0, \infty)$. We show that

$$\lim_{j\to\infty} (x_j - x) = 0 \text{ in } E.$$

For $t \in [0, 1/n]$, $F_n(x_j)(t) = F(x)(t) = u_0$, so

$$(x_j(t) - F_n(x_j)(t)) - (x(t) - F_n(x)(t)) = x_j(t) - x(t),$$

and since (2.8) holds uniformly for $t \in [0, 1/n]$, we have

$$\lim_{j\to\infty} \sup\{|x_j(t) - x(t)| : t \in [0, 1/n]\} = 0. \tag{2.9}$$

Now if $\overline{x}$ is the extension of x defined by

$$\overline{x} = \begin{cases} x(t) \text{ for } t \in [0, 1/n] \\ x\left(\frac{1}{n}\right) \text{ for } t > \frac{1}{n} \end{cases}$$

and $\overline{x}_j$ is the extension of x_j defined similarly, then, by (2.9) and the continuity of F, we have

$$\lim_{j\to\infty}(F(\overline{x}_j) - F(\overline{x})) = 0 \text{ in } E. \tag{2.10}$$

In particular, for $t \in [1/n, 2/n]$, (2.10) and the definition of F_n give

$$\begin{aligned} &\limsup\nolimits_{j\to\infty}\{|F_n(x_j)(t) - F_n(x)(t)| : t \in [1/n, 2/n]\} \\ &\quad = \limsup\nolimits_{j\to\infty}\{|F(\overline{x}_j)(t) - F(\overline{x})(t)| : t \in [0, 1/n]\} = 0. \end{aligned} \tag{2.11}$$

Now (2.8) and (2.11) imply

$$\limsup_{j\to\infty}\{|x_j(t) - x(t)| : t \in [1/n, 2/n]\} = 0. \tag{2.12}$$

By induction, assume that for some k,

$$\limsup_{j\to\infty}\{|x_j(t) - x(t)| : t \in [(k-1)/n, k/n]\} = 0. \tag{2.13}$$

Repeating the extension argument above, this also implies

$$\limsup_{j\to\infty}\{|F_n(x_j)(t) - F_n(x)(t)| : t \in [k/n, (k+1)/n]\} = 0. \tag{2.14}$$

Now (2.8) and (2.14) imply

$$\limsup_{j\to\infty}\{|x_j(t) - x(t)| : t \in [k/n, (k+1)/n]\} = 0.$$

Thus, we showed by induction that

$$\limsup_{j\to\infty}\{|x_j(t) - x(t)| : t \in [(k-1)/n, k/n]\} = 0, \forall k,$$

which implies that $\lim_{j\to\infty} p_m(x_j - x) = 0, \forall m$, or $\lim_{j\to\infty}(x_j - x) = 0$ in E.

To prove the surjectivity of $I - F_n$ let $z \in E$ and consider the equation

$$x = F_n(x) + z. \tag{2.15}$$

Let $x \in E$ be defined by induction as follows:

$$x(t) = u_0 + z(t), \text{ for } t \in [0, 1/n]. \tag{2.16}$$

This x is continuous on $[0, 1/n]$ because z is continuous on $[0, \infty)$ (by "x is continuous on $[0, 1/n]$" we mean that x is continuous on $(0, 1/n)$, it is continuous at 0 from the right, and it is continuous at $1/n$ from the left). Now let x_1 be a continuous extension of x from $[0, 1/n]$ to $[0, \infty)$, given by Tietze's theorem. Let

$$x(t) = F(x)\left(t - \frac{1}{n}\right) + z(t), \text{ for } t \in (1/n, 2/n]. \tag{2.17}$$

Since $t - \frac{1}{n} \in (0, 1/n]$, we have $F(x)\left(t - \frac{1}{n}\right) = F(x_1)\left(t - \frac{1}{n}\right)$, and therefore x is continuous on $(1/n, 2/n]$, because $F(x_1)$ and z are continuous on $[0, \infty)$. To see that x is also continuous at $1/n$, we have

$$\begin{aligned}\lim_{t\to 1/n, t>1/n} x(t) &= \lim_{t\to 1/n, t>1/n}\left(F(x_1)\left(t-\frac{1}{n}\right)+z(t)\right)\\ &= F(x_1)(0)+z\left(\frac{1}{n}\right)\\ &= F(x)(0)+z\left(\frac{1}{n}\right)\\ &= u_0+z\left(\frac{1}{n}\right)\end{aligned}$$

from the continuity of $F(x_1)$ and z, and

$$\lim_{t\to 1/n, t<1/n} x(t) = u_0+z\left(\frac{1}{n}\right)$$

from (2.16). Now x is constructed on $[0,2/n]$ and is continuous on $[0,2/n]$. By induction, assume x is defined and continuous on $[0,k/n]$. Let x_k be a continuous extension of x from $[0,k/n]$ to $[0,\infty)$, given by Tietze's theorem. Define

$$x(t) = F(x)\left(t-\frac{1}{n}\right)+z(t), \text{ for } t\in (k/n,(k+1)/n], \tag{2.18}$$

Since $t-\frac{1}{n}\in(0,k/n]$, we have $F(x)\left(t-\frac{1}{n}\right) = F(x_k)\left(t-\frac{1}{n}\right)$, and therefore x is continuous on $(k/n,(k+1)/n]$, because $F(x_k)$ and z are continuous on $[0,\infty)$. To see that x is also continuous at k/n, we have

$$\begin{aligned}\lim_{t\to k/n, t>k/n} x(t) &= \lim_{t\to k/n, t>k/n}\left(F(x_k)\left(t-\frac{1}{n}\right)+z(t)\right)\\ &= F(x_k)\left(\frac{k-1}{n}\right)+z\left(\frac{k}{n}\right)\end{aligned}$$

from the continuity of $F(x_k)$ and z, and

$$\lim_{t\to k/n, t<k/n} x(t) = F(x_k)\left(\frac{k-1}{n}\right)+z\left(\frac{k}{n}\right)$$

from (2.18). Now this x, constructed inductively, satisfies (2.18) and is continuous on $[0,\infty)$.

Now since E is a Fréchet space there exists a sequence $\{U_n\}_n$ of closed, convex neighborhoods of 0 with

$$U_n\subseteq \overline{B}\left(0,\frac{1}{n}\right);$$

here $\overline{B}\left(0,\frac{1}{n}\right)$ is the closed ball with center 0 and radius $\frac{1}{n}$. Note

$$diam\, U_n \le \frac{2}{n}.$$

Also since U_n is a neighborhood of 0,

$$\forall n\in\mathbf{N},\ \exists \epsilon_n>0 \text{ with } \overline{B}(0,\epsilon_n)\subseteq U_n \tag{2.19}$$

and $\lim_{n\to\infty}\epsilon_n = 0$ (Without loss of generality assume $\epsilon_{n+1}\le\epsilon_n$ for all $n\in\mathbf{N}$).

We now show that F_n converges to F in E, uniformly for $x \in E$. Since F is compact, $F(E)$ is relatively compact in $C([0,\infty),\mathbf{R}^n)$. Then $F(E)$ is relatively compact in $C([0,m],\mathbf{R}^n)$, $\forall m$. By the Arzelà-Ascoli theorem, $F(E)$ is bounded and equicontinuous in $C([0,m],\mathbf{R}^n)$, $\forall m$, i.e. $\forall m, \exists M_m > 0$ such that $p_m(F(x)) \leq M_m, \forall x \in E$, and $\forall \epsilon > 0, \exists \delta_m > 0$ such that if $t, s \in [0,m]$ with $|t-s| < \delta_m$, then

$$|F(x)(t) - F(x)(s)| < \epsilon, \forall x \in E. \tag{2.20}$$

Let $\epsilon > 0$ and let n be sufficiently large, such that $\frac{1}{n} < \delta_m$. We have

$$F_n(x)(t) - F(x)(t) = \begin{cases} F(x)(0) - F(x)(t), \text{ if } t \in [0,1/n] \\ F(x)\left(t - \frac{1}{n}\right) - F(x)(t), \text{ if } t \geq \frac{1}{n}, \end{cases} \tag{2.21}$$

therefore $|F_n(x)(t) - F(x)(t)| = |F(x)(t) - F(x)(s)|$, where

$$s = \begin{cases} 0, \text{ if } t \in [0,1/n) \\ t - \frac{1}{n}, \text{ if } t \in [1/n,\infty), \end{cases}$$

so $|t-s| < \frac{1}{n} < \delta_m$. Then (2.20) and (2.21) show that for n sufficiently large, we have

$$|F_n(x)(t) - F(x)(t)| < \epsilon, \forall t \in [0,m], \forall x \in E, \forall m,$$

or

$$p_m(F_n(x) - F(x)) < \epsilon, \forall x \in E, \forall m. \tag{2.22}$$

Thus F_n converges to F in $C([0,\infty),\mathbf{R}^n)$, uniformly for $x \in E$. Consequently

$$\lim_{n\to\infty} \sup\{d(F_n(x), F(x)) : \; x \in E\} \;=\; 0. \tag{2.23}$$

To finish the proof, we show that there exists a subsequence $\{G_n\}_n$ of $\{F_n\}_n$, such that for all n

$$G_n(x) - F(x) \in U_n, \forall x \in E, \tag{2.24}$$

since in that case $\{U_n\}_n$ and $\{G_n\}_n$ satisfy the hypotheses in Theorem 2.1, and so $Fix(F)$ is an R_δ set. For this, apply (2.23) to construct $\{G_n\}_n$ inductively. There exists n_1 such that for $n \geq n_1$

$$d(F_n(x), F(x)) < \epsilon_1, \forall x \in E.$$

There exists $n_2 > n_1$ such that for $n \geq n_2$

$$d(F_n(x), F(x)) < \epsilon_2, \forall x \in E.$$

By induction, there exists $n_k > n_{k-1}$ such that for $n \geq n_k$

$$d(F_n(x), F(x)) < \epsilon_k, \forall x \in E.$$

Now define $G_k := F_{n_k}, \forall k$, and $\{G_k\}_k$ is such that for all k

$$d(G_k(x), F(x)) < \epsilon_k, \forall x \in E.$$

This together with (2.19) guarantees that (2.24) holds. □

We remark that in application

$$F : C([0,\infty),\mathbf{R}^n) \to C([0,\infty),\mathbf{R}^n)$$

is usually continuous, and completely continuous but it is rarely compact. As a result we would like to relax the compactness assumption on F in Theorem 2.2. In applications we usually encounter the nonlinear operator equation

$$y(t) = L\,Fy(t) \quad \text{for} \quad t \in [0,\infty); \tag{2.25}$$

here L is an affine map. We will assume the following conditions are satisfied:

$$L\,F : C([0,\infty),\mathbf{R}^n) \to C([0,\infty),\mathbf{R}^n) \tag{2.26}$$

$$\exists u_0 \in \mathbf{R}^n \quad \text{with} \quad L\,F(x)(0) = u_0, \quad \text{for all} \quad x \in C([0,\infty),\mathbf{R}^n) \tag{2.27}$$

$$\begin{cases} \forall \epsilon > 0,\ \forall x, y \in C([0,\infty),\mathbf{R}^n), \ \text{ if } \ x(t) = y(t)\ \forall t \in [0,\epsilon] \\ \text{then } \ L\,F\,(x)(t) = L\,F\,(y)(t)\ \forall t \in [0,\epsilon] \end{cases} \tag{2.28}$$

and

$$\begin{cases} \exists \text{ a continuous function } \ \phi : [0,\infty) \to [0,\infty) \\ \text{such that } \ |y(t)| \le \phi(t) \ \text{ for } \ t \in [0,\infty), \ \text{ for} \\ \text{all solutions } \ y \in C([0,\infty),\mathbf{R}^n) \ \text{ to (2.25).} \end{cases} \tag{2.29}$$

Let $\epsilon > 0$ be given and let $\tau_\epsilon : \mathbf{R}^n \to [0,1]$ be the Urysohn function for

$$\big(\overline{B}(0,1)\,,\ \mathbf{R}^n \setminus B(0,1+\epsilon)\big)$$

such that

$$\tau_\epsilon(x) = 1 \ \text{ if } \ |x| \le 1 \quad \text{and} \quad \tau_\epsilon(x) = 0 \ \text{ if } \ |x| \ge 1+\epsilon.$$

Let the operator F_ϵ be defined by

$$F_\epsilon\,(y)(t) = \tau_\epsilon\left(\frac{y(t)}{\phi(t)+1}\right) F(y)(t); \ \text{ here } \ y \in C([0,\infty),\mathbf{R}^n).$$

Consider the operator equation

$$y(t) = L\,F_\epsilon\,y(t) \quad \text{for} \quad t \in [0,\infty). \tag{2.30}$$

Let S_F denote the solution set of (2.25) and S_{F_ϵ} the solution set of (2.30). Our next result will be particularly useful in applications, as we will see in Section 4.

Theorem 2.3. *Suppose* (2.26)–(2.29) *hold. Let* $\epsilon > 0$ *be given and assume the following conditions are satisfied:*

$$\begin{cases} |w(t)| \le \phi(t) \ \textit{ for } \ t \in [0,\infty), \ \textit{ for any possible} \\ \textit{solution } \ w \in C([0,\infty),\mathbf{R}^n) \ \textit{ to (2.30)} \end{cases} \tag{2.31}$$

and

$$L\,F_\epsilon : C([0,\infty),\mathbf{R}^n) \to C([0,\infty),\mathbf{R}^n) \ \textit{ is continuous and compact.} \tag{2.32}$$

Then S_F *is an* R_δ *set.*

Remark. If, for example,

$$\begin{cases} L : C([0,\infty),\mathbf{R}^n) \to C([0,\infty),\mathbf{R}^n) \ \text{ is completely continuous} \\ \text{and } \ F \ \text{ maps bounded sets in } \ C([0,\infty),\mathbf{R}^n) \ \text{ into} \\ \text{bounded sets in } \ C([0,\infty),\mathbf{R}^n) \end{cases}$$

then it is clear that

$$L\,F_\epsilon : C([0,\infty),\mathbf{R}^n) \to C([0,\infty),\mathbf{R}^n) \ \text{ is compact.}$$

PROOF: Notice (2.29) and (2.31) guarantee that $S_F = S_{F_\epsilon}$; to see this notice if $y \in S_{F_\epsilon}$ then (2.31) implies that $|y(t)| \le \phi(t)$ for $t \in [0,\infty)$, so

$$\tau_\epsilon\left(\frac{y(t)}{\phi(t)+1}\right) = 1 \quad \text{since} \quad \left|\frac{y(t)}{\phi(t)+1}\right| \le 1,$$

and so we have $y(t) = L\,F_\epsilon y(t) = L\,F y(t)$ i.e. $y \in S_F$. Next notice Theorem 2.2 guarantees that S_{F_ϵ} is an R_δ set. □

3. $Fix(F)$ WHEN $E = L^p_{loc}([0,\infty), \mathbf{R}^n)$, $1 < p < \infty$

Recall $L^p_{loc}([0,\infty), \mathbf{R}^n)$ is a Fréchet space with the topology given by the complete family of seminorms $\{p_m\}_{m\ge 1}$ (here $p_m(y) = \left(\int_0^m |y(t)|^p\,dt\right)^{\frac{1}{p}}$), or, equivalently, by the distance d defined by

$$d(x,y) = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{p_m(x-y)}{1+p_m(x-y)},$$

for $x, y \in L^p_{loc}([0,\infty), \mathbf{R}^n)$.

In this section we consider an operator

$$F : L^p_{loc}([0,\infty), \mathbf{R}^n) \to L^p_{loc}([0,\infty), \mathbf{R}^n).$$

Define the sequence of operators $\{F_n\}_n$,

$$F_n : L^p_{loc}([0,\infty), \mathbf{R}^n) \to L^p_{loc}([0,\infty), \mathbf{R}^n),$$

as follows:

$$F_n(x)(t) = F(x)(r_n(t)), \text{ for } x \in L^p_{loc}([0,\infty), \mathbf{R}^n) \text{ and } t \ge 0, \tag{3.1}$$

where

$$r_n(t) = \begin{cases} 0, \text{ if } t \in [0, 1/n]; \\ t - \frac{1}{n}, \text{ if } t > 1/n. \end{cases} \tag{3.2}$$

Theorem 3.1. *Let $F : L^p_{loc}([0,\infty), \mathbf{R}^n) \to L^p_{loc}([0,\infty), \mathbf{R}^n)$ be a continuous, compact map. Also assume that the following conditions hold:*
(i) $\exists u_0 \in \mathbf{R}^n$ with $F(x)(0) = u_0$, for all $x \in L^p_{loc}([0,\infty), \mathbf{R}^n)$;
(ii) $\forall \epsilon > 0$, $\forall x, y \in L^p_{loc}([0,\infty), \mathbf{R}^n)$, if $x(t) = y(t)$ for a.e. $t \in [0,\epsilon]$, then $F(x)(t) = F(y)(t)$, for a.e. $t \in [0,\epsilon]$ (i.e. F is an abstract Volterra operator);

Then $Fix(F)$ is an R_δ set.

PROOF: Let $E = X = L^p_{loc}([0,\infty), \mathbf{R}^n)$. Consider the sequence $\{F_n\}_n$ defined by $(3.1) - (3.2)$. We show that there exists a sequence $\{U_n\}_n$ of closed convex sets in E and there exists a subsequence $\{G_n\}_n$ of $\{F_n\}_n$ such that $\{U_n\}_n$ and $\{G_n\}_n$ satisfy conditions (2.3) – (2.6) in Theorem 2.1.

First, we show that $\forall n$, $I - F_n$ is a homeomorphism from E onto E. To see that $I - F_n$ is continuous, let $\epsilon > 0$. Since F is continuous, $\exists \delta > 0$ such that if $x, y \in E$

with $p_m(x-y)<\delta$, then $p_m(F(x)-F(y))<\epsilon$, $\forall m$. Then, if $x,y\in E$ are such that $p_m(x-y)<\delta$, $\forall m$, we also have

$$\begin{aligned} p_m^p(F_n(x)-F_n(y)) &= \int_0^m |F_n(x)(t)-F_n(y)(t)|^p dt \\ &= \int_{1/n}^m |F_n(x)(t)-F_n(y)(t)|^p dt \\ &= \int_{1/n}^m \left|F(x)\left(t-\frac{1}{n}\right)-F(y)\left(t-\frac{1}{n}\right)\right|^p dt \\ &= \int_0^{m-1/n} |F(x)(t)-F(y)(t)|^p dt \\ &\le \int_0^m |F(x)(t)-F(y)(t)|^p dt \\ &= p_m^p(F(x)-F(y))<\epsilon^p,\ \forall m. \end{aligned}$$

Therefore, F_n is continuous, and thus $I-F_n$ is continuous.

The fact that $I-F_n$ is one-to-one follows exactly like in the proof of Theorem 2.2.

To see that $(I-F_n)^{-1}$ is continuous, let $\{x_j\}_j$ be a sequence in E. Let $x\in E$ be such that

$$\lim_{j\to\infty} p_m((x_j-F_n(x_j))-(x-F_n(x)))=0, \forall m. \tag{3.3}$$

For $t\in[0,1/n]$ we have $F_n(x_j)(t)=F(x_j)(0)=u_0=F_n(x)(t)$, so (3.3) implies that

$$\lim_{j\to\infty} p_m((x_j-x)\chi_{[0,1/n]})=0, \forall m,$$

where $\chi_{[0,1/n]}$ is the characteristic function of the interval $[0,1/n]$. Since F is continuous, it also follows that

$$\lim_{j\to\infty} p_m((F(x_j)-F(x))\chi_{[0,1/n]})=0, \forall m. \tag{3.4}$$

For $t\in[1/n,2/n]$, we have $t-\frac{1}{n}\in[0,1/n]$ and $F_n(x_j)(t)=F(x_j)\left(t-\frac{1}{n}\right)$, $F_n(x)(t)=F(x)\left(t-\frac{1}{n}\right)$, therefore (3.3) and (3.4) give

$$\lim_{j\to\infty} p_m((F(x_j)-F(x))\chi_{[0,2/n]})=0, \forall m,$$

and therefore

$$\lim_{j\to\infty} p_m((x_j-x)\chi_{[0,2/n]})=0, \forall m.$$

Assume by induction that for some positive integer k we have

$$\lim_{j\to\infty} p_m((x_j-x)\chi_{[0,(k-1)/n]})=0, \forall m. \tag{3.5}$$

By the continuity of F, we also have

$$\lim_{j\to\infty} p_m((F(x_j)-F(x))\chi_{[0,(k-1)/n]})=0, \forall m. \tag{3.6}$$

Then, as above, (3.5) and (3.6) imply that

$$\lim_{j\to\infty} p_m((x_j-x)\chi_{[0,k/n]})=0, \forall m. \tag{3.7}$$

Therefore, (3.7) is true for all k, which shows that

$$\lim_{j\to\infty} p_m(x_j - x) = 0, \forall m,$$

and therefore $(I - F_n)^{-1}$ is continuous.

To prove the surjectivity of $I - F_n$ let $z \in E$ and consider the equation

$$x = F_n(x) + z. \tag{3.8}$$

Let $x \in E$ be defined by induction as follows:

$$x(t) = u_0 + z(t), \text{ for } t \in [0, 1/n),$$
$$x(t) = F(x)\left(t - \frac{1}{n}\right) + z(t), \text{ for } t \in [1/n, 2/n).$$

Given $x(t)$ for $t \in [(k-1)/n, k/n)$, define

$$x(t) = F(x)\left(t - \frac{1}{n}\right) + z(t), \text{ for } t \in [k/n, (k+1)/n).$$

Then this x is in E and it satisfies the equation $x = F_n(x) + z$.

Construct $\{U_n\}_n$ as in Theorem 2.2, and the proof of Theorem 3.1 is complete if we show that F_n converges to F uniformly for $x \in E$. Since F is compact, $F(E)$ is relatively compact in $L^p_{loc}([0,\infty), \mathbf{R}^n)$, which implies that $F(E)$ is also relatively compact in $L^p([0,m], \mathbf{R}^n)$, $\forall m$. Therefore, $\forall m$, $\exists M_m > 0$ such that

$$p_m(F(x)) \le M_m, \forall x \in E \tag{3.9}$$

and

$$\int_0^m |F(x)(t+\tau) - F(x)(t)|^p dt \to 0 \text{ as } \tau \to 0, \text{ uniformly for } x \in E. \tag{3.10}$$

Thus we have

$$\begin{aligned} p_m^p(F_n(x) - F(x)) &= \int_0^m |F_n(x)(t) - F(x)(t)|^p dt \\ &= \int_0^{1/n} |F(x)(0) - F(x)(t)|^p dt \\ &+ \int_{1/n}^m \left| F(x)\left(t - \frac{1}{n}\right) - F(x)(t)\right|^p dt. \end{aligned}$$

The first term is bounded above by $\frac{1}{n}(|u_0|^p + M_m^p)$, therefore its limit is 0 as $n \to \infty$, uniformly for $x \in E$. The second term has the limit 0 as $n \to \infty$, uniformly for $x \in E$, since (3.10) holds. Therefore,

$$\lim_{n\to\infty} p_m(F_n(x) - F(x)) = 0, \text{ uniformly for } x \in E, \forall m,$$

and the proof is complete. □

Next consider the operator equation

$$y(t) = L\,F y(t) \quad \text{for a.e.} \quad t \in [0, \infty); \tag{3.11}$$

here L is an affine map. We will assume the following conditions are satisfied:

$$L\,F : L^p_{loc}([0,\infty), \mathbf{R}^n) \to L^p_{loc}([0,\infty), \mathbf{R}^n) \tag{3.12}$$

$$\exists u_0 \in \mathbf{R}^n \text{ with } L\,F(x)(0) = u_0, \text{ for all } x \in L^p_{loc}([0,\infty),\mathbf{R}^n) \tag{3.13}$$

$$\left\{ \begin{array}{l} \forall \epsilon > 0,\ \forall x, y \in L^p_{loc}([0,\infty),\mathbf{R}^n), \text{ if } x(t) = y(t) \text{ for a.e.} \\ t \in [0,\epsilon] \text{ then } L\,F\,(x)(t) = L\,F\,(y)(t) \text{ for a.e. } t \in [0,\epsilon] \end{array} \right. \tag{3.14}$$

and

$$\left\{ \begin{array}{l} \exists \text{ a continuous function } \phi : [0,\infty) \to [0,\infty) \\ \text{such that } \int_0^t |y(s)|^p\,ds \le \phi(t) \text{ for } t \in [0,\infty), \text{ for} \\ \text{all solutions } y \in L^p_{loc}([0,\infty),\mathbf{R}^n) \text{ to (3.11).} \end{array} \right. \tag{3.15}$$

Let $\epsilon > 0$ be given and let $\tau_\epsilon : \mathbf{R}^n \to [0,1]$ be as in Section 2. Let the operator F_ϵ be defined by

$$F_\epsilon\,(y)(t) = \tau_\epsilon \left(\frac{\int_0^t |y(s)|^p\,ds}{\phi(t)+1} \right) F(y)(t); \quad \text{here } y \in L^p_{loc}([0,\infty),\mathbf{R}^n).$$

Consider the operator equation

$$y(t) = L\,F_\epsilon\,y(t) \quad \text{for a.e. } t \in [0,\infty). \tag{3.16}$$

Let S_F denote the solution set of (3.11) and S_{F_ϵ} the solution set of (3.16).

Theorem 3.2. *Suppose* (3.12)–(3.15) *hold. Let $\epsilon > 0$ be given and assume the following conditions are satisfied:*

$$\left\{ \begin{array}{l} \int_0^t |w(s)|^p\,ds \le \phi(t) \text{ for } t \in [0,\infty), \text{ for any possible} \\ \text{solution } w \in L^p_{loc}([0,\infty),\mathbf{R}^n) \text{ to (3.16)} \end{array} \right. \tag{3.17}$$

and

$$\left\{ \begin{array}{l} L\,F_\epsilon : L^p_{loc}([0,\infty),\mathbf{R}^n) \to L^p_{loc}([0,\infty),\mathbf{R}^n) \\ \text{is continuous and compact.} \end{array} \right. \tag{3.18}$$

Then S_F is an R_δ set.

PROOF: Notice (3.15) and (3.17) guarantee that $S_F = S_{F_\epsilon}$; to see this notice if $y \in S_{F_\epsilon}$ then $\int_0^t |y(s)|^p\,ds \le \phi(t)$ for $t \in [0,\infty)$, so

$$\tau_\epsilon \left(\frac{\int_0^t |y(s)|^p\,ds}{\phi(t)+1} \right) = 1,$$

and so $y \in S_F$. Theorem 3.1 guarantees that S_{F_ϵ} is an R_δ set. □

4. Applications

In this section we will use the theorems in Section 2 to establish existence results for the integral equation

$$y(t) = h(t) + \int_0^t k(t,s)\,g(s,y(s))ds \quad \text{for } t \in [0,\infty). \tag{4.1}$$

Throughout this section we assume $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. We begin by presenting a result for (4.1) based on Theorem 2.2.

Theorem 4.1. *Assume that $g : [0,\infty) \times \mathbf{R} \to \mathbf{R}$ is a locally integrably bounded L^q-Carathéodory function, i.e. for each compact subinterval I of $[0,\infty)$, the following three conditions hold:*

$$\text{the map } t \mapsto g(t,y) \text{ is measurable for all } y \in \mathbf{R}, \tag{4.2}$$

$$\text{the map } y \mapsto g(t,y) \text{ is continuous for almost all } t \in I, \tag{4.3}$$

$$\begin{cases} \text{there exists } \mu \in L^q(I,\mathbf{R}) \text{ such that} \\ |g(t,y)| \le \mu(t), \text{ for almost all } t \in I. \end{cases} \tag{4.4}$$

Suppose also that

$$h \in C([0,\infty),\mathbf{R}), \tag{4.5}$$

$$k_t(s) = k(t,s) \in L^p([0,t],\mathbf{R}), \forall t \in [0,m], \forall m \in \mathbf{N}, \tag{4.6}$$

and

$$\begin{cases} \forall t,t' \in [0,\infty), \\ \int_0^{t^*} |k_t(s) - k_{t'}(s)|^p ds \to 0 \text{ as } t \to t', \\ \text{where } t^* = \min\{t,t'\} \end{cases} \tag{4.7}$$

hold. Then the solution set of equation (4.1) *is an R_δ set.*

PROOF: Let $E = C([0,\infty),\mathbf{R})$, and let $F : E \to E$ be defined by

$$F(y)(t) = h(t) + \int_0^t k(t,s)\, g(s,y(s))ds \text{ for } t \in [0,\infty).$$

From [5], conditions (4.2) – (4.7) guarantee that F is well defined, F is a Volterra operator such that $F(y)(0) = h(0)$, $\forall y \in C([0,\infty),\mathbf{R})$, and the restriction $F : C([0,m],\mathbf{R}) \to C([0,m],\mathbf{R})$ is continuous. In fact, $F : E \to E$ is continuous, because if $\{y_j\}_{j\in\mathbf{N}}$ is a sequence in E and $y_0 \in C([0,\infty),\mathbf{R})$ is such that $y_j \to y_0$ in $C([0,\infty),\mathbf{R})$ as $j \to \infty$, then $y_j \to y_0$ in $C([0,m],\mathbf{R})$ as $j \to \infty$, for all m. Since $F : C([0,m],\mathbf{R}) \to C([0,m],\mathbf{R})$ is continuous, we then have that $F(y_j) \to F(y_0)$ in $C([0,m],\mathbf{R})$ as $j \to \infty$, for all m. This implies that $F(y_j) \to F(y_0)$ in $C([0,\infty),\mathbf{R})$ as $j \to \infty$.

We next show $F : E \to E$ is compact. Let $\{y_j\}_{j\in\mathbf{N}}$ be a sequence in E and consider the sequence $\{F(y_j)\}_{j\in\mathbf{N}}$ in $F(E)$. The restriction $F : C([0,m],\mathbf{R}) \to C([0,m],\mathbf{R})$, is compact, so $F(E|_{[0,m]})$ is relatively compact in $C([0,m],\mathbf{R})$; here $E|_{[0,m]} = \{y|_{[0,m]} : y \in E\}$. For $m = 1$, there exists a subsequence N_1 of $\mathbf{N}$, and there exists a $z_1 \in C([0,1],\mathbf{R})$, such that

$$F(y_j)|_{[0,1]} \to z_1 \text{ in } C([0,1],\mathbf{R}) \text{ as } j \to \infty \text{ in } N_1.$$

Now consider the sequence $\{F(y_j)\}_{j\in N_1}$, restricted to $[0,2]$. Since $F(E|_{[0,2]})$ is relatively compact in $C([0,2],\mathbf{R})$, there exists a subsequence N_1 of N_2, and there exists a $z_2 \in C([0,2],\mathbf{R})$, such that

$$F(y_j)|_{[0,2]} \to z_2 \text{ in } C([0,2],\mathbf{R}) \text{ as } j \to \infty \text{ in } N_2.$$

In addition,

$$z_2|_{[0,1]} = z_1 \text{ on } [0,1].$$

By induction, assume the sequence $\{F(y_j)\}_{j \in N_k}$ and $z_k \in C([0,k],\mathbf{R})$ are found such that $N_k \subseteq N_{k-1} \subseteq ... \subseteq N_1 \subseteq \mathbf{N}$,

$$F(y_j)|_{[0,k]} \to z_k \text{ in } C([0,k],\mathbf{R}) \text{ as } j \to \infty \text{ in } N_k,$$

and

$$z_k|_{[0,1]} = z_{k-1} \text{ on } [0,k-1].$$

Since $F(E|_{[0,k+1]})$ is relatively compact in $C([0,k+1],\mathbf{R})$, there exists a subsequence N_{k+1} of N_k, and there exists a $z_{k+1} \in C([0,k+1],\mathbf{R})$, such that

$$F(y_j)|_{[0,k+1]} \to z_{k+1} \text{ in } C([0,k+1],\mathbf{R}) \text{ as } j \to \infty \text{ in } N_{k+1}.$$

In addition,

$$z_{k+1}|_{[0,k]} = z_k \text{ on } [0,k].$$

Now define $z \in C[0,\infty)$ by

$$z(t) = z_k(t),\ t \in [k-1,k),\ k = 1,2,\dots .$$

The induction above shows that the sequence $\{F(y_j)\}_{j \in \mathbf{N}}$ contains a subsequence which converges in $C([0,\infty),\mathbf{R})$ to $z \in C[0,\infty)$. Therefore $F(E)$ is relatively compact in $C([0,\infty),\mathbf{R})$, and the operator $F : E \to E$ is compact. Now apply Theorem 2.2. $\square$

In Theorem 4.1 notice assumption (4.4) is very restrictive. In our next theorem we remove this "global" condition and replace with a "local" one. Our proof is based on Theorem 2.3.

Theorem 4.3. *Assume that*

$$g : [0,\infty) \times \mathbf{R} \to \mathbf{R}$$

is a locally L^q-Carathéodory function, i.e. for each compact subinterval I of $[0,\infty)$, the following three conditions hold:

(4.8) $$\text{the map } t \mapsto g(t,y) \text{ is measurable for all } y \in \mathbf{R},$$

(4.9) $$\text{the map } y \mapsto g(t,y) \text{ is continuous for almost all } t \in I,$$

(4.10) $$\begin{cases} \text{for all } r > 0 \text{ there exists } \mu_r \in L^q(I,\mathbf{R}) \text{ such that} \\ |y| < r \text{ implies that } |g(t,y)| \le \mu_r(t), \text{ for almost} \\ \text{all } t \in I. \end{cases}$$

Suppose also that

(4.11) $$h \in BC([0,\infty),\mathbf{R}),$$

(4.12) $$k_t(s) = k(t,s) \in L^p([0,t],\mathbf{R}), \forall t \in [0,m], \forall m \in \mathbf{N},$$

(4.13) $$\begin{cases} \forall t,t' \in [0,\infty), \\ \int_0^{t^*} |k_t(s) - k_{t'}(s)|^p ds \to 0 \text{ as } t \to t', \\ \text{where } t^* = \min\{t,t'\}, \end{cases}$$

$$(4.14)\qquad \begin{cases} \textit{there exists an } \alpha \in L^1_{loc}([0,\infty),\mathbf{R}), \textit{ and there exists} \\ \textit{a nondecreasing, continuous function } \psi : [0,\infty) \to [0,\infty), \\ \textit{such that } |k(t,s)g(s,y(s))| \le \alpha(s)\psi(|y(s)|), \quad \textit{a.e. } t \in [0,\infty), \\ \textit{a.e. } s \in [0,t], \ \forall y \in C([0,\infty),\mathbf{R}), \end{cases}$$

and

$$(4.15)\qquad \int_0^m \alpha(s)ds < \int_{|h|_\infty}^\infty \frac{ds}{\psi(s)}, \ \forall m \in \mathbf{N};$$

here $|h|_\infty = \sup_{t\in[0,\infty)} |h(t)|$. *Then the solution set of equation* (4.1) *is an* R_δ *set.*

PROOF: Let

$$\phi(t) = I^{-1}\left(\int_0^t \alpha(s)\,ds\right) \quad \text{for } \ t \in [0,\infty),$$

where

$$I(z) = \int_{|h|_\infty}^z \frac{ds}{\psi(s)}.$$

Let $\epsilon > 0$ be given and let $\tau_\epsilon : \mathbf{R} \to [0,1]$ be the Urysohn function for

$$\left(\overline{B}(0,1)\,,\, \mathbf{R}\setminus B(0,1+\epsilon)\right)$$

such that

$$\tau_\epsilon(x) = 1 \ \text{ if } \ |x| \le 1 \quad \text{and} \quad \tau_\epsilon(x) = 0 \ \text{ if } \ |x| \ge 1+\epsilon.$$

Let the operator L and F be given by

$$L\,y(t) = h(t) + \int_0^t k(t,s)\,y(s)\,ds, \ \ F\,(y)(t) = g(t,y(t)),$$

and the operator F_ϵ be defined by

$$F_\epsilon\,(y)(t) = \tau_\epsilon\left(\frac{y(t)}{\phi(t)+1}\right)\,F(y)(t).$$

Associate with (4.1) we consider the equation

$$(4.17)\qquad y(t) = h(t) + \int_0^t k(t,s)\,\tau_\epsilon\left(\frac{y(s)}{\phi(s)+1}\right)\,g(s,y(s))ds \ \text{ for } \ t\in[0,\infty).$$

Essentially the same reasoning as in Theorem 4.2 guarantees that

$$\begin{cases} L\,F_\epsilon : C([0,\infty),\mathbf{R}) \to C([0,\infty),\mathbf{R}) \\ \text{is continuous and completely continuous.} \end{cases}$$

If we show

$$(4.18)\qquad \begin{cases} |y(t)| \le \phi(t) \ \text{ for } \ t \in [0,\infty) \ \text{ for any} \\ \text{possible solution } \ y \in C([0,\infty),\mathbf{R}) \ \text{ to (4.1)}, \end{cases}$$

and

$$(4.19)\qquad \begin{cases} |y(t)| \le \phi(t) \ \text{ for } \ t \in [0,\infty) \ \text{ for any} \\ \text{possible solution } \ y \in C([0,\infty),\mathbf{R}) \ \text{ to (4.17)}, \end{cases}$$

then we can apply Theorem 2.3 to complete the proof.

It remains to show (4.18) and (4.19). Let $y \in C([0,\infty),\mathbf{R})$ be any solution to (4.1). Then

$$|y(t)| \le |h|_\infty + \int_0^t \alpha(s)\,\psi(|y(s)|)\,ds \quad \text{for} \quad t \in [0,\infty).$$

Let

$$w(t) = |h|_\infty + \int_0^t \alpha(s)\,\psi(|y(s)|)\,ds \quad \text{for} \quad t \in [0,\infty).$$

Then

$$w'(t) = \alpha(t)\,\psi(|y(t)|) \le \alpha(t)\,\psi(w(t))$$

so

$$\int_{|h|_\infty}^{w(t)} \frac{ds}{\psi(s)} \le \int_0^t \alpha(s)\,ds \quad \text{for} \quad t \in [0,\infty).$$

Consequently

$$|y(t)| \le w(t) \le \phi(t) \quad \text{for} \quad t \in [0,\infty),$$

so (4.18) holds. Let $y \in C([0,\infty),\mathbf{R})$ be any solution to (4.17). Then since $\tau_\epsilon : \mathbf{R} \to [0,1]$ we have

$$|y(t)| \le |h|_\infty + \int_0^t \alpha(s)\,\psi(|y(s)|)\,ds \quad \text{for} \quad t \in [0,\infty),$$

and as above we obtain

$$|y(t)| \le w(t) \le \phi(t) \quad \text{for} \quad t \in [0,\infty).$$

Thus (4.19) is true. □

Remark. If ψ in (4.14) has at most linear growth, then one could replace (4.11) with

$$h \in C([0,\infty),\mathbf{R}),$$

and delete assumption (4.15) and the result in Theorem 4.3 is again true. The proof is similar to that in Theorem 4.3, the only difference is that the ϕ in (4.18) (and (4.19)) is constructed from Gronwall's inequality (i.e. construct ϕ from Gronwall's inequality and

$$|y(t)| \le |h(t)| + \int_0^t \alpha(s)\,\psi(|y(s)|)\,ds \quad \text{for} \quad t \in [0,\infty);$$

note ψ is at most linear growth).

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Manuscript received September 13, 2000

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