Journal of Nonlinear and Convex Analysis
Volume 1, Number 2, 2000, 213–231

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

NONLINEAR STRONG ERGODIC THEOREMS FOR COMMUTATIVE NONEXPANSIVE SEMIGROUPS ON STRICTLY CONVEX BANACH SPACES[1]

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Abstract. In this paper, we study nonlinear ergodic properties for a commutative semigroup of nonexpansive mappings in a strictly convex Banach space E. We prove that if S is a commutative semigroup, $\mathcal{S} = \{T(t) : t \in S\}$ is a nonexpansive semigroup on a nonempty closed convex subset X of E, K is a compact subset of X such that $T(t)(X) \subset K$ for all $t \in S$ and $\{\lambda_\alpha\}$ is any bounded net of linear functionals on the Banach space of all bounded real-valued functions on S such that $\lim_\alpha \lambda_\alpha(1) = 1$ and $\lim_\alpha \|\lambda_\alpha - r_s^*\lambda_\alpha\| = 0$ for every $s \in S$, then $\int T(h+t)x d\lambda_\alpha(t)$ converges strongly to a common fixed point of $T(t), t \in S$ uniformly in $h \in S$. Various applications of our main theorems will be given.

1. Introduction

Let C be a nonempty closed convex subset of a real Banach space E. Then a mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. We also denote by $N(C)$ the set of all nonexpansive mappings of C into itself. For any $x \in C$, the ω-limit set of x is defined by

$$\omega(x) = \{z \in C : z = \lim_{i\to\infty} T^{n_i}x \quad \text{with} \quad n_i \to \infty \quad \text{as} \quad i \to \infty\}.$$

Edelstein [12] obtained the following nonlinear ergodic theorem for nonexpansive mappings with compact domains in a strictly convex Banach space: Let C be a nonempty compact convex subset of a strictly convex Banach space and let T be a nonexpansive mapping of C into itself. Let $x \in C$. Then, for any $\xi \in \overline{\text{co}}\,\omega(x)$, the Cesàro mean $S_n(\xi) = (1/n)\sum_{k=0}^{n-1} T^k\xi$ converges strongly to a fixed point of T, where $\overline{\text{co}}A$ is the closure of the convex hull of A. See also Dafermos and Slemrod [10] for a one-parameter nonexpansive semigroup in a strictly convex Banach space. The first nonlinear ergodic theorem for nonexpansive mappings with bounded domains was established in the framework of a Hilbert space by Baillon [4]: Let C be a nonempty bounded closed convex subset of a Hilbert space and let T be a nonexpansive mapping of C into itself. Then, for any $x \in C$, the Cesàro mean $S_n(x) = (1/n)\sum_{k=0}^{n-1} T^k x$ converges weakly to a fixed point of T. Bruck [7] extended Baillon's theorem in [4] to a uniformly convex Banach space whose norm is Fréchet differentiable. Brézis

[1]This research is supported by an NSERC-grant and by Grant-in-Aid for General Scientific Research No. 12640157, the Ministry of Education, Science, Sports and Culture, Japan.

2000 *Mathematics Subject Classification.* Primary 47H09, 49M05.

Key words and phrases. Fixed point, iteration, nonexpansive mapping, nonexpansive semigroup, strong convergence.

and Browder [5] also proved a nonlinear strong ergodic theorem for nonexpansive mappings of odd-type in a Hilbert space (see also Reich [19]).

Recently, the first and third authors [2] improved Edelstein's theorem in [12] by using Bruck [7, 8] and [1]: For any $x \in C$, the Cesàro mean $S_n(x)$ converges strongly to a fixed point of T. Furthermore, they [3] also obtained a nonlinear strong ergodic theorem for a one-parameter nonexpansive semigroup on a compact convex subset of a strictly convex Banach space.

In this paper, we shall study nonlinear strong ergodic properties for a commutative semigroup of nonexpansive mappings in a strictly convex Banach space E. Our main theorem (Theorem 4.2) implies that if $\mathcal{S} = \{T(t) : t \in S\}$ is a commutative semigroup of nonexpansive mappings on a nonempty closed convex subset X of E and K is a compact subset of X such that $T(t)(X) \subset K$ for each $t \in S$, then there is a nonexpansive mapping Q from X onto $F(\mathcal{S})$, the fixed point set of $\mathcal{S}$, such that $QT(t) = T(t)Q = Q$ for each $t \in S$ and $Qx \in \overline{\text{co}}\{T(s)x : s \in S\}$ for each $x \in X$. Furthermore, Qx is the strong limit of $\int T(h+t)x d\lambda_\alpha(t)$, where $\{\lambda_\alpha\}$ is a strongly regular net of functionals on $B(S)$, the space of all bounded real-valued functions on S. Various applications (including sumability method with respect to a strongly regular matrix due to Lorentz [18]) will be given in Section 5. This improves the results of Atsushiba and Takahashi in [2, 3].

2. Preliminaries

Throughout this paper, we assume that a Banach space E is real and S is a commutative semigroup with identity unless other specified. In this case, $(S, \leq)$ is a directed system when the binary relation $\leq$ on S is defined by $a \leq b$ if and only if there is $c \in S$ with $a + c = b$.

We denote by E^* the dual space of E and by $\mathbb{N}$ the set of all positive integers. In addition, we denote by $\mathbb{R}^+$ and $\mathbb{Z}^+$ the sets of all nonnegative real numbers and all nonnegative integers, respectively. We also denote by $\langle y, x^* \rangle$ the value of $x^* \in E^*$ at $y \in E$. For a subset A of E, co$A$ and $\overline{\text{co}}A$ mean the convex hull of A and the closure of convex hull of A, respectively. We denote by $S_1(E)$ the unit sphere in E with center 0. Let $B(S)$ be the Banach space of all bounded real-valued functions on S with the supremum norm. Then, for each $s \in S$ and $f \in B(S)$, we can define $r_s f \in B(S)$ by $(r_s f)(t) = f(t+s)$ for all $t \in S$. We also denote by r_s^* the conjugate operator of r_s. Let D be a subspace of $B(S)$ and let μ be an element of D^*. Then, we denote by $\mu(f)$ the value of μ at $f \in D$. Sometimes, $\mu(f)$ will be also denoted by $\mu_t(f(t))$ or $\int f(t) d\mu(t)$. When D contains 1, a linear functional μ on D is called a mean on D if $\|\mu\| = \mu(1) = 1$. Further, let D be r_s-invariant, i.e., $r_s(D) \subset D$ for every $s \in S$. Then, a mean μ on D is invariant if $\mu(r_s f) = \mu(f)$ for all $s \in S$ and $f \in D$. For $s \in S$, we can define the point evaluation δ_s by $\delta_s(f) = f(s)$ for every $f \in B(S)$. A convex combination of point evaluations is called a finite mean on S. A finite mean μ on S is also a mean on any subspace D of $B(S)$ containing 1.

The following definition which was introduced by Takahashi [22] is crucial in the nonlinear ergodic theory for abstract semigroups (see also [15]). Let f be a function of S into E such that the weak closure of $\{f(t) : t \in S\}$ is weakly compact. Let D be a subspace of $B(S)$ containing 1 and r_s-invariant for every $s \in S$. Assume that for each $x^* \in E^*$, the function $t \mapsto \langle f(t), x^* \rangle$ is in D. Then, for any $\mu \in D^*$ there exists a unique element $f_\mu \in E$ such that

$$\langle f_\mu, x^* \rangle = \int \langle f(t), x^* \rangle d\mu(t)$$

for all $x^* \in E^*$. If μ is a mean on D, then f_μ is contained in $\overline{\text{co}}\{f(t) : t \in S\}$ (for example, see [16, 17, 22]). Sometimes, f_μ will be denoted by $\int f(t) d\mu(t)$.

Let C be a subset of a Banach space E. Then, a family $\mathcal{S} = \{T(s) : s \in S\}$ of mappings of C into itself is called a nonexpansive semigroup on C if it satisfies the following conditions:

(i) $T(s+t) = T(s)T(t)$ for all $s, t \in S$;
(ii) $\|T(s)x - T(s)y\| \le \|x - y\|$ for all $x, y \in C$ and $s \in S$.

We denote by $F(\mathcal{S})$ the set of common fixed points of $T(t), t \in S$, that is, $F(\mathcal{S}) = \bigcap_{t \in S} F\big(T(t)\big)$. If C is a compact convex subset of a strictly convex Banach space E and S is commutative, then we know that $F(\mathcal{S})$ is nonempty.

Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that for each $x \in C$, $\{T(t)x : t \in S\}$ is contained in a weakly compact, convex subset of E. Let D be a subspace of $B(S)$ containing 1 with the property that the function $t \mapsto \langle T(t)x, x^* \rangle$ is an element of D for each $x \in C$ and $x^* \in E^*$, and let μ be a mean on D. Following [20], we also write $T_\mu x$ instead of $\int T(t)x \, d\mu(t)$ for $x \in C$. We remark that $T_\mu x = x$ for each $x \in F(\mathcal{S})$.

For a mapping T of C into itself and $\varepsilon > 0$, we define the set $F_\varepsilon(T)$ to be

$$F_\varepsilon(T) = \{x \in C : \|Tx - x\| \le \varepsilon\}.$$

A Banach space E is said to be strictly convex if $\|x + y\|/2 < 1$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \ne y$. In a strictly convex Banach space, we have that if

$$\|x\| = \|y\| = \|(1-\lambda)x + \lambda y\|$$

for $x, y \in E$ and $\lambda \in (0,1)$, then $x = y$.

3. Some Lemmas

The following lemma was proved by Bruck [7, Remark] and [8, Lemma 2.1](see also [2]).

Lemma 3.1 (Bruck)**.** Let E be a strictly convex Banach space and let C be a nonempty compact convex subset of E. Then, for any $n \in \mathbb{N}$, there exists a strictly increasing continuous, convex function $\gamma_n : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\gamma(0) = 0$ and

$$\gamma_n\left(\left\|\sum_{i=1}^n \lambda_i T y_i - T\left(\sum_{i=1}^n \lambda_i y_i\right)\right\|\right) \le \max_{1 \le i,j \le n} \big(\|y_i - y_j\| - \|Ty_i - Ty_j\|\big) \qquad (1)$$

for every T in $N(C)$, $\{\lambda_i\}_{i=1}^n$ in $\mathbb{R}^+$ with $\sum_{i=1}^n \lambda_i = 1$ and $\{y_i\}_{i=1}^n$ in C.

The following two lemmas will be useful for us (see also [13, 15] and [1, Lemma 3.1]).

Lemma 3.2. Let C be a nonempty compact convex subset of a strictly convex Banach space E and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. Let $x \in C$. Then, for any finite mean μ on S and $\varepsilon > 0$, there exists $w_0 = w_0(\mu, \varepsilon) \in S$ such that

$$\left\| \int T(h+s+w)x d\mu(s) - T(h)\left(\int T(s+w)x d\mu(s)\right) \right\| < \varepsilon$$

for every $h \in S$ and $w \geq w_0$.

Proof. Let μ be a finite mean on S and suppose

$$\mu = \sum_{i=1}^{n} a_i \delta_{s_i} \quad (a_i \geq 0,\ \sum_{i=1}^{n} a_i = 1).$$

From Lemma 3.1, for each $n \in \mathbb{N}$, there exists a strictly increasing continuous, convex function γ_n from $\mathbb{R}^+$ into $\mathbb{R}^+$ with $\gamma_n(0) = 0$ which satisfies (1). Let $\varepsilon > 0$. Since γ_n^{-1} is continuous and $\gamma_n^{-1}(0) = 0$, there exists $\delta > 0$ with $\gamma_n^{-1}(\delta) < \varepsilon$. Since

$$\|T(h+t+s_i)x - T(h+t+s_j)x\| \leq \|T(t+s_i)x - T(t+s_j)x\|,$$

for every $h, t \in S$, the $\lim_{t\to\infty} \|T(t+s_i)x - T(t+s_j)x\|$ exists for every $i, j \in \{1, 2, \ldots, n\}$. Then, there exists $t_1 = t_1(\varepsilon, i, j) \in S$ such that

$$0 \leq \|T(t+s_i)x - T(t+s_j)x\| - \|T(h+t+s_i)x - T(h+t+s_j)x\| < \delta$$

for every $t \geq t_1$ and $h \in S$. Let $w_0 \in S$ such that $w_0 \geq t_1(\varepsilon, i, j)$ for all $i, j \in \{1, 2, \ldots, n\}$. So, it follows from Lemma 3.1 that

$$\begin{aligned}
&\left\| \sum_{i=1}^{n} a_i T(h)T(w+s_i)x - T(h)\left(\sum_{i=1}^{n} a_i T(w+s_i)x\right) \right\| \\
&\leq \gamma_n^{-1}\Big(\max_{1\leq i,j\leq n} \big(\|T(w+s_i)x - T(w+s_j)x\| - \|T(h)T(w+s_i)x - T(h)T(w+s_j)x\|\big)\Big) \\
&< \gamma_n^{-1}(\delta) < \varepsilon
\end{aligned}$$

for every $h \in S$ and $w \geq w_0$. □

Lemma 3.3. Let C be a nonempty compact convex subset of a strictly convex Banach space E. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C, let $x \in C$ and let $\{\mu_\alpha : \alpha \in I\}$ and $\{\lambda_\beta : \beta \in J\}$ be nets of finite means on S such that

$$\lim_\alpha \|\mu_\alpha - r_t^* \mu_\alpha\| = 0 \quad \text{and} \quad \lim_\beta \|\lambda_\beta - r_t^* \lambda_\beta\| = 0 \quad \text{for every } t \in S. \tag{$*$}$$

Then, there exist nets $\{p_\alpha : \alpha \in I\}$ and $\{q_\beta : \beta \in J\}$ in S such that for any $z \in F(\mathcal{S})$,

$$\lim_\alpha \left\| \int T(p_\alpha + t)x d\mu_\alpha(t) - z \right\| = \lim_\beta \left\| \int T(q_\beta + t)x d\lambda_\beta(t) - z \right\|. \tag{2}$$

Proof. Let $\varepsilon > 0$. From Lemma 3.2, for $\alpha \in I$ and $\beta \in J$, there exist $p_\alpha, q_\beta \in S$ such that

$$\sup_{h\in S} \left\| \int T(h)T(w+p_\alpha+t)x d\mu_\alpha(t) - T(h)\left(\int T(w+p_\alpha+t)x d\mu_\alpha(t)\right) \right\| < \varepsilon$$

and

$$\sup_{h\in S}\left\|\int T(h)T(w+q_\beta+s)xd\lambda_\beta(s)-T(h)\left(\int T(w+q_\beta+s)xd\lambda_\beta(s)\right)\right\|<\varepsilon$$

for every $w \in S$. Fix $z \in F(\mathcal{S})$ and consider

$$L=\left\|\int T(p_\alpha+t)xd\mu_\alpha(t)-z\right\|,$$

$$I_1=\left\|\int T(p_\alpha+t)xd\mu_\alpha(t)-\iint T(p_\alpha+t+q_\beta+s)xd\lambda_\beta(s)d\mu_\alpha(t)\right\|,$$

$$I_2=\left\|\iint T(p_\alpha+t+q_\beta+s)xd\lambda_\beta(s)d\mu_\alpha(t)-z\right\|,$$

$$J_1^{(2)}=\left\|\iint T(p_\alpha+t+q_\beta+s)xd\lambda_\beta(s)d\mu_\alpha(t)-\int T(p_\alpha+t)\left(\int T(q_\beta+s)xd\lambda_\beta(s)\right)d\mu_\alpha(t)\right\|$$

and

$$J_2^{(2)}=\left\|\int T(p_\alpha+t)\left(\int T(q_\beta+s)xd\lambda_\beta(s)\right)d\mu_\alpha(t)-z\right\|.$$

Then, we have $L \le I_1 + I_2$ and $I_2 \le J_1^{(2)} + J_2^{(2)}$. Suppose

$$\mu_\alpha=\sum_{i=1}^n a_i\delta_{t_i}\quad (a_i\ge 0,\ \sum_{i=1}^n a_i=1)\quad\text{and}\quad \lambda_\beta=\sum_{j=1}^m b_j\delta_{s_j}\quad (b_j\ge 0,\ \sum_{j=1}^m b_j=1). \tag{3}$$

Then, we have

$$J_1^{(2)}\le\sum_{i=1}^n a_i\left\|\int T(p_\alpha+t_i)T(q_\beta+s)xd\lambda_\beta(s)-T(p_\alpha+t_i)\left(\int T(q_\beta+s)xd\lambda_\beta(s)\right)\right\|$$
$$\le\sup_{h\in S}\left\|\int T(h)T(q_\beta+s)xd\lambda_\beta(s)-T(h)\left(\int T(q_\beta+s)xd\lambda_\beta(s)\right)\right\|<\varepsilon.$$

Since $z \in F(\mathcal{S})$, we obtain

$$J_2^{(2)}\le\sum_{i=1}^n a_i\left\|T(p_\alpha+t_i)\left(\int T(q_\beta+s)xd\lambda_\beta(s)\right)-z\right\|$$
$$\le\left\|\int T(q_\beta+s)xd\lambda_\beta(s)-z\right\|.$$

Then, we have

$$I_2\le J_1^{(2)}+J_2^{(2)}<\varepsilon+\left\|\int T(q_\beta+s)xd\lambda_\beta(s)-z\right\|.$$

On the other hand, from (3), we obtain

$$I_1 = \left\|\int T(p_\alpha+t)xd\mu_\alpha(t)-\sum_{j=1}^m b_j\int T(p_\alpha+t+q_\beta+s_j)xd\mu_\alpha(t)\right\|$$

$$\leq \sum_{j=1}^{m} b_j \left\| \int T(p_\alpha + t)xd\mu_\alpha(t) - \int T(p_\alpha + t)xd(r^*_{q_\beta+s_j}\, \mu_\alpha)(t) \right\|$$
$$\leq \sum_{j=1}^{m} b_j \sup_{g\in S} \|T(g)x\| \, \|\mu_\alpha - r^*_{q_\beta+s_j}\, \mu_\alpha\|.$$

Therefore, from $\lim_\alpha I_1 = 0$, we have

$$\overline{\lim_\alpha} \left\| \int T(p_\alpha + t)xd\mu_\alpha(t) - z \right\| = \overline{\lim_\alpha}\, L \leq \overline{\lim_\alpha}\, (I_1 + I_2)$$
$$\leq \varepsilon + \left\| \int T(q_\beta + s)xd\lambda_\beta(s) - z \right\|.$$

Then, we have

$$\overline{\lim_\alpha} \left\| \int T(p_\alpha + t)xd\mu_\alpha(t) - z \right\| \leq \varepsilon + \underline{\lim_\beta} \left\| \int T(q_\beta + s)xd\lambda_\beta(s) - z \right\|.$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\overline{\lim_\alpha} \left\| \int T(p_\alpha + t)xd\mu_\alpha(t) - z \right\| \leq \underline{\lim_\beta} \left\| \int T(q_\beta + s)xd\lambda_\beta(s) - z \right\|.$$

Similarly, we have

$$\overline{\lim_\beta} \left\| \int T(q_\beta + s)xd\lambda_\beta(s) - z \right\| \leq \underline{\lim_\alpha} \left\| \int T(p_\alpha + t)xd\mu_\alpha(t) - z \right\|.$$

Therefore, we have

$$\lim_\alpha \left\| \int T(p_\alpha + t)xd\mu_\alpha(t) - z \right\| = \lim_\beta \left\| \int T(q_\beta + t)xd\lambda_\beta(t) - z \right\|. \qquad \square$$

Repeating the above argument, we have the following.

Remark 3.4. In Lemma 3.3, take nets $\{p_\alpha{}'\}$ and $\{q_\beta{}'\}$ in S such that $p_\alpha{}' \geq p_\alpha$ and $q_\beta{}' \geq q_\beta$. Then, we can see

$$\lim_\alpha \left\| \int T(p_\alpha{}' + t)xd\mu_\alpha(t) - z \right\| = \lim_\beta \left\| \int T(q_\beta{}' + t)xd\lambda_\beta(t) - z \right\|$$

for every $z \in F(\mathcal{S})$.

From [2, 7, 8], we have the following lemmas.

Lemma 3.5. Let E be a strictly convex Banach space and let C be a nonempty compact convex subset of E. For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any T in $N(C)$,

$$\overline{\mathrm{co}} F_\delta(T) \subset F_\varepsilon(T).$$

Lemma 3.6. Let E be a strictly convex Banach space and let C be a nonempty compact convex subset of E. Then,

$$\lim_{n\to\infty} \sup_{\substack{y\in C\\ T\in N(C)}} \left\| \frac{1}{n}\sum_{i=1}^{n} T^i y - T\left(\frac{1}{n}\sum_{i=1}^{n} T^i y\right) \right\| = 0.$$

Using Lemmas 3.5 and 3.6, we have the following lemma (see also [2, 8, 15, 21]).

Lemma 3.7. Let C be a nonempty compact convex subset of a strictly convex Banach space E, let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C and let $x \in C$. Let $\{\mu_\alpha : \alpha \in I\}$ be a net of finite means on S such that

$$\lim_\alpha \|\mu_\alpha - r_t^* \mu_\alpha\| = 0 \quad \text{for every} \quad t \in S. \tag{$*$}$$

Then, for any $\varepsilon > 0$ and $t \in S$, there exists $\alpha_0(\varepsilon, t) \in I$ such that

$$\left\| \int T(s+p)x d\mu_\alpha(s) - T(t)\Big(\int T(s+p)x d\mu_\alpha(s) \Big) \right\| < \varepsilon$$

for all $\alpha \geq \alpha_0(\varepsilon, t)$ and $p \in S$.

Proof. Let $\varepsilon > 0$ and $t \in S$. From Lemma 3.5, there exists $\delta > 0$ such that

$$\overline{\text{co}} F_\delta(U) \subset F_{\varepsilon/3}(U) \tag{4}$$

for every U in $N(C)$. From Lemma 3.6, there exits $n_1 \in \mathbb{N}$ such that

$$\begin{aligned} &\sup_{s\in S} \left\| \frac{1}{n} \sum_{i=1}^n T(it+s)x - T(t)\Big(\frac{1}{n} \sum_{i=1}^n T(it+s)x \Big) \right\| \\ &= \sup_{s\in S} \left\| \frac{1}{n} \sum_{i=1}^n \big(T(t)\big)^i T(s)x - T(t)\Big(\frac{1}{n} \sum_{i=1}^n \big(T(t)\big)^i T(s)x \Big) \right\| < \delta \end{aligned}$$

for every $n \geq n_1$. So, it follows

$$\frac{1}{n} \sum_{i=1}^n T(it+s)x \in F_\delta(T(t)) \subset \overline{\text{co}} F_\delta(T(t)) \tag{5}$$

for every $s \in S$ and $n \geq n_1$. Let $n \geq n_1$. Then, we have, for $p \in S$ and $\alpha \in I$,

$$\begin{aligned} &\left\| \int T(s+p)x d\mu_\alpha(s) - T(t) \int T(s+p)x d\mu_\alpha(s) \right\| \\ \leq\ & \left\| \int T(s+p)x d\mu_\alpha(s) - \int \frac{1}{n} \sum_{i=1}^n T(it+s+p)x d\mu_\alpha(s) \right\| \\ &+ \left\| \int \frac{1}{n} \sum_{i=1}^n T(it+s+p)x d\mu_\alpha(s) - T(t)\Big(\int \frac{1}{n} \sum_{i=1}^n T(it+s+p)x d\mu_\alpha(s) \Big) \right\| \\ &+ \left\| T(t)\Big(\int \frac{1}{n} \sum_{i=1}^n T(it+s+p)x d\mu_\alpha(s) \Big) - T(t)\Big(\int T(s+p)x d\mu_\alpha(s) \Big) \right\| \\ \leq\ & 2 \left\| \int T(s+p)x d\mu_\alpha(s) - \int \frac{1}{n} \sum_{i=1}^n T(it+s+p)x d\mu_\alpha(s) \right\| \\ &+ \left\| \int \frac{1}{n} \sum_{i=1}^n T(it+s+p)x d\mu_\alpha(s) - T(t)\Big(\int \frac{1}{n} \sum_{i=1}^n T(it+s+p)x d\mu_\alpha(s) \Big) \right\| \\ =\ & 2I_1 + I_2, \end{aligned}$$

and

$$\begin{aligned} I_1 &= \left\| \int T(s+p)x d\mu_\alpha(s) - \int \frac{1}{n} \sum_{i=1}^{n} T(it+s+p)x d\mu_\alpha(s) \right\| \\ &\leq \frac{1}{n} \sum_{i=1}^{n} \left\| \int T(s+p)x d\mu_\alpha(s) - \int T(it+s+p)x d\mu_\alpha(s) \right\| \\ &= \frac{1}{n} \sum_{i=1}^{n} \left\| \int T(s+p)x d(\mu_\alpha - r_{it}^* \mu_\alpha)(s) \right\| \\ &\leq \frac{1}{n} \sum_{i=1}^{n} \sup_{z \in C} \|z\| \|\mu_\alpha - r_{it}^* \mu_\alpha\|. \end{aligned}$$

From the assumption of the net $\{\mu_\alpha : \alpha \in I\}$, there exists $\alpha_1 \in I$ such that $\|\mu_\alpha - r_{it}^*\mu_\alpha\| < \dfrac{\varepsilon}{3\sup_{z\in C}\|z\|}$ for every $\alpha \geq \alpha_1$ and $i \in \{1, 2, \dots n\}$. So, $I_1 < \varepsilon/3$ for every $\alpha \geq \alpha_1$ and $p \in S$. Next we prove that there exists $\alpha_2 \in I$ such that $\int (1/n) \sum_{i=1}^{n} T(it+s+p)x d\mu_\alpha(s) \in \overline{\text{co}} F_\delta(T(t))$ for every $p \in S$ and $\alpha \geq \alpha_2$. If not, we have, for each $\alpha_2 \in I$,

$$\int \frac{1}{n} \sum_{i=1}^{n} T(it+s+p')x d\mu_{\alpha'}(s) \notin \overline{\text{co}} F_\delta\big(T(t)\big).$$

for some $p' \in S$ and $\alpha' \geq \alpha_2$. From the separation theorem, there exists $y_0^* \in E^*$ such that

$$\int \Big\langle \frac{1}{n} \sum_{i=1}^{n} T(it+s+p')x, y_0^* \Big\rangle d\mu_{\alpha'}(s) < \inf\{\langle z, y_0^*\rangle : z \in \overline{\text{co}} F_\delta\big(T(t)\big)\}.$$

Then, from (5), we obtain

$$\begin{aligned} \inf\{\langle z, y_0^*\rangle : z \in \overline{\text{co}} F_\delta(T(t))\} &\leq \inf_{s\in S} \Big\langle \frac{1}{n} \sum_{i=1}^{n} T(it+s+p')x, y_0^* \Big\rangle \\ &\leq \int \Big\langle \frac{1}{n} \sum_{i=1}^{n} T(it+s+p')x, y_0^* \Big\rangle d\mu_{\alpha'}(s) \\ &< \inf\{\langle z, y_0^*\rangle : z \in \overline{\text{co}} F_\delta(T(t))\}. \end{aligned}$$

This is a contradiction. Hence, from (4), there exists $\alpha_2 \in I$ such that

$$\int \frac{1}{n} \sum_{i=1}^{n} T(it+s+p)x d\mu_\alpha(s) \in \overline{\text{co}} F_\delta(T(t)) \subset F_{\varepsilon/3}(T(t)) \tag{6}$$

for every $p \in S$ and $\alpha \geq \alpha_2$. Then, from (6), we obtain $I_2 < \varepsilon/3$ for every $p \in S$ and $\alpha \geq \alpha_2$. Let $\alpha_0 \in I$ with $\alpha_0 \geq \alpha_1$ and $\alpha_0 \geq \alpha_2$. Then, we obtain

$$\left\| \int T(s+p)x d\mu_\alpha(s) - T(t)\left(\int T(s+p)x d\mu_\alpha(s) \right) \right\| \leq 2I_1 + I_2 < \varepsilon$$

for every $\alpha \geq \alpha_0$ and $p \in S$. This completes the proof. □

4. Nonlinear strong ergodic theorems

In this section, we establish our main strong mean ergodic theorem in a strictly convex Banach space. Using Lemmas 3.3 and 3.7, we can show the following lemma which is crucial to prove the main theorem (Theorem 4.2).

Lemma 4.1. Let E be a strictly convex Banach space, let X be a nonempty closed convex subset of E and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on X. Assume $\bigcup_{t\in S} T(t)(X) \subset K$ for some compact subset K of X. Let D be a subspace of $B(S)$ such that $1 \in D$, D is r_s-invariant for each $s \in S$ and the function $t \mapsto \langle T(t)x, x^*\rangle$ is an element of D for each $x \in X$ and $x^* \in E^*$. Let $\{\mu_\alpha : \alpha \in I\}$ be a net of finite means on S such that

$$\lim_\alpha \|\mu_\alpha - r_s^*\mu_\alpha\| = 0 \quad \text{for every } s \in S.$$

Then, for any $x \in X$, $\int T(p+t)x d\mu_\alpha(t)$ converges strongly to a common fixed point y_0 of $T(t), t \in S$ uniformly in $p \in S$. Furthermore, y_0 is independent of $\{\mu_\alpha : \alpha \in I\}$ and for any invariant mean μ on D, $y_0 = T_\mu x = \int T(t)x d\mu(t)$.

Proof. Let $x \in X$. From Mazur's theorem, $C = \overline{\text{co}}\big(\{x\} \cup \bigcup_{t\in S} T(t)(X)\big)$ is a compact subset of X. We see that $C = \overline{\text{co}}\big(\{x\} \cup \bigcup_{t\in S} T(t)(X)\big)$ is convex and invariant under $T(t), t \in S$. Thus, we may assume that $\mathcal{S} = \{T(t) : t \in S\}$ is a nonexpansive semigroup on a compact convex subset of X.

Let $\{\mu_\alpha : \alpha \in I\}$ and $\{\lambda_\beta : \beta \in J\}$ be nets of finite means on S such that

$$\lim_\alpha \|\mu_\alpha - r_t^*\mu_\alpha\| = 0 \quad \text{and} \quad \lim_\beta \|\lambda_\beta - r_t^*\lambda_\beta\| = 0 \tag{$*$}$$

for each $t \in S$. It follows from Lemma 3.7 that for each $h \in S$,

$$\lim_\alpha \sup_p \left\| \int T(p+t)x d\mu_\alpha(t) - T(h)\left(\int T(p+t)x d\mu_\alpha(t)\right)\right\| = 0. \tag{7}$$

Further, by Lemma 3.3, we can take a net $\{p_\alpha\}$ in S such that for any $z \in F(\mathcal{S})$,

$$\lim_\alpha \left\| \int T(p_\alpha + t)x d\mu_\alpha(t) - z\right\| \tag{8}$$

exists. Let $\{\Phi_\alpha\} = \left\{\int T(p_\alpha + t)x d\mu_\alpha(t) : \alpha \in I\right\}$. Then, we first prove that Φ_α converges strongly to a common fixed point of $T(t), t \in S$. From the compactness, $\{\Phi_\alpha\}$ must contain a subnet which converges strongly to a point in C. So, let $\{\Phi_{\alpha_\gamma}\}$ be a subnet of $\{\Phi_\alpha\}$ such that $\lim_\gamma \Phi_{\alpha_\gamma} = y_0 \in C$. From (7), we have, for any $h \in S$,

$$\begin{aligned} 0 = \lim_\alpha \|\Phi_\alpha - T(h)\Phi_\alpha\| &= \lim_\gamma \|\Phi_{\alpha_\gamma} - T(h)\Phi_{\alpha_\gamma}\| \\ &= \|y_0 - T(h)y_0\| \end{aligned}$$

and hence y_0 is a common fixed points of $T(t), t \in S$. So, from (8), we have

$$\lim_\alpha \|\Phi_\alpha - y_0\| = \lim_\gamma \|\Phi_{\alpha_\gamma} - y_0\| = 0.$$

This implies that $\Phi_\alpha \to y_0$. Next we prove that $\int T(h+t)x d\mu_\alpha(t)$ converges strongly to $y_0 \in F(\mathcal{S})$ uniformly in h. In the above argument, take a net $\{p_\alpha{}' : \alpha \in I\}$ in S such that $p_\alpha{}' \geq p_\alpha$ for each $\alpha \in I$. Then, repeating the above argument, we see that $\Phi_\alpha{}' = \int T(p_\alpha{}' + t)xd\mu_\alpha(t)$ converges strongly to a common fixed point y_1 of $T(t), t \in S$. We show $y_0 = y_1$. From Lemma 3.3 and Remark 3.4, we know

$$\lim_\alpha \left\| \int T(p_\alpha{}' + t)xd\mu_\alpha(t) - z \right\| = \lim_\alpha \left\| \int T(p_\alpha + t)xd\mu_\alpha(t) - z \right\| \tag{9}$$

for every $z \in F(\mathcal{S})$. Suppose $y_0 \neq y_1$. Then, Φ_α does not converge strongly to y_1. Since y_0 and y_1 are common fixed points of $T(t), t \in S$, from (9), we have

$$0 \leq \lim_\alpha \|\Phi_\alpha - y_1\| = \lim_\alpha \|\Phi_\alpha{}' - y_1\| = 0$$

and hence $\Phi_\alpha \to y_1$. This is a contradiction. So, we have $y_0 = y_1 \in F(\mathcal{S})$. Since $\{p_\alpha{}'\}$ is an arbitrary net in S such that $p_\alpha{}' \geq p_\alpha$ for each $\alpha \in I$, we have that $\int T(h + p_\alpha + t)xd\mu_\alpha(t)$ converges strongly to y_0 uniformly in $h \in S$. Let $\varepsilon > 0$. Then, there exists $\alpha_0 \in I$ such that

$$\left\| \int T(h + p_\alpha + s)xd\mu_\alpha(s) - y_0 \right\| < \frac{\varepsilon}{2} \tag{10}$$

for every $\alpha \geq \alpha_0$ and $h \in S$. Suppose

$$\mu_{\alpha_0} = \sum_{k=1}^m b_k \delta_{s_k} \quad (b_k \geq 0, \sum_{k=1}^m b_k = 1).$$

Put $\mu_0 = \mu_{\alpha_0}$ and $p_0 = p_{\alpha_0}$. From (10), we have

$$\begin{aligned}
&\left\| \iint T(h + t + p_0 + s)xd\mu_0(s)d\lambda_\beta(t) - y_0 \right\| \\
&= \left\| \iint T(h + t + p_0 + s)xd\mu_0(s)d\lambda_\beta(t) - \int y_0 d\lambda_\beta(t) \right\| \\
&\leq \sup_{t,h \in S} \left\| \int T(h + t + p_0 + s)xd\mu_0(s) - y_0 \right\| < \frac{\varepsilon}{2}
\end{aligned}$$

for every $h \in S$ and $\beta \in J$. Since $\{\lambda_\beta\}$ satisfies $(*)$, there exists β_1 such that

$$\|\lambda_\beta - r^*_{p_0 + s_k} \lambda_\beta\| < \frac{\varepsilon}{2 \max\{1, M\}}$$

for every $k \in \{1, 2, \ldots, m\}$ and $\beta \geq \beta_1$, where $M = \sup_{g \in S} \|T(g)x\|$. Then, we have

$$\begin{aligned}
&\left\| \int T(h + t)xd\lambda_\beta(t) - \iint T(h + t + p_0 + s)xd\mu_0(s)d\lambda_\beta(t) \right\| \\
&= \left\| \int T(h + t)xd\lambda_\beta(t) - \sum_{k=1}^m b_k \int T(h + t + p_0 + s_k)xd\lambda_\beta(t) \right\| \\
&\leq \sum_{k=1}^m b_k \left\| \int T(h + t)xd\lambda_\beta(t) - \int T(h + t)d(r^*_{p_0 + s_k} \lambda_\beta)(t) \right\| \\
&\leq \sum_{k=1}^m b_k \, M \, \|\lambda_\beta - r^*_{p_0 + s_k} \lambda_\beta\| < \frac{\varepsilon}{2}
\end{aligned}$$

for every $\beta \geq \beta_1$ and $h \in S$. Therefore,

$$\begin{aligned} &\left\| \int T(h+t)xd\lambda_\beta(t) - y_0 \right\| \\ &\leq \left\| \int T(h+t)xd\lambda_\beta(t) - \iint T(h+t+p_0+s)xd\mu_0(s)d\lambda_\beta(t) \right\| \\ &\quad + \left\| \iint T(h+t+p_0+s)xd\mu_0(s)d\lambda_\beta(t) - y_0 \right\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for every $\beta \geq \beta_1$ and $h \in S$. Hence, $\int T(h+t)xd\lambda_\beta(t)$ converges strongly to y_0 uniformly in $h \in S$. Since $\{\lambda_\beta : \beta \in J\}$ and $\{\mu_\alpha : \alpha \in I\}$ are arbitrary nets of finite means on S such that

$$\lim_\beta \|\lambda_\beta - r_t^*\lambda_\beta\| = 0 \quad \text{and} \quad \lim_\beta \|\mu_\alpha - r_t^*\mu_\alpha\| = 0,$$

for every $t \in S$, we see that such an element y_0 of $F(\mathcal{S})$ is independent of $\{\lambda_\beta : \beta \in J\}$ and $\{\mu_\alpha : \alpha \in I\}$. Finally, we prove that for any invariant mean μ on D, $y_0 = T_\mu x$.

Let $AP(S)$ denote the space of all almost periodic functions on S, i.e., all $f \in B(S)$ such that $RO(f) = \{r_s f : s \in S\}$ is relatively compact in the supremum norm topology of $B(S)$. Then, $AP(S)$ is a closed subalgebra of $B(S)$ invariant under translations. Since S is commutative, $B(S)$, and hence $AP(S)$ has an invariant mean. In fact, $AP(S)$ has a unique invariant mean m. To see this, let Σ denote the spectrum of the Banach algebra $AP(S)$, i.e., the set of nonzero multiplicative linear functionals on $AP(S)$ with the relative weak*-topology from $AP(S)^*$. Then, Σ is a compact Hausdorff space, $\{\delta_s : s \in S\}$ is dense in Σ and Σ is a commutative compact topological semigroup with multiplications $\langle \theta_1 + \theta_2, f\rangle = \iint f(s+t)d\theta_1(s)d\theta_2(t)$, $\theta_1, \theta_2 \in \Sigma$. Furthermore, the Banach algebras $AP(S)$ and $C(\Sigma)$ (bounded continuous real-valued functions on Σ) are isometrically isomorphic via the Gelfand transform $\sigma : f \mapsto \hat{f}$, and m is an invariant mean on $AP(S)$ if and only if $(\sigma^{-1})^* m = \hat{m}$ is an invariant mean on $C(\Sigma)$. It follows from [9, Corollary 2.5, p.23] that $C(\Sigma)$ has a unique invariant mean. Hence $AP(S)$ also has a unique invariant mean.

We next show that for each $x \in X$ and $x^* \in E^*$, the function $f(t) = \langle T(t)x, x^*\rangle$ is in $AP(S)$. Indeed, let Y be the norm closure of $\{T(t)x : t \in S\}$. Then, Y is compact. For each $y \in Y$, let $h_y(t) = \langle T(t)y, x^*\rangle$. Then, for $a \in S$, $r_a f(t) = f(t+a) = \langle T(t)T(a)x, x^*\rangle$ and hence $\{r_a f : a \in S\} \subset \{h_y : y \in Y\}$. Now, the map $y \mapsto h_y$ is continuous from Y into $B(S)$ by nonexpansiveness of each $T(t), t \in S$. Hence $f \in AP(S)$.

Let $\{\mu_\beta\}$ be a net of finite means on S such that $\lim_\beta \|\mu_\beta - r_s^*\mu_\beta\| = 0$ for all $s \in S$. Such a net always exists since S is commutative (see [11]). Now let μ be a weak*-cluster point of $\{\mu_\beta\}$ in D^*. Then, μ is an invariant mean on D. Let $x \in X, x^* \in E^*$, and $f(t) = \langle T(t)x, x^*\rangle$. We will show that $\langle \mu, f\rangle = \langle m, f\rangle$, where m is the unique invariant mean on $AP(S)$. If $\langle \mu, f\rangle \neq \langle m, f\rangle$, by Hahn-Banach extension theorem, there is a mean $\tilde{\mu}$ on $B(S)$ such that $\tilde{\mu}$ extends μ. Let ν be any invariant mean on $B(S)$, $\nu \odot \tilde{\mu} \in B(S)^*$ be defined by $\langle \nu \odot \tilde{\mu}, h\rangle = \langle \nu, \tilde{\mu} \cdot h\rangle$, where $(\tilde{\mu} \cdot h)(t) = \langle \tilde{\mu}, r_t h\rangle, t \in S$. Then, as readily checked $\nu \odot \tilde{\mu}$ is also an invariant mean on $B(S)$, and $\langle \nu \odot \tilde{\mu}, f\rangle = \langle \tilde{\mu}, f\rangle = \langle \mu, f\rangle \neq \langle m, f\rangle$. Consequently, the restriction of $\nu \odot \tilde{\mu}$ to $AP(S)$ is an invariant mean on $AP(S)$ different from m, which contradict the uniqueness of m on $AP(S)$. So, $\langle \mu, f\rangle = \langle m, f\rangle$. Consequently we have

$$\int \langle T(t)x, x^*\rangle d\mu_\beta(t) \to \langle \mu, f\rangle = \int \langle T(t)x, x^*\rangle d\mu(t) = \langle T_\mu x, x^*\rangle.$$

On the other hand, we obtain

$$\int T(t)x d\mu_\beta(t) \to y_0.$$

Hence, we obtain $y_0 = T_\mu x$. □

Let D be a subspace of $B(S)$ containing 1 and r_s-invariant for every $s \in S$. Then, a net $\{\mu_\alpha : \alpha \in I\}$ of linear functionals on D is called strongly regular [15] if it satisfies the following conditions:

(a) $\sup_\alpha \|\mu_\alpha\| < +\infty$;

(b) $\lim_\alpha \mu_\alpha(1) = 1$;

(c) $\lim_\alpha \|\mu_\alpha - r_s^* \mu_\alpha\| = 0$ for every $s \in S$.

A remarkable result of Day [11] shows that for any commutative semigroup S, there is always a strongly regular net of finite means on $B(S)$ and hence on D.

Theorem 4.2. Let E be a strictly convex Banach space, let X be a nonempty a closed convex subset of E and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on X. Assume $\bigcup_{t\in S} T(t)(X) \subset K$ for some compact subset K of X. Let D be a subspace of $B(S)$ such that $1 \in D$, D is r_s-invariant for each $s \in S$ and the function $t \mapsto \langle T(t)x, x^*\rangle$ is an element of D for each $x \in X$ and $x^* \in E^*$. Let $\{\lambda_\alpha : \alpha \in A\}$ be a strongly regular net of continuous linear functionals on D and let $x \in X$. Then, $\int T(h+t)x d\lambda_\alpha(t)$ converges strongly to a common fixed point y_0 of $T(t), t \in S$ uniformly in $h \in S$. Further, such an element y_0 of $F(\mathcal{S})$ is independent of $\{\lambda_\alpha\}$ and for any invariant mean μ on D, $y_0 = T_\mu x = \int T(t)x d\mu(t)$. In this case, putting $Qx = \lim_\alpha \int T(t)x d\lambda_\alpha(t)$ for each $x \in X$, Q is a nonexpansive mapping of X onto $F(\mathcal{S})$ such that $QT(t) = T(t)Q = Q$ for every $t \in S$ and $Qx \in \overline{\text{co}}\{T(s)x : s \in S\}$ for every $x \in X$.

Proof. Let $\{\lambda_\alpha : \alpha \in A\}$ be a strongly regular net of continuous linear functionals on D and let $\{\mu_\beta : \beta \in B\}$ be a net of finite means on S such that

$$\lim_\beta \|\mu_\beta - r_t^* \mu_\beta\| = 0 \quad \text{for every } t \in S. \tag{$*$}$$

From Lemma 4.1, we have that $\int T(h+t)x d\mu_\beta(t)$ converges strongly to a common fixed point y_0 of $T(t), t \in S$ uniformly in $h \in S$. Let $\varepsilon > 0$ and let μ be an invariant mean on D. From Lemma 4.1, we also know $y_0 = T_\mu x$. Further, there exists β_1 such that

$$\left\| \int T(h+t)x d\mu_\beta(t) - T_\mu x \right\| < \frac{\varepsilon}{\sup_\alpha \|\lambda_\alpha\|}$$

for all $\beta \geq \beta_1$ and $h \in S$. Suppose

$$\mu_{\beta_1} = \sum_{i=1}^{n} b_i \delta_{t_i} \quad (b_i \geq 0,\ \sum_{i=1}^{n} b_i = 1) \tag{11}$$

and put $\mu_1 = \mu_{\beta_1}$. Then, we have

$$\left\| \int T(h+t)x d\mu_1(t) - \ T_\mu x \right\| < \frac{\varepsilon}{\sup_\alpha \|\lambda_\alpha\|} \tag{12}$$

for every $h \in S$. Since $\{\lambda_\alpha\}$ is strongly regular, there exists α_0 such that

$$|1 - \lambda_\alpha(1)| < \frac{\varepsilon}{\max\{1, \|T_\mu x\|\}}$$

and

$$\|\lambda_\alpha - r_{t_i}^* \lambda_\alpha\| < \frac{\varepsilon}{\max\{1, M\}} \tag{13}$$

for every $i \in \{1, 2, \cdots, n\}$ and $\alpha \geq \alpha_0$, where $M = \sup_{g \in S} \|T(g)x\|$. Then, we have

$$\begin{aligned} \left\| T_\mu x - \int T_\mu x d\lambda_\alpha(s) \right\| &= \sup_{x^* \in S_1(E^*)} \left| \langle T_\mu x, x^* \rangle - \int \langle T_\mu x, x^* \rangle d\lambda_\alpha(s) \right| \\ &\leq \sup_{x^* \in S_1(E^*)} \left| \langle T_\mu x, x^* \rangle \right| \cdot |1 - \lambda_\alpha(1)| < \varepsilon \end{aligned}$$

for every $\alpha \geq \alpha_0$ and from (12),

$$\begin{aligned} &\left\| \iint T(h+s+t)x d\mu_1(t) d\lambda_\alpha(s) - \int T_\mu x d\lambda_\alpha(s) \right\| \\ &\leq \|\lambda_\alpha\| \cdot \sup_{s,h \in S} \left\| \int T(h+s+t)x d\mu_1(t) - T_\mu x \right\| < \varepsilon \end{aligned}$$

for every $h \in S$ and $\alpha \in A$. Thus, we obtain

$$\left\| \iint T(h+s+t)x d\mu_1(t) d\lambda_\alpha(s) - T_\mu x \right\| < \varepsilon + \varepsilon = 2\varepsilon$$

for every $h \in S$ and $g\alpha \geq \alpha_0$. On the other hand, from (11) and (13), we have

$$\begin{aligned} &\left\| \int T(h+s)x d\lambda_\alpha(s) - \iint T(h+s+t)x d\mu_1(t) d\lambda_\alpha(s) \right\| \\ &= \left\| \int T(h+s)x d\lambda_\alpha(s) - \sum_{i=1}^{n} b_i \int T(h+s+t_i)x d\lambda_\alpha(s) \right\| \\ &\leq \sum_{i=1}^{n} b_i \left\| \int T(h+s)x d\lambda_\alpha(s) - \int T(h+s+t_i)x d\lambda_\alpha(s) \right\| \\ &= \sum_{i=1}^{n} b_i \left\| \int T(h+s)x d(\lambda_\alpha - r_{t_i}^* \lambda_\alpha)(s) \right\| \\ &\leq \sum_{i=1}^{n} b_i \|\lambda_\alpha - r_{t_i}^* \lambda_\alpha\| \cdot M < \varepsilon \end{aligned}$$

for every $h \in S$ and $\alpha \geq \alpha_0$. Therefore, we obtain

$$\begin{aligned}
&\left\|\int T(h+s)xd\lambda_\alpha(s) - T_\mu x\right\| \\
&\leq \left\|\int T(h+s)xd\lambda_\alpha(s) - \iint T(h+s+t)xd\mu_1(t)d\lambda_\alpha(s)\right\| \\
&\qquad + \left\|\iint T(h+s+t)xd\mu_1(t)d\lambda_\alpha(s) - T_\mu x\right\| \\
&< \varepsilon + 2\varepsilon = 3\varepsilon
\end{aligned}$$

for every $h \in S$ and $\alpha \geq \alpha_0$. Then, $\int T(h+t)xd\lambda_\alpha(t)$ converges strongly to a common fixed point y_0 of $T(t), t \in S$ uniformly in h. Further, such an element y_0 is independent of $\{\lambda_\alpha\}$ and $y_0 = T_\mu x$ for any invariant mean μ on D. If $Qx = \lim_\alpha \int T(t)xd\lambda_\alpha(t)$ for each $x \in X$, then Q is a nonexpansive mapping of X onto $F(\mathcal{S})$ such that $QT(t) = T(t)Q = Q$ for every $t \in S$ and $Qx \in \overline{\text{co}}\{T(s)x : s \in S\}$ for every $x \in X$. □

Using Lemma 4.1, we also have the following result.

Theorem 4.3. Let E, X, D and $\mathcal{S} = \{T(t) : t \in S\}$ be as in Theorem 4.2. Assume $\bigcup_{t\in S} T(t)(X) \subset K$ for some compact subset K of X. Then, $T(t)x$ is strongly convergent if and only if

$$T(s+t)x - T(t)x \to 0 \quad \text{for every } s \in S. \tag{14}$$

In this case, the limit point of $\{T(t)x : t \in S\}$ is a common fixed point of $T(t), t \in S$.

Proof. It is trivial to show the "only if" part. Let $\{\mu_\alpha : \alpha \in A\}$ be a net of finite means on S such that

$$\lim_\alpha \|\mu_\alpha - r_t^*\mu_\alpha\| = 0 \qquad \text{for every } t \in S. \tag{$*$}$$

Then, from Lemma 4.1, $\lim_\alpha \int T(h+t)xd\mu_\alpha(t)$ converges strongly to $y_0 \in F(\mathcal{S})$ uniformly in $h \in S$. Let $\varepsilon > 0$. Then, there exists α_0 such that $\left\|\int T(h+t)xd\mu_\alpha(t) - y_0\right\| < \varepsilon/2$ for every $\alpha \geq \alpha_0$ and $h \in S$. Put $\mu_{\alpha_0} = \sum_{i=1}^n a_i\delta_{s_i}$ $(a_i \geq 0,\ \sum_{i=1}^n a_i = 1)$. From (14), there exists $t_0 \in S$ such that $\|T(t+s_i)x - T(t)x\| < \varepsilon/2$ for every $t \geq t_0$ and $i = 1, 2, \ldots, n$. Then, we obtain

$$\begin{aligned}
\|T(t)x - y_0\| &= \left\|\int T(t)xd\mu_{\alpha_0}(s) - y_0\right\| \\
&\leq \left\|\int T(t+s)xd\mu_{\alpha_0}(s) - y_0\right\| + \left\|\int [\,T(t+s)x - T(t)x\,]\,xd\mu_{\alpha_0}(s)\right\| \\
&< \frac{\varepsilon}{2} + \sum_{i=1}^n a_i \left\|T(t+s_i)x - T(t)x\right\| < \varepsilon
\end{aligned}$$

for every $t \geq t_0$. This implies that $\lim_t T(t)x = y_0 \in F(\mathcal{S})$. □

5. Applications

We now apply Theorem 4.2 to obtain other strong nonlinear ergodic theorems with compact domains (for related results, see [2, 3]).

Theorem 5.1 ([2])**.** Let X be a nonempty closed convex subset of a strictly convex Banach space E. Let T be a nonexpansive mapping of X into itself such that $T(X)$ is relatively compact. Then, for any $x \in X$, $(1/n)\sum_{i=0}^{n-1} T^{i+k}x$ converges strongly to some $y \in F(T)$, as $n \to \infty$, uniformly in $k \in \mathbb{Z}^+$.

Proof. Let $S = \mathbb{Z}^+, \mathcal{S} = \{T^i : i \in S\},\ D = B(S)$ and $\lambda_n(f) = (1/n)\sum_{i=0}^{n-1} f(i)$ for all $n \in \mathbb{N}$ and $f \in D$. Then, $\{\lambda_n : n \in \mathbb{N}\}$ is a sequence of means. Further, we have

$$\begin{aligned}\|\lambda_n - r_1^*\lambda_n\| &= \sup_{\|f\|\le 1} |(\lambda_n - r_1^*\lambda_n)(f)| \\ &= \frac{1}{n}\sup_{\|f\|\le 1} |f(0) - f(n)| \le \frac{2}{n} \to 0,\end{aligned}$$

as $n \to \infty$ and hence for $k \ge 2$,

$$\begin{aligned}\|\lambda_n - r_k^*\lambda_n\| &\le \|r_k^*\lambda_n - r_{k-1}^*\lambda_n\| + \cdots + \|r_1^*\lambda_n - \lambda_n\| \\ &\le k\|\lambda_n - r_1^*\lambda_n\| \to 0,\end{aligned}$$

as $n \to \infty$. Therefore, we obtain Theorem 5.1 by using Theorem 4.2. □

Theorem 5.2. Let E, X, T be as in Theorem 5.1. Then, for each $x \in X$, $(1 - s)\sum_{i=0}^{\infty} s^i T^{i+k}x$ converges strongly to some $y \in F(T)$, as $s \uparrow 1$, uniformly in $k \in \mathbb{Z}^+$.

Proof. Let $S = \mathbb{Z}^+, \mathcal{S} = \{T^i : i \in S\},\ D = B(S)$ and $\lambda_s(f) = (1 - s)\sum_{i=0}^{\infty} s^i f(i)$ for every $s \in (0,1)$ and $f \in D$. Then, $\{\lambda_s : s \in (0,1)\}$ is a net of means. Further, we have, $\|\lambda_s - r_k^*\lambda_s\| \to 0$ for every $k \in \mathbb{Z}^+$. Indeed, we have, for any $k \ge 2$,

$$\begin{aligned}\|\lambda_s - r_k^*\lambda_s\| &= \sup_{\|f\|\le 1} |(\lambda_s - r_k^*\lambda_s)(f)| \\ &= \sup_{\|f\|\le 1} \left|(1-s)\sum_{i=0}^{k-1} s^i f(i) + (1-s)\sum_{i=k}^{\infty} s^i f(i) - (1-s)\sum_{i=0}^{\infty} s^i f(i+k)\right| \\ &= \sup_{\|f\|\le 1} \left|(1-s)\sum_{i=0}^{k-1} s^i f(i) + (1-s)\sum_{i=0}^{\infty} s^{i+k} f(i+k) \right. \\ &\qquad\qquad \left. -(1-s)\sum_{i=0}^{\infty} s^i f(i+k)\right| \\ &\le (1-s)\sum_{i=0}^{k-1} s^i\|f\| + (1-s)\sum_{i=0}^{\infty} s^i |s^k - 1|\|f\| \\ &= 2(1-s^k)\|f\| \to 0,\end{aligned}$$

as $s \to 1$. Therefore, we obtain Theorem 5.2 by using Theorem 4.2. □

Let $Q = \{q_{n,m}\}_{n,m\in\mathbb{Z}^+}$ be a matrix satisfying the following conditions:

(a) $\sup_{n\in\mathbb{Z}^+}\sum_{m=0}^{\infty}|q_{n,m}|<\infty$;

(b) $\lim_{n\to\infty}\sum_{m=0}^{\infty}q_{n,m}=1$;

(c) $\lim_{n\to\infty}\sum_{m=0}^{\infty}|q_{n,m+1}-q_{n,m}|=0$.

Then, according to Lorentz [18], Q is called a strongly regular matrix. If Q is a strongly regular matrix, then for each $m\in\mathbb{Z}^+$, we have that $|q_{n,m}|\to 0$, as $n\to\infty$ (see also [15]).

Theorem 5.3. Let E, X and T be as in Theorem 5.1. Let $Q=\{q_{n,m}\}_{n,m\in\mathbb{Z}^+}$ be a strongly regular matrix. Then, for any $x\in X$, $\sum_{m=0}^{\infty}q_{n,m}T^{m+k}x$ converges strongly to some $y\in F(T)$, as $n\to\infty$, uniformly in $k\in\mathbb{Z}^+$.

Proof. Let $S=\mathbb{Z}^+, \mathcal{S}=\{T^i : i\in S\},\ D=B(S)$ and $\lambda_n(f)=\sum_{m=0}^{\infty}q_{n,m}f(m)$ for each $n\in\mathbb{N}$ and $f\in D$. Then, $\{\lambda_n : n\in\mathbb{N}\}$ is a sequence of means. Further, we have $\|\lambda_n-r_k^*\lambda_n\|\to 0$ for every $k\in\mathbb{Z}^+$. Indeed, we have that

$$\begin{aligned}\|\lambda_n-r_1^*\lambda_n\| &= \sup_{\|f\|\le 1}|(\lambda_n-r_1^*\lambda_n)(f)|\\ &= \sup_{\|f\|\le 1}\left|\sum_{m=0}^{\infty}q_{n,m}\{f(m)-f(m+1)\}\right|\\ &= \sup_{\|f\|\le 1}\left|q_{n,0}f(0)+\sum_{m=0}^{\infty}q_{n,m+1}f(m+1)-\sum_{m=0}^{\infty}q_{n,m}f(m+1)\right|\\ &\le \sum_{m=0}^{\infty}|q_{n,m+1}-q_{n,m}|+|q_{n,0}|\to 0,\end{aligned}$$

as $n\to\infty$ and hence for $k\ge 2$,

$$\begin{aligned}\|\lambda_n-r_k^*\lambda_n\| &\le \|r_k^*\lambda_n-r_{k-1}^*\lambda_n\|+\cdots+\|r_1^*\lambda_n-\lambda_n\|\\ &\le k\|\lambda_n-r_1^*\lambda_n\|\to 0,\end{aligned}$$

as $n\to\infty$. So, using Theorem 4.2, we obtain Theorem 5.3. □

Theorem 5.4. Let X be a nonempty closed convex subset of a strictly convex Banach space E. Let U and T be nonexpansive mappings of X into itself with $UT=TU$. Assume $\big(U(X)\cup T(X)\big)\subset K$ for some compact subset K of X. Then, for each $x\in X$, $(1/n^2)\sum_{i,j=0}^{n-1}U^{i+k}T^{j+h}x$ converges strongly to some $y\in F(U)\cap F(T)$, as $n\to\infty$, uniformly in $k,h\in\mathbb{Z}^+$.

Proof. Let $S=\mathbb{Z}^+\times\mathbb{Z}^+, \mathcal{S}=\{U^iT^j : (i,j)\in S\},\ D=B(S)$ and $\lambda_n(f)=(1/n^2)\sum_{i,j=0}^{n-1}f(i,j)$ for each $n\in\mathbb{N}$ and $f\in D$. Then, $\{\lambda_n : n\in\mathbb{N}\}$ is a sequence of means. Further, we have that for each $(l,m)\in S$,

$$\|\lambda_n-r_{(l,m)}^*\lambda_n\| = \sup_{\|f\|\le 1}|(\lambda_n-r_{(l,m)}^*\lambda_n)(f)|$$

$$
\begin{aligned}
&= \sup_{\|f\|\le 1} \left| \frac{1}{n^2}\sum_{i,j=0}^{n-1} f(i,j) - \frac{1}{n^2}\sum_{i,j=0}^{n-1} f(i+l, j+m) \right| \\
&\le \frac{1}{n^2}\{l\cdot n + m(n-l) + l\cdot n + m(n-l)\} \\
&= \frac{1}{n^2}\{2n(l+m) - 2ml\} \to 0,
\end{aligned}
$$

as $n \to \infty$. Therefore, using Theorem 4.2, we obtain Theorem 5.4. □

Let X be a closed convex subset of a Banach space E and let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a family of nonexpansive mappings of X into itself. Then, $\mathcal{S}$ is called a one-parameter nonexpansive semigroup on X if it satisfies the following conditions: $T(0) = I$, $T(t+s) = T(t)T(s)$ for all $t, s \in \mathbb{R}^+$ and $T(t)x$ is continuous in $t \in \mathbb{R}^+$ for each $x \in X$.

Theorem 5.5 ([3])**.** Let X be a nonempty compact convex subset of a strictly convex Banach space E and let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a one-parameter nonexpansive semigroup on X. Then, for any $x \in X$, $(1/s)\int_0^s T(t+k)x dt$ converges strongly to some $y \in F(\mathcal{S})$, as $s \to \infty$, uniformly in $k \in \mathbb{R}^+$.

Proof. Let $S = \mathbb{R}^+$, $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ and let D be the Banach space $C(S)$ of all bounded continuous functions on S with the supremum norm. Define $\lambda_s(f) = (1/s)\int_0^s f(t)dt$ for every $s > 0$ and $f \in D$. Then, $\{\lambda_s : 0 < s < \infty\}$ is a net of means. Further, we obtain that for any k with $0 < k < \infty$,

$$
\begin{aligned}
\|\lambda_s - r_k^*\lambda_s\| &= \sup_{\|f\|\le 1} \left| \frac{1}{s}\int_0^s f(t)dt - \frac{1}{s}\int_0^s f(t+k)dt \right| \\
&= \frac{1}{s}\sup_{\|f\|\le 1} \left| \int_0^s f(t)dt - \int_k^{s+k} f(t)dt \right| \\
&= \frac{1}{s}\sup_{\|f\|\le 1} \left| \int_0^k f(t)dt - \int_s^{s+k} f(t)dt \right| \\
&\le \frac{1}{s}\sup_{\|f\|\le 1} \left(\int_0^k |f(t)|dt + \int_s^{s+k} |f(t)|dt \right) \\
&= \frac{2k}{s} \to 0,
\end{aligned}
$$

as $s \to \infty$. Therefore, using Theorem 4.2, we obtain Theorem 5.5. □

Theorem 5.6. Let $E, X, \mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be as in Theorem 5.5. Then, for any $x \in X$, $r\int_0^\infty e^{-rt}T(t+k)x dt$ converges strongly to some $y \in F(\mathcal{S})$, as $r \downarrow 0$, uniformly in $k \in \mathbb{R}^+$.

Proof. Let $S = \mathbb{R}^+$, $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ and $D = C(S)$. Define $\lambda_r(f) = r\int_0^\infty e^{-rt}f(t)dt$ for each $r > 0$ and $f \in D$. Then, $\{\lambda_r : 0 < r < \infty\}$ is a net of means. Further, we have that for each s with $0 < s < \infty$,

$$
\|\lambda_r - r_s^*\lambda_r\| = \sup_{\|f\|\le 1} \left| r\int_0^\infty e^{-rt}f(t)dt - r\int_0^\infty e^{-rt}f(s+t)dt \right|
$$

$$= \sup_{\|f\|\le 1} \left| r\int_0^s e^{-rt} f(t)dt + r\left(1-e^{rs}\right)\int_s^\infty e^{-rt} f(t)dt \right|$$
$$\le rs + |1-e^{rs}| \to 0,$$

as $r \downarrow 0$. Therefore, using Theorem 4.2, we obtain Theorem 5.6. □

Let $Q = \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be a function satisfying the following conditions:

(a) $\sup_{s\in\mathbb{R}^+} \int_0^\infty |Q(s,t)|dt < \infty$;

(b) $\lim_{s\to\infty} \int_0^\infty Q(s,t)dt = 1$;

(c) $\lim_{s\to\infty} \int_0^\infty |Q(s,t+h) - Q(s,t)|dt = 0 \quad$ for every $\quad h \in \mathbb{R}^+$.

Then, Q is called a strongly regular kernel.

Theorem 5.7. Let $E, X, \mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be as in Theorem 5.5. Let $Q : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be a strongly regular kernel. Then, for any $x \in X$, $\int_0^\infty Q(s,t)T(t+h)xdt$ converges strongly to some $y \in F(\mathcal{S})$, as $s \to \infty$, uniformly in $h \in \mathbb{R}^+$.

Proof. Let $S = \mathbb{R}^+$, $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ and $D = C(S)$. Define $\lambda_s(f) = \int_0^\infty Q(s,t)f(t)dt$ for every $s > 0$ and $f \in D$. Then, $\{\lambda_s : 0 < s < \infty\}$ is a net of means. Further, we have that for each h with $0 < h < \infty$,

$$\begin{aligned}
\|\lambda_s - r_h^*\lambda_s\| &= \sup_{\|f\|\le 1} |(\lambda_s - r_h^*\lambda_s)(f)| \\
&= \sup_{\|f\|\le 1} \left| \int_0^\infty Q(s,t)f(t)dt - \int_0^\infty Q(s,t)f(t+h)dt \right| \\
&= \sup_{\|f\|\le 1} \left| \int_0^h Q(s,t)f(t)dt + \int_0^\infty Q(s,t+h)f(t+h)dt \right. \\
&\qquad \left. - \int_0^\infty Q(s,t)f(t+h)dt \right| \\
&\le \left| \int_0^h Q(s,t)dt \right| + \left| \int_0^\infty |Q(s,t+h) - Q(s,t)|\, dt \right| \to 0,
\end{aligned}$$

as $s \to \infty$. Therefore, using Theorem 4.2, we obtain Theorem 5.7. □

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Manuscript received March 25, 2000
revised August 2, 2000

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