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VECTOR-VALUED SET-VALUED VARIANTS OF KY FAN'S INEQUALITY

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Dedicated to
the memory of Professor Kensuke Tanaka

Abstract. We prove two variants of Fan's type inequality for vector-valued multifunctions in topological vector spaces with respect to a cone preorder in the target space. The main tool for their proofs is a new two-function result, based on a two-function result of Simons, which in turn is proved here directly by the classical scalar Fan inequality. As a consequence of our results, this new two-function result is equivalent to the scalar Fan inequality.

1. Introduction

Fan's inequality is one of the main tools in the nonlinear and convex analysis, equivalent to Brouwer's fixed point theorem, Knaster-Kuratowski-Mazurkiewicz theorem, etc. As an analytical instrument, in many situations it is more appropriate and applicable than the other main theorems in nonlinear analysis. We refer to [2] for various type equivalent theorems in nonlinear analysis.

In this paper we prove two kinds of vector-valued Fan's type inequality for multifunctions. One of them (Theorem 3.1) generalizes the main result of Ansari-Yao in [1], namely, the existence result in the so-called there Generalized Vector Equilibrium Problem. The generalization is in the sense that our Theorem 3.1 contains Fan's inequality in its full generality (for lower semicontinuous functions), while their result cantains it only for continuous functions.

Our proofs are quite different from that one in [1] and are based on the classical scalar Fan inequality. More precisely, in the proofs we use a new two-function result (see Theorem 2.3) which is a slightly more general form of a two-function result of Simons [6, Corollary 1.6] and, as a consequence of our results, it implies the classical Fan inequality. Our two-function result follows from another two-function result of Simons [6, Theorem 1.2]. The latter is used in [6] to derive Fan's inequality, while here, conversely, we derive it directly by Fan's inequality. For a simple proof of the classical Fan inequality, based on Brouwer's fixed point theorem and continuous partition of unity, we refer to [3].

The proofs of the main results (Theorems 4.1–4.2) use Theorem 2.3 for special scalar functions possessing semicontinuity and convexity properties, inherited by the semicontinuity and convexity properties of multifunctions.

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2. Fan's inequality and a new two-function result

Firstly we recall the classical scalar Fan inequality and prove that it implies a two-function result of Simons (namely [6, Theorem 1.2]), which is used in the sequel to prove the main tool for proving the multivalued versions of Fan's inequality (Theorems 4.1–4.2).

Theorem 2.1. **(Fan)** *Let X be a nonempty compact convex subset of a topological vector space and $f : X \times X \to \mathbf{R}$ be quasiconcave in its first variable and lower semicontinuous in its second variable. Then*

$$\min_{y \in X} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} f(x, x).$$

Theorem 2.2. **(Simons [6, Theorem 1.2])** *Let Z be a nonempty compact convex subset of a topological vector space, $f : Z \times Z \to \mathbf{R}$ lower semicontinuous in its second variable, $g : Z \times Z \to \mathbf{R}$ quasiconcave in its first variable, and $f \leq g$ on $Z \times Z$. Then*

$$\min_{y \in Z} \sup_{x \in Z} f(x, y) \leq \sup_{z \in Z} g(z, z).$$

Proof. Define the function co$f$ as a quasiconcave envelope of f with respect to the first variable:

$$\mathrm{co}f(x, y) := \sup\{ \min_{i \in \{1,\dots,n\}} f(x_i, y) : x = \sum_{i=1}^{n} \lambda_i x_i, x_i \in Z, \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1, n \in \mathbf{N}\},$$

where $\mathbf{N}$ is the set of the natural numbers. This function satisfies the conditions of Fan's inequality and applying the latter, we obtain the result. ∎

Now we present our main tool for proving the main results in this paper (Theorems 4.1 and 4.2). Its proof is similar to that of [6, Corollary 1.6].

Theorem 2.3. *Let X be a nonempty compact convex subset of a topological vector space, $a : X \times X \to \mathbf{R}$ lower semicontinuous in its second variable, $b : X \times X \to \mathbf{R}$ quasiconvex in its second variable, and*

$$x, y \in X \quad \text{and } a(x, y) > 0 \Rightarrow b(y, x) < 0.$$

Suppose that $\inf_{x \in X} b(x, x) \geq 0$. Then there exists $z \in X$ such that $a(x, z) \leq 0$ for all $x \in X$.

Proof. Define

$$f(x, y) = 1 \text{ if } a(x, y) > 0 \text{ and } f(x, y) = 0 \text{ otherwise.}$$

Analogically define

$$g(x, y) = 1 \text{ if } b(y, x) < 0 \text{ and } g(x, y) = 0 \text{ otherwise.}$$

These functions satisfy the conditions of Theorem 2.2, and applying it, we obtain the result. ∎

3. Definitions and auxiliary results

Further let E and Y be topological vector spaces and $F, C : E \to 2^Y$ two multivalued mappings and let for every $x \in E$, $C(x)$ be a closed convex cone with nonempty interior. We introduce two types of cone-semicontinuity of set-valued mappings, which are regarded as extensions of the ordinary lower semicontinuity for real-valued functions; see [4].

Definition 3.1. Let $\hat{x} \in E$. The multifunction F is called $C(\hat{x})$*-upper semicontinuous* at x_0, if for every $y \in C(\hat{x}) \cup (-C(\hat{x}))$ such that $F(x_0) \subset y + \text{int}C(\hat{x})$, there exists an open $U \ni x_0$ such that $F(x) \subset y + \text{int}C(\hat{x})$ for every $x \in U$.

Definition 3.2. Let $\hat{x} \in E$. The multifunction F is called $C(\hat{x})$*-lower semicontinuous* at x_0, if for every open V such that $F(x_0) \cap V \neq \emptyset$, there exists an open $U \ni x_0$ such that $F(x) \cap (V + \text{int}C(x_0)) \neq \emptyset$ for every $x \in U$.

Remark 3.1. In the two definitions above, the notions for single-valued functions are equivalent to the ordinary notion of lower semicontinuity of real-valued ones, whenever $Y = \mathbf{R}$ and $C = [0, \infty)$. When the cone $C(\hat{x})$ consists only of the zero of the space, the notion in Definition 3.2 coincides with that of lower semicontinuous set-valued mapping. Moreover, it is equivalent to the cone-lower semicontinuity defined in [4], based on the fact that $V + \text{int}C(\hat{x}) = V + C(\hat{x})$; see [7, Theorem 2.2].

Proposition 3.1. *If for some $x_0 \in E$, $A \subset \text{int}C(x_0)$ is a compact subset and multivalued mapping $W(\cdot) := Y \setminus \{\text{int}C(\cdot)\}$ has a closed graph, then there exists an open set $U \ni x_0$ such that $A \subset C(x)$ for every $x \in U$. In particular C is lower semicontinuous .*

Proof. Assume the contrary. Then there exists a net $\{x_i\}$ converging to x_0 such that for every i there exists $a_i \in A \setminus C(x_i)$. Since A is compact, we may assume that $a_i \to a \in A$. Since W has a closed graph, it follows that $a \in W(x_0)$, which is a contradiction. ∎

Denote $B(x) = (\text{int}C(x)) \cap (2S \setminus \overline{S})$ (an open base of $\text{int}C(x)$), where S is a neighborhood of 0 in Y, and define the functions

$$h(k, x, y) = \inf\{t : y \in tk - C(x)\}.$$

Note that $h(k, x, \cdot)$ is positively homogeneous and subadditive for every fixed $x \in E$ and $k \in \text{int}C(x)$. Moreover, we use the following notations

$$h(k, y) = \inf\{t : y \in tk - C\},$$

and $B = \text{int}C \cap (2S \setminus \overline{S})$, where C is a convex closed cone with nonempty interior and S is a neighborhood of 0 in Y. Note again that $h(k, \cdot)$ is positively homogeneous and subadditive for every fixed $k \in \text{int}C$.

We shall say that (F, X), where X is a subset of E, has property (P), if

(P) *for every $x \in X$ there exists an open $U \ni x$ such that the set $F(U \cap X)$ is precompact in Y, that is, $\overline{F(U \cap X)}$ is compact.*

Lemma 3.1. *Suppose that multifunction* $W : E \to 2^Y$ *defined as* $W(x) = Y \setminus \mathrm{int}C(x)$ *has a closed graph. If the multifunction* F *is* $(-C(x))$*-upper semicontinuous at* x *for each* $x \in E$*, then the function* $\varphi_1|_X$ *(the restriction of*

$$\varphi_1(x) := \inf_{k\in B(x)} \sup_{y\in F(x)} h(k,x,y).$$

to the set X*) is upper semicontinuous, if* (F,X) *satisfies the property* (P)*. If the mapping* C *is constant-valued, then* φ_1 *is upper semicontinuous.*

Proof. Assume that (F,X) has property (P). Let $\varepsilon > 0$ and $x_0 \in X$ be given. By the definition of φ_1 there exists $k_0 \in B(x_0)$ such that

$$\sup_{y\in F(x_0)} h(k_0,x_0,y) < \varphi_1(x_0) + \varepsilon.$$

Since $\sup_{y\in F(x_0)} h(k_0,x_0,y) = \inf\{t : F(x_0) \subset tk_0 - C(x_0)\}$, we can take

$$\inf\{t : F(x_0) \subset tk_0 - C(x_0)\} < t_0 < \varphi_1(x_0) + \varepsilon.$$

Since F is $(-C(x_0))$-upper semicontinuous at x_0, there exists an open $U \ni x_0$ such that

$$F(x) \subset t_0k_0 - \mathrm{int}C(x_0) \quad \text{for every } x \in U.$$

By Proposition 3.1 and property (P), for $t_0 < t' < \varphi_1(x_0) + \varepsilon$, there exists an open $U_1 \subset U$ such that

$$F(x) \subset t'k_0 - \mathrm{int}C(x) \quad \text{and} \quad k_0 \in B(x) \qquad \text{for every } x \in U_1 \cap X.$$

Then

$$\begin{aligned}
\varphi_1(x) &= \inf_{k\in B(x)} \sup_{y\in F(x)} h(k,x,y)\\
&\le \sup_{y\in t'k_0 - C(x)} h(k_0,x,y)\\
&= t'h(k_0,x,k_0) + \sup_{y\in -C(x)} h(k_0,x,y)\\
&\le t'\\
&\le \varphi_1(x_0) + \varepsilon.
\end{aligned}$$

The proof of the second statement (when C is constant-valued) is similar, but in this case there is no need to use Proposition 3.1 and property (P). ∎

Definition 3.3. The multivalued mapping $F : E \to 2^Y$ is called *C-properly quasiconvex* if for every two points $x_1, x_2 \in X$ and every $\lambda \in [0,1]$ we have either

$$\begin{aligned}
F(\lambda x_1 + (1-\lambda)x_2) &\subset F(x_1) - C \qquad \text{or}\\
F(\lambda x_1 + (1-\lambda)x_2) &\subset F(x_2) - C.
\end{aligned}$$

If $-F$ is C-properly quasiconvex, then F is called *C-properly quasiconcave*, which is equivalent to $(-C)$-properly quasiconvex mapping.

Remark 3.2. The above definition is exactly that of *type (v) properly quasiconvex* mapping in [5, Definition 3.6] and that of *C-quasiconvex-like* multifunction in [1].

Lemma 3.2. *If the multifunction $F : E \to 2^Y$ is C-properly quasiconvex, then the function*

$$\psi_1(x) := \inf_{k \in B} \sup_{y \in F(x)} h(k, y)$$

is quasiconvex.

Proof. By definition, for every $\lambda \in [0, 1]$ and every $x_1, x_2 \in X$ we have: either

$$F(\lambda x_1 + (1 - \lambda)x_2) \subset F(x_1) - C$$

or

$$F(\lambda x_1 + (1 - \lambda)x_2) \subset F(x_2) - C.$$

Assume that $F(\lambda x_1 + (1 - \lambda)x_2) \subset F(x_1) - C$. Then

$$\begin{aligned}
\psi_1(\lambda x_1 + (1 - \lambda)x_2) &:= \inf_{k \in B} \sup\{h(k, y) : y \in F(\lambda x_1 + (1 - \lambda)x_2)\} \\
&\leq \inf_{k \in B} \sup\{h(k, y) : y \in F(x_1) - C\} \\
&= \inf_{k \in B} \sup_{\substack{y \in F(x_1) \\ c \in C}} h(k, y - c) \\
&\leq \inf_{k \in B} \sup_{\substack{y \in F(x_1) \\ c \in C}} (h(k, y) + h(k, -c)) \quad \left(\begin{array}{c}\text{by subadditivity of} \\ h(k, \cdot)\end{array}\right) \\
&\leq \psi_1(x_1) \\
&\leq \max\{\psi_1(x_1), \psi_1(x_2)\}.
\end{aligned}$$

Analogously we proceed in the second case, when $F(\lambda x_1 + (1 - \lambda)x_2) \subset F(x_2) - C$. ∎

Lemma 3.3. *Suppose that the multifunction F is $-C(x)$-lower semicontinuous for each $x \in E$ and the multifunction $W : E \to 2^Y$ defined by $W(x) = Y \setminus \mathrm{int}C(x)$ has a closed graph. Then the function $\varphi_2|_X$ (the restriction of*

$$\varphi_2(x) := \inf_{k \in B(x)} \inf_{y \in F(x)} h(k, x, y)$$

to the set X) is upper semicontinuous, if (F, X) satisfies the property (P). If the mapping C is constant-valued, then φ_2 is upper semicontinuous.

Proof. Let $\varepsilon > 0$ and $x_0 \in E$ be given. By the definition of φ_2, for $t_0 \in (\varphi_2(x_0), \varphi_2(x_0) + \varepsilon)$ there exists $k_0 \in B(x_0), k_0 \in \mathrm{int}C(x_0)$, and $z_0 \in F(x_0)$ such that $z_0 - t_0 k_0 \in -\mathrm{int}C(x_0)$. By Proposition 3.1, there exists an open set $U_1 \ni x_0$ such that

$$z_0 - t_0 k_0 \in -\mathrm{int}C(x) \quad \text{and} \quad k_0 \in \mathrm{int}C(x) \quad \text{for every } x \in U_1.$$

Therefore

$$h(k_0, x, z_0) \leq t_0 \qquad \text{for every } x \in U_1. \tag{3.1}$$

Let $\gamma < \varepsilon/2$. By $(-C(x_0))$-lower semicontinuity of F, there exists an open set $U_2 \subset U_1, x_0 \in U_2$ such that

$$G(x) := F(x) \cap [z_0 + \gamma k_0 - \mathrm{int}C(x_0)] \neq \emptyset \qquad \text{for every } x \in U_2. \tag{3.2}$$

Hence

$$G(U_2 \cap X) \subset z_0 + \gamma k_0 - \mathrm{int}C(x_0)$$

and

$$\overline{G(U_2 \cap X)} \subset z_0 + 2\gamma k_0 - \mathrm{int}C(x_0).$$

By Proposition 3.1 there exists an open $U_3 \subset U_2, U_3 \ni x_0$ such that

$$\overline{G(U_2 \cap X)} \subset z_0 + 2\gamma k_0 - \mathrm{int}C(x) \quad \text{for every } x \in U_3.$$

This implies

$$F(x) \cap (z_0 + 2\gamma k_0 - \mathrm{int}C(x)) \neq \emptyset \quad \text{for every } x \in U_3 \cap X.$$

Take $x \in U_3 \cap X$ and $y_x \in F(x) \cap (z_0 + 2\gamma k_0 - \mathrm{int}C(x))$. Therefore $y_x = z_0 + 2\gamma k_0 + c_x$, where $c_x \in -\mathrm{int}C(x)$. We obtain

$$\begin{aligned}
\varphi_2(x_0) + \varepsilon &\geq t_0 \\
&\geq h(k_0, x, z_0) \quad \text{(by (3.1))} \\
&= h(k_0, x, y - 2\gamma k_0 - c_x) \\
&\geq h(k_0, x, y) - h(k_0, x, 2\gamma k_0) - h(k_0, x, c_x) \left(\begin{array}{c} \text{by subadditivity of} \\ h(k_0, x, \cdot) \end{array} \right) \\
&\geq h(k_0, x, y) - 2\gamma \\
&\geq \varphi_2(x) - \varepsilon.
\end{aligned}$$

Hence

$$\varphi_2(x_0) + 2\varepsilon \geq \varphi_2(x) \qquad \text{for every } x \in U_3 \cap X.$$

The proof of the second statement (when C is constant-valued) is similar, but in this case there is no need to use Proposition 3.1 and property (P). ∎

Definition 3.4. The multifunction $F : E \to 2^Y$ is called *C-quasiconvex*, if the set

$$\{x \in E : F(x) \cap (a - C) \neq \emptyset\}$$

is convex for every $a \in Y$. If $-F$ is C-quasiconvex, then F is called *C-quasiconcave*, which is equivalent to $(-C)$-quasiconvex mapping.

Remark 3.3. The above definition is exactly that of *Ferro type* (-1)-*quasiconvex* mapping in [5, Definition 3.5].

Lemma 3.4. *If F is C-quasiconvex, then for every $k \in B$ the function*

$$\psi_2(x; k) := \inf\{h(k, y) : y \in F(x)\}$$

is quasiconvex.

Proof. By the definition of $\psi_2(\cdot\,; k)$, for every $\varepsilon > 0$ and $x_1, x_2 \in E$ there exist $z_i \in F(x_i), t_i \in \mathbf{R}$ such that

$$z_i - t_i k \in -C, \tag{3.3}$$

and

$$t_i < \psi_2(x_i; k) + \varepsilon, i = 1, 2. \tag{3.4}$$

Since $s_1k - C \subset s_2k - C$ for $s_1 \le s_2$, by (3.3), we have

$$z_i \in t_ik - C \subset \max\{t_1, t_2\}k - C.$$

Hence, by the C-quasiconvexity of F, for every $\lambda \in [0, 1]$ there exists $y \in F(\lambda x_1 + (1-\lambda)x_2)$ such that $y \in \max\{t_1, t_2\}k - C$, which means

$$\begin{aligned} h(k, y) &\le \max\{t_1, t_2\} \\ &< \max\{\psi_2(x_1; k), \psi_2(x_2; k)\} + \varepsilon \end{aligned}$$

by (3.4) and since, the definition, we have

$$\psi_2(\lambda x_1 + (1-\lambda)x_2; k) = \inf\{h(k, y) : y \in F(\lambda x_1 + (1-\lambda)x_2)\},$$

and $\varepsilon > 0$ is arbitrarily small, we obtain

$$\psi_2(\lambda x_1 + (1-\lambda)x_2; k) \le \max\{\psi_2(x_1; k), \psi_2(x_2; k)\}.$$

∎

4. Main results

Now we state the main results in this paper. The following theorem is a generalization of that one in [1] in the sense that condition (iii) is more general and allows us to recover the classical Fan inequality, when Y is the real line. The result in [1] recovers it only for continuous functions.

Theorem 4.1. *Let K be a nonempty convex subset of a topological vector space E, Y be a topological vector space. Let $F : K \times K \to 2^Y$ be a multifunction. Assume that*

(i) *$C : K \to 2^Y$ is a multifunction such that for every $x \in K, C(x)$ is a closed convex cone in Y with $\text{int}C(x) \neq \emptyset$;*
(ii) *$W : K \to 2^Y$ is a multifunction defined as $W(x) = Y \setminus \text{int}C(x)$, and the graph of W is closed in $K \times Y$;*
(iii) *for every $x, y \in K$, $F(\cdot, y)$ is $C(x)$-upper semicontinuous at x with closed values on K and if the mapping C is not constant-valued, then the mapping $F(\cdot, y)$ maps the compact subsets of K into precompact subsets of Y;*
(iv) *there exists a multifunction $G : K \times K \to 2^Y$ such that*
   (a) *for every $x \in K, G(x, x) \not\subset \text{int}C(x)$,*
   (b) *for every $x, y \in K, F(x, y) \subset \text{int}C(x)$ implies $G(x, y) \subset \text{int}C(x)$,*
   (c) *$G(x, \cdot)$ is $C(x)$-properly quasiconcave on K for every $x \in X$,*
   (d) *$G(x, y)$ is compact, if $G(x, y) \subset \text{int}C(x)$;*
(v) *there exists a nonempty compact convex subset D of K such that for every $x \in K \setminus D$, there exists $y \in D$ with $F(x, y) \subset \text{int}C(x)$.*

Then, the solutions set

$$S = \{x \in K : F(x, y) \not\subset \text{int}C(x), \quad \textit{for all } y \in K\}$$

is a nonempty and compact subset of D.

Proof. Put

$$a(x, y) := -\inf_{k \in B(y)} \sup_{z \in -F(y,x)} h(k, y, z), \quad b(x, y) := \inf_{k \in B(x)} \sup_{z \in -G(x,y)} h(k, x, z).$$

It is easy to check that

$$a(x,y) > 0 \quad \text{if and only if } F(y,x) \subset \text{int}C(y)$$

by using the compactness of $\overline{F(x,y)}$, and also

$$b(y,x) \ < 0 \ \text{ if } G(y,x) \subset \text{int}C(y)$$

by using condition (d), and then

$$a(x,x) \le 0, \quad b(x,x) \ge 0.$$

Denote

$$S_y := \{x \in D : F(x,y) \not\subset \text{int}C(x)\}. \tag{4.1}$$

Since $a(y,\cdot)$ is lower semicontinuous (by Lemma 3.1), the set S_y is closed. Let Y_0 be a finite subset of K. Denote by Z the closed convex hull of $Y_0 \cup D$. Obviously Z is compact and convex. Lemmas 3.1, 3.2 and condition (iv) (b) show that the conditions of Theorem 2.3 are satisfied.

Now we apply Theorem 2.3 and obtain a point $z \in Z$ such that

$$a(y,z) \le 0 \qquad \text{for every } \ y \in Z$$

which means

$$F(z,y) \not\subset \text{int}C(z) \qquad \text{for every} \quad y \in Z. \tag{4.2}$$

The conditions (v) and (4.2) imply that $z \in D$. Relation (4.1) implies that

$$\cap\{S_y : y \in Y_0\} \ne \emptyset.$$

So we proved that the family $\{S_y : y \in K\}$ has finite intersection property. Since D is compact,

$$\cap\{S_y : y \in K\} \ne \emptyset,$$

which means that there exists $x_0 \in K$ such that

$$F(x_0,y) \not\subset \text{int}C(x_0) \ \text{ for every } y \in K.$$

So we proved that S is nonempty, and since S is a closed subset of D, the proof is completed. ∎

Theorem 4.2. *Let K be a nonempty convex subset of a topological vector space E, Y a topological vector space, and $F : K \times K \to 2^Y$ a multifunction. Assume that*

(i) *$C : K \to 2^Y$ is a multifunction such that for every $x \in K, C(x)$ is a closed convex cone in Y with* $\text{int}C(x) \ne \emptyset$*;*
(ii) *$W : K \to 2^Y$ is a multifunction defined as $W(x) = Y \setminus \text{int}C(x)$ and the graph of W is closed in $K \times Y$;*
(iii) *for every $x, y \in K, F(\cdot,y)$ is $C(x)$-lower semicontinuous with closed values on K and if the mapping C is not constant-valued, then the mapping $F(\cdot,y)$, for every $y \in K$, maps the compact subsets of K into precompact subsets of Y;*
(iv) *there exists a multifunction $G : K \times K \to 2^Y$ such that*
  (a) *for every $x \in K$, $G(x,x) \cap \text{int}C(x) = \emptyset$,*
  (b) *for every $x, y \in K$, $F(x,y) \cap \text{int}C(x) \ne \emptyset$ implies $G(x,y) \cap \text{int}C(x) \ne \emptyset$,*
  (c) *$G(x,\cdot)$ is $C(x)$-quasiconcave on K for every $x \in K$;*

(v) *there exists a nonempty compact convex subset* D *of* K *such that for every* $x \in K \setminus D$, *there exists* $y \in D$ *with* $F(x,y) \cap \mathrm{int}C(x) \neq \emptyset$.

Then, the solutions set

$$S = \{x \in K : F(x,y) \cap \mathrm{int}C(x) = \emptyset, \quad \text{for all } y \in K\}$$

is a nonempty and compact subset of D.

Proof. Put

$$a(x,y) := -\inf_{k \in B(y)} \inf_{z \in -F(y,x)} h(k,y,z), \quad b(x,y) := \inf_{z \in -G(x,y)} h(k(x),x,z),$$

where the function k is any fixed selection of the multivalued mapping $x \mapsto \mathrm{int}C(x)$, i.e., $k(x) \in \mathrm{int}C(x)$ for every $x \in K$.

It is easy to check that

$$a(x,y) > 0 \quad \text{if and only if } F(y,x) \cap (\mathrm{int}C(y)) \neq \emptyset,$$

$$b(y,x) < 0 \quad \text{if and only if } G(y,x) \cap (\mathrm{int}C(y)) \neq \emptyset,$$

$$a(x,x) \leq 0, \quad b(x,x) \geq 0.$$

Lemmas 3.3, 3.4 and condition (iv) (b) show that the conditions of Theorem 2.3 are satisfied. Further the proof is the same as that of Theorem 4.1, but in this case

$$S_y := \{x \in D : F(x,y) \cap (\mathrm{int}C(x)) = \emptyset\}.$$

∎

Remark 4.1. As a corollary from any of Theorems 4.1 and 4.2, when $Y = \mathbf{R}$, $C = [0,\infty)$, we obtain that any of Theorems 4.1 and 4.2 implies the scalar Fan inequality (Theorem 2.1). Indeed, under the assumptions of Theorem 2.1, we apply any of Theorems 4.1 and 4.2 to the function $f - \sup_{z \in Z} f(z,z)$. Since those theorems are based on Theorem 2.3, we conclude that Theorem 2.3 is equivalent to the scalar Fan inequality.

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