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OPTIMIZATION OF FUZZY FEEDBACK CONTROL DETERMINED BY PRODUCT-SUM-GRAVITY METHOD

TAKASHI MITSUISHI, JUN KAWABE*, KATSUMI WASAKI, AND YASUNARI SHIDAMA

Abstract. The fuzzy feedback control discussed here is a nonlinear feedback control, in which the feedback laws are determined by if-then type fuzzy rules through product-sum-gravity method. In this paper, we show that a set of admissible fuzzy controllers is compact metrizable with respect to an appropriate topology on fuzzy membership functions. Using this compactness, we show the existence of a fuzzy controller which minimizes (maximizes) the integral cost (benefit) function of the feedback system.

1. Introduction

In 1965, Zadeh [7] introduced the notion of fuzziness, and then Mamdani [2] has applied it to the field of control theory using what is called Mamdani method. This method is one of the ways to numerically represent the control given by human language and sensitivity, and it has been applied in various practical control plants. However, unlike the theory of classical control and modern control, systematized considerations have not yet been discussed sufficiently.

The authors have been trying to give a mathematical framework in fuzzy feedback control theory, and to establish the automatic and computational determination of fuzzy membership functions, which give optimal controls in some fuzzy feedback control systems (see [4] and [6]). Especially, in [6] we studied some properties of fuzzy controllers constructed by Mamdani method, and showed the existence of an optimal fuzzy feedback control, in which the feedback laws are given by the amount of operation from the fuzzy controllers.

Since the control given by Mamdani method might not change smoothly, it is pointed out that his method is not necessarily appropriate to express the human intuition. Then, Mizumoto [5] recently proposed the product-sum-gravity method by replacing minimum with product and maximum with summation in Mamdani method. The fuzzy feedback control discussed here is a nonlinear feedback control, in which the feedback laws are determined by if-then type fuzzy control rules through not Mamdani but product-sum-gravity method.

In section 2, for the convenience of the reader, we briefly explain fuzzy control methods (Mamdani method and product-sum-gravity method) which are widely used in practical applications. In section 3, we introduce the notion of an admissible

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fuzzy controller, and show that the set of such controllers is compact metrizable for an appropriate topology on fuzzy membership functions. In section 4, using this compactness, we show the existence of an optimal fuzzy controller in the nonlinear fuzzy feedback control system determined by product-sum-gravity method.

Throughout this paper, $\mathbb{R}^n$ denotes the finite dimensional Banach space of all n-tuple vectors $x = (x_1, x_2, \ldots, x_n)$ $(x_1, x_2, \ldots, x_n \in \mathbb{R})$ with the norm $\|x\| := \max_{1 \leq i \leq n} |x_i|$. For any $a, b \in \mathbb{R}$, we set lattice operations $a \vee b := \max(a, b)$ and $a \wedge b := \min(a, b)$.

2. Fuzzy controller

In this section, for the convenience of the reader, we briefly explain some of fuzzy control methods which are widely used in practical applications:

Assume the following if-then type fuzzy control rules are given.

(2.1)

$$\begin{array}{ll} \text{Rule } 1 & : \text{if } x_1 \text{ is } A_{11} \text{ and } x_2 \text{ is } A_{12} \text{ and } \ldots \text{ and } x_n \text{ is } A_{1n} \text{ then } y \text{ is } B_1 \\ \text{Rule } 2 & : \text{if } x_1 \text{ is } A_{21} \text{ and } x_2 \text{ is } A_{22} \text{ and } \ldots \text{ and } x_n \text{ is } A_{2n} \text{ then } y \text{ is } B_2 \\ & \qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ \text{Rule } m & : \text{if } x_1 \text{ is } A_{m1} \text{ and } x_2 \text{ is } A_{m2} \text{ and } \ldots \text{ and } x_n \text{ is } A_{mn} \text{ then } y \text{ is } B_m \end{array}$$

Let $\mu_{A_{ij}}$ $(i = 1, \ldots, m; j = 1, \ldots, n)$ be fuzzy membership functions defined on $[a, b]$ of the fuzzy set A_{ij}. Let μ_{B_i} $(i = 1, \ldots, m)$ be fuzzy membership functions defined on $[c, d]$ of the fuzzy set B_i. For simplicity, we write "if" and "then" parts in the rules by the following notation:

$$\mathcal{A}_i := (\mu_{A_{i1}}, \ldots, \mu_{A_{in}}) \ (i = 1, \ldots, m),$$
$$\mathcal{A} := (\mathcal{A}_1, \ldots, \mathcal{A}_m),$$
$$\mathcal{B} := (\mu_{B_1}, \ldots, \mu_{B_m}).$$

Then, the if-then type fuzzy control rules (2.1) is called a *fuzzy controller*, and is denoted by $(\mathcal{A}, \mathcal{B})$. In the rules in (2.1), $x = (x_1, x_2, \ldots, x_n) \in [a, b]^n$ is called an *input information* given to the fuzzy controller $(\mathcal{A}, \mathcal{B})$, and $y \in [c, d]$ is called an *control variable*.

(A) Mamdani Method: Mamdani method is widely used in fuzzy controls because of its simplicity and comprehensibility [2]. In this method, when an input information $x = (x_1, x_2, \ldots, x_n) \in [a, b]^n$ is given to the fuzzy controller $(\mathcal{A}, \mathcal{B})$, then one can obtain the amount of operation $\rho_{\mathcal{AB}}(x)$ from the controller through the following calculation:

- Procedure 1: The agreement degree of each rule is calculated by

$$\alpha_{\mathcal{A}_i}(x) := \bigwedge_{j=1}^{n} \mu_{A_{ij}}(x_j) \quad (i = 1, \ldots, m).$$

- Procedure 2: The inference result of each rule is calculated by

$$\beta_{B_i \mathcal{A}_i}(x, y) := \alpha_{\mathcal{A}_i}(x) \wedge \mu_{B_i}(y) \quad (i = 1, \ldots, m).$$

• Procedure 3: The inference result of the entire rule is calculated by

$$\gamma_{\mathcal{AB}}(x,y) := \bigvee_{i=1}^{m} \beta_{B_i\mathcal{A}_i}(x,y).$$

• Procedure 4: The center of gravity of the inference result is calculated by

$$\rho_{\mathcal{AB}}(x) := \frac{\int_c^d y\gamma_{\mathcal{AB}}(x,y)dy}{\int_c^d \gamma_{\mathcal{AB}}(x,y)dy}.$$

This method is also called a *min-max-gravity method.* We say that a fuzzy controller $(\mathcal{A},\mathcal{B})$ is *constructed by Mamdani method* if the amount of operation $\rho_{\mathcal{AB}}$ is constructed by the procedures (1)–(4) above.

(B) Product-Sum-Gravity Method: Since min-max-gravity method uses minimum and maximum operations, the value of the agreement degree and the center of gravity might not change smoothly. In addition, it is pointed out that this method is not necessarily appropriate to express the human intuition. Then, Mizumoto [5] recently proposed the product-sum-gravity method by replacing minimum with product and maximum with summation.

• Procedure 1: The agreement degree of each rule is calculated by

$$\alpha_{\mathcal{A}_i}(x) := \prod_{j=1}^{n} \mu_{A_{ij}}(x_j) \quad (i = 1,\ldots,m).$$

• Procedure 2. The inference result of each rule is calculated by

$$\beta_{B_i\mathcal{A}_i}(x,y) := \alpha_{\mathcal{A}_i}(x)\cdot\mu_{B_i}(y) \quad (i = 1,\ldots,m).$$

• Procedure 3: The inference result of the entire rule is calculated by

$$\gamma_{\mathcal{AB}}(x,y) := \sum_{i=1}^{m} \beta_{B_i\mathcal{A}_i}(x,y).$$

• Procedure 4: The center of gravity of the inference result is calculated by

$$\rho_{\mathcal{AB}}(x) := \frac{\int_c^d y\gamma_{\mathcal{AB}}(x,y)dy}{\int_c^d \gamma_{\mathcal{AB}}(x,y)dy}.$$

We say that a fuzzy controller $(\mathcal{A},\mathcal{B})$ is *constructed by product-sum-gravity method* if the amount of operation $\rho_{\mathcal{AB}}$ is constructed by the procedures (1)–(4) above.

In the following sections, we only consider fuzzy controllers constructed by product-sum-gravity method. See [6] for some results concerning fuzzy controllers constructed by Mamdani method.

3. Compactness of a set of admissible fuzzy controllers

In this section, we show that a set of admissible fuzzy controllers is compact and metrizable with respect to an appropriate topology on fuzzy membership functions.

Let $C[a,b]$ be the Banach space of all continuous real functions on $[a,b]$ with the norm $\|\mu\| := \max_{s\in[a,b]} |\mu(s)|$. Denote by $L^1[c,d]$ the Banach space of all Lebesgue measurable real functions μ on $[c,d]$ such that $\int_c^d |\mu(s)|ds < \infty$. We also denote by $L^\infty[c,d]$ the Banach space of all Lebesgue measurable, essentially bounded real functions on $[c,d]$.

Let $\Delta_{ij} > 0$ $(1 \le i \le m; 1 \le j \le n)$. We consider the following two sets of fuzzy membership functions.

$$F_{\Delta_{ij}} := \left\{ \mu \in C[a,b] \quad : \quad \begin{array}{l} 0 \le \mu(s) \le 1 \text{ for } \forall s \in [a,b], \\ |\mu(s) - \mu(s')| \le \Delta_{ij}|s - s'| \text{ for } \forall s, s' \in [a,b] \end{array} \right\},$$

and

$$G := \{\mu \in L^\infty[c,d] : 0 \le \mu(s) \le 1 \text{ a.e. } s \in [c,d]\}.$$

The set $F_{\Delta_{ij}}$, which is more restrictive than G, contains triangular, trapezoidal and bell-shaped fuzzy membership functions with gradients less than positive value Δ_{ij}. Consequently, if $\Delta_{ij} > 0$ is taken large enough, $F_{\Delta_{ij}}$ contains almost all fuzzy membership functions which are used in practical applications. In section 4, we shall assume that the fuzzy membership functions $\mu_{A_{ij}}$ in "if" parts of the rules (2.1) belong to the set $F_{\Delta_{ij}}$. On the other hand, we shall also assume that the fuzzy membership functions μ_{B_i} in "then" parts of the rules (2.1) belong to the set G.

In the following, we endow the space $F_{\Delta_{ij}}$ with the norm topology on $C[a,b]$ and endow the space G with the weak topology $\sigma(L^\infty, L^1)$ on $L^\infty[c,d]$.

Put

$$\mathcal{F} := \prod_{i=1}^{m} \left\{ \prod_{j=1}^{n} F_{\Delta_{ij}} \right\} \times G^m,$$

where G^m denotes the m times Cartesian product of G. Then, each element $(\mathcal{A}, \mathcal{B})$ of $\mathcal{F}$ can be viewed as a fuzzy controller given by the if-then type fuzzy control rules (2.1).

Proposition 1. *$\mathcal{F}$ is compact and metrizable with respect to the product topology on $C[a,b]^{mn} \times G^m$.*

Proof. Since, for each $i = 1, \ldots, m$ and $j = 1, \ldots, n$, $F_{\Delta_{ij}}$ is a closed, bounded and equicontinuous subset of $C[a,b]$, by Ascoli-Arzelà theorem (see, e.g., Theorem IV.6.7 of [1]), $F_{\Delta_{ij}}$ is a compact subset of the Banach space $C[a,b]$.

Next we prove compactness and metrizability of G: Since $L^1[c,d]$ is separable, by Theorem V.4.2 (Alaoglu theorem) and Theorem V.5.1 of [1], the closed unit sphere of $L^\infty[c,d]$ is compact and metrizable for the weak topology $\sigma(L^\infty, L^1)$ on $L^\infty[c,d]$. Hence we have only to show that G is sequentially closed for $\sigma(L^\infty, L^1)$.

Let $\{\mu_n\} \subset G$ converges to μ for $\sigma(L^\infty, L^1)$. Then, we have

$$0 \le \int_c^d \alpha(s)\mu(s)ds \le \int_c^d \alpha(s)ds \tag{3.1}$$

for every $\alpha \in L^1[c,d]$ with $\alpha(s) \ge 0$ a.e. $s \in [c,d]$. Consequently, it easily follows from (3.1) that $0 \le \mu(s) \le 1$ a.e. $s \in [c,d]$. Hence, $\mu \in G$ and thus G is closed for $\sigma(L^\infty, L^1)$.

By Tychonoff theorem

$$\mathcal{F} = \prod_{i=1}^{m} \left\{ \prod_{j=1}^{n} F_{\Delta_{ij}} \right\} \times G^m$$

is compact metrizable for the product topology. □

In the procedure 4 of product-sum-gravity method, the amount of operation $\rho_{\mathcal{AB}}$ is obtained through the gravity calculation

$$\rho_{\mathcal{AB}}(x) := \frac{\int_c^d y\gamma_{\mathcal{AB}}(x,y)dy}{\int_c^d \gamma_{\mathcal{AB}}(x,y)dy}, \quad x = (x_1, \ldots, x_n) \in [a,b]^n.$$

To avoid making the denominator of the expression above equal to 0, for any $\delta > 0$, we consider the set

$$\mathcal{F}_\delta := \left\{ (\mathcal{A}, \mathcal{B}) \in \mathcal{F} : \int_c^d \gamma_{\mathcal{AB}}(x,y)dy \ge \delta \text{ for all } x \in [a,b]^n \right\},$$

which is a slight modification of $\mathcal{F}$. If δ is taken small enough, it is possible to consider $\mathcal{F} = \mathcal{F}_\delta$ for practical applications. We say that an element $(\mathcal{A}, \mathcal{B})$ of $\mathcal{F}_\delta$ is an *admissible* fuzzy controller. Then, we have the following

Proposition 2. *Let $\delta > 0$. The set $\mathcal{F}_\delta$ of all admissible fuzzy controllers is compact and metrizable with respect to the product topology on $C[a,b]^{mn} \times G^m$.*

Proof. We first note that a sequence $\{(\mathcal{A}^k, \mathcal{B}^k)\} \subset \mathcal{F}$ converges to $(\mathcal{A}, \mathcal{B}) \in \mathcal{F}$ for the product topology if and only if, for each $i = 1, \ldots, m$,

$$\|\alpha_{\mathcal{A}_i^k} - \alpha_{\mathcal{A}_i}\|_\infty := \sup_{x \in [a,b]^n} |\alpha_{\mathcal{A}_i^k}(x) - \alpha_{\mathcal{A}_i}(x)| \to 0$$

and

$$\mu_{B_i^k} \to \mu_{B_i} \text{ for the weak topology } \sigma(L^\infty, L^1) \text{ on } L^\infty[c,d].$$

Assume that a sequence $\{(\mathcal{A}^k, \mathcal{B}^k)\}$ in $\mathcal{F}_\delta$ converges to $(\mathcal{A}, \mathcal{B}) \in \mathcal{F}$. Fix $x \in [a,b]^n$. Then, we have

$$\begin{aligned} \int_c^d \gamma_{\mathcal{A}^k\mathcal{B}^k}(x,y)dy &= \sum_{i=1}^{m} \alpha_{\mathcal{A}_i^k}(x) \int_c^d \mu_{B_i^k}(y)dy \\ &\to \sum_{i=1}^{m} \alpha_{\mathcal{A}_i}(x) \int_c^d \mu_{B_i}(y)dy = \int_c^d \gamma_{\mathcal{AB}}(x,y)dy, \end{aligned}$$

which implies $\int_c^d \gamma_{\mathcal{AB}}(x,y)dy \geq \delta$ and hence $(\mathcal{A},\mathcal{B}) \in \mathcal{F}_\delta$. Therefore, $\mathcal{F}_\delta$ is a closed subset of $\mathcal{F}$, and hence by Proposition 1 it is compact and metrizable. □

4. Existence of optimal fuzzy feedback control

In this section, using the compactness of a set of admissible fuzzy controllers, we show the existence of an optimal fuzzy feedback control.

Let $f(x,u) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be a vector valued Lipschitz continuous function with the Lipschitz constant L_f:

$$\|f(x,u) - f(x',u')\| \leq L_f(\|x - x'\| + |u - u'|) \tag{4.1}$$

for all $(x,u),(x',u') \in \mathbb{R}^n \times \mathbb{R}$.

In addition, assume that there exists a constant $M_f > 0$ such that

$$\|f(x,u)\| \leq M_f\,(\|x\| + |u| + 1) \tag{4.2}$$

for all $(x,u) \in \mathbb{R}^n \times \mathbb{R}$.

Consider a system given by the following state equation:

$$\dot{x}(t) = f(x(t),u(t)), \tag{4.3}$$

where $x(t)$ is the state, and the control function $u(t)$ of the system is given by the state feedback

$$u(t) = \rho(x(t)),$$

where $\rho : \mathbb{R}^n \to \mathbb{R}$ is a continuous function. For $r > 0$, $B_r := \{x \in \mathbb{R}^n : \|x\| \leq r\}$ denotes a bounded set containing all possible initial states x_0 of the system. Let T be a final time. Then, we have

Proposition 3. (1) *Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz continuous function and $x_0 \in B_r$. Then, the state equation*

$$\dot{x}(t) = f(x(t),\rho(x(t))) \tag{4.4}$$

has a unique solution $x(t,x_0,\rho)$ on $[0,T]$ with the initial condition $x(0) = x_0$ such that the mapping

$$(t,x_0) \in [0,T] \times B_r \mapsto x(t,x_0,\rho)$$

is continuous.

(2) *For any $r_2 > 0$, denote by Φ the set of all Lipschitz continuous functions $\rho : \mathbb{R}^n \to \mathbb{R}$ with the Lipschitz constant L_ρ satisfying*

$$\sup_{u \in \mathbb{R}^n} |\rho(u)| \leq r_2. \tag{4.5}$$

Then, the following (a) *and* (b) *hold.*

(a) *For any $t \in [0,T], x_0 \in B_r$ and $\rho \in \Phi$, we have*

$$\|x(t,x_0,\rho)\| \leq r_1, \tag{4.6}$$

where

$$r_1 = e^{M_f T} r + (e^{M_f T} - 1)(r_2 + 1). \tag{4.7}$$

(b) *Let $\rho_1, \rho_2 \in \Phi$. Then, for any $t \in [0,T]$ and $x_0 \in B_r$, we have*

$$\|x(t,x_0,\rho_1) - x(t,x_0,\rho_2)\| \leq \frac{e^{L_f\,(1+L_{\rho_1})t} - 1}{1+L_{\rho_1}} \sup_{u\in[-r_1,r_1]^n} |\rho_1(u) - \rho_2(u)|. \tag{4.8}$$

Proof. (1) See [3] for the existence and the uniqueness of the solution. The joint continuity of the solution can be readily proved.

(2) We first prove (a): Let $x(t) = x(t,x_0,\rho)$ be a solution of (4.4) on $[0,T]$ with the initial condition $x(0) = x_0$. Then

$$x(t) = x_0 + \int_0^t f(x(s),\rho(x(s)))ds, \quad 0 \leq \forall t \leq T.$$

By (4.2) and (4.5), we have

$$\begin{aligned}\|x(t)\| &\leq \|x_0\| + \int_0^t \|f(x(s),\rho(x(s)))\|ds\\ &\leq r + \int_0^t M_f(\|x(s)\| + |\rho(x(s))| + 1)ds\\ &\leq r + \int_0^t M_f(\|x(s)\| + r_2 + 1)ds,\end{aligned}$$

and this implies that

$$\|x(t)\| + r_2 + 1 \leq (r + r_2 + 1) + \int_0^t M_f(\|x(s)\| + r_2 + 1)ds$$

for $0 \leq \forall t \leq T$. By Grownwall inequality (see, e.g., Theorem 2.1.6 of [3]), we have

$$\|x(t)\| + r_2 + 1 \leq (r + r_2 + 1)e^{M_f T}.$$

Thus

$$\|x(t)\| \leq re^{M_f T} + (e^{M_f T} - 1)(r_2 + 1),$$

and the proof of (a) is complete.

Next we prove (b): Let $\rho_1, \rho_2 \in \Phi$. Let $x_1(t) = x(t,x_0,\rho_1)$ and $x_2(t) = x(t,x_0,\rho_2)$ be solutions of (4.4) for the feedback ρ_1 and ρ_2, respectively. Then, we have

$$\|x_1(t) - x_2(t)\| \leq \int_0^t \|f(x_1(s),\rho_1(x_1(s))) - f(x_2(s),\rho_2(x_2(s)))\|ds.$$

Since $\|x_2(s)\| \leq r_1$ by (4.6), it follows from (4.1) that

$$\begin{aligned}\|f(x_1(s),\rho_1(x_1(s))) - f(x_2(s),\rho_2(x_2(s)))\| &\leq L_f(1+L_{\rho_1})\|x_1(s) - x_2(s)\|\\ &\quad + L_f \sup_{u\in[-r_1,r_1]^n} |\rho_1(u) - \rho_2(u)|.\end{aligned}$$

Put $\alpha = \sup_{u\in[-r_1,r_1]^n} |\rho_1(u) - \rho_2(u)|$, then we have

$$\|x_1(t) - x_2(t)\| \leq \int_0^t L_f(1+L_{\rho_1})\left\{\|x_1(s) - x_2(s)\| + \frac{\alpha}{1+L_{\rho_1}}\right\}ds.$$

Adding $\alpha/(1+L_{\rho_1})$ to both sides and using Grownwall inequality again, we have

$$\|x_1(t) - x_2(t)\| + \frac{\alpha}{1+L_{\rho_1}} \leq \frac{\alpha}{1+L_{\rho_1}} e^{L_f(1+L_{\rho_1})t},$$

which implies (4.8). □

Let $(\mathcal{A},\mathcal{B})$ be a fuzzy controller given by the if-then type fuzzy control rules (2.1). We say that the system (4.3) is a *fuzzy feedback system* if the control function $u(t)$ is given by the state feedback

$$u(t) = \rho_{\mathcal{A}\mathcal{B}}(x(t)),$$

where $\rho_{\mathcal{A}\mathcal{B}}(x(t))$ is the amount of operation from the fuzzy controller $(\mathcal{A},\mathcal{B})$ for an input information $x(t)$.

In the following, fix $r > 0$, $r_2 > 0$ and a final time T of the control. Put r_1 be the constant determined by (4.7). We also fix $\Delta_{ij} > 0$ $(1 \le i \le m; 1 \le j \le n)$ and $\delta > 0$.

We define two sets of fuzzy membership functions by the same way as in Section 3.

$$F_{\Delta_{ij}} := \left\{ \mu \in C[-r_1, r_1] \quad : \quad \begin{array}{l} 0 \le \mu(s) \le 1 \text{ for } \forall s \in [-r_1, r_1], \\ |\mu(s) - \mu(s')| \le \Delta_{ij}|s - s'| \text{ for } \forall s, s' \in [-r_1, r_1]) \end{array} \right\},$$

and

$$G := \{\mu \in L^\infty[-r_2, r_2] : 0 \le \mu(s) \le 1 \text{ a.e. } s \in [-r_2, r_2]\}.$$

We assume that the fuzzy membership functions $\mu_{A_{ij}}$ in "if" parts of the rules (2.1) belong to the set $F_{\Delta_{ij}}$. On the other hand, we also assume that the fuzzy membership functions μ_{B_i} in "then" parts of the rules (2.1) belong to the set G.

We endow the space $F_{\Delta_{ij}}$ with the norm topology on $C[-r_1, r_1]$ and endow the space G with the weak topology $\sigma(L^\infty, L^1)$ on $L^\infty[-r_2, r_2]$.

Put

$$\mathcal{F} := \prod_{i=1}^{m} \left\{ \prod_{j=1}^{n} F_{\Delta_{ij}} \right\} \times G^m,$$

and

$$\mathcal{F}_\delta := \left\{ (\mathcal{A},\mathcal{B}) \in \mathcal{F} : \int_{-r_2}^{r_2} \gamma_{\mathcal{A}\mathcal{B}}(x, y)dy \ge \delta \text{ for all } x \in [-r_1, r_1]^n \right\}. \tag{4.9}$$

In this paper, for any admissible fuzzy controller $(\mathcal{A},\mathcal{B}) \in \mathcal{F}_\delta$, the amount of operation $\rho_{\mathcal{A}\mathcal{B}} : [-r_1, r_1]^n \to \mathbb{R}$ is constructed by product-sum-gravity method:

$$\rho_{\mathcal{A}\mathcal{B}}(x) := \frac{\int_{-r_2}^{r_2} y\gamma_{\mathcal{A}\mathcal{B}}(x, y)dy}{\int_{-r_2}^{r_2} \gamma_{\mathcal{A}\mathcal{B}}(x, y)dy},$$

where

$$\alpha_{\mathcal{A}_i}(x) := \prod_{j=1}^{n} \mu_{A_{ij}}(x_j) \quad (i = 1, \ldots, m),$$

$$\beta_{B_i\mathcal{A}_i}(x, y) := \alpha_{\mathcal{A}_i}(x) \cdot \mu_{B_i}(y) \quad (i = 1, \ldots, m),$$

and

$$\gamma_{\mathcal{A}\mathcal{B}}(x, y) := \sum_{i=1}^{m} \beta_{B_i\mathcal{A}_i}(x, y)$$

for each $x = (x_1, \ldots, x_n) \in [-r_1, r_1]^n$ and $y \in [-r_2, r_2]$. Then, we have

Proposition 4. *Let* $(\mathcal{A},\mathcal{B}) \in \mathcal{F}_\delta$. *Then, the following* (1) *and* (2) *hold.*
(1) $\rho_{\mathcal{AB}}$ *is Lipschitz continuous on* $[-r_1, r_1]^n$.
(2) $|\rho_{\mathcal{AB}}(x)| \le r_2$ *for all* $x \in [-r_1, r_1]^n$.

Proof. (1) For any $x = (x_1, \dots, x_n)$, $x' = (x'_1, \dots, x'_n) \in [-r_1, r_1]^n$ and any $i = 1, \dots, m$, since $0 \le \mu_{A_{ij}}(x_j), \mu_{A_{ij}}(x'_j) \le 1$, we have

$$\begin{aligned}|\alpha_{\mathcal{A}_i}(x) - \alpha_{\mathcal{A}_i}(x')| &= \left|\prod_{j=1}^n \mu_{A_{ij}}(x_j) - \prod_{j=1}^n \mu_{A_{ij}}(x'_j)\right| \\ &\le \varDelta_i \sum_{j=1}^n |x_j - x'_j| \le n\varDelta_i \|x - x'\|,\end{aligned}$$

where $\varDelta_i = \max_{1\le j\le n} \varDelta_{ij}$. Consequently, noting that $0 \le \mu_{B_i}(y) \le 1$ a.e., we have

$$|\beta_{B_i\mathcal{A}_i}(x, y) - \beta_{B_i\mathcal{A}_i}(x', y)| \le n\varDelta_i \|x - x'\|$$

for a.e. $y \in [-r_2, r_2]$ and hence

$$|\gamma_{\mathcal{AB}}(x, y) - \gamma_{\mathcal{AB}}(x', y)| \le \varDelta \|x - x'\|, \tag{4.10}$$

where $\varDelta = n \sum_{i=1}^m \varDelta_i$.

Put

$$g(x) = \int_{-r_2}^{r_2} y\gamma_{\mathcal{AB}}(x, y)dy \quad \text{and} \quad h(x) = \int_{-r_2}^{r_2} \gamma_{\mathcal{AB}}(x, y)dy.$$

Then, for any $x, x' \in [-r_1, r_1]^n$, by (4.10) we have

$$|g(x) - g(x')| \le r_2^2 \varDelta \|x - x'\| \tag{4.11}$$

and

$$|h(x) - h(x')| \le 2r_2 \varDelta \|x - x'\|. \tag{4.12}$$

Since $h(x) \ge \delta$, $|g(x)| \le mr_2^2$ and $|h(x)| \le 2mr_2$ for all $x \in [-r_1, r_1]^n$, it follows from (4.11) and (4.12) that

$$\begin{aligned}|\rho_{\mathcal{AB}}(x) - \rho_{\mathcal{AB}}(x')| &\le \frac{|g(x) - g(x')||h(x')| + |h(x) - h(x')||g(x')|}{\delta^2} \\ &\le \frac{4mr_2^3\varDelta}{\delta^2}\|x - x'\|,\end{aligned}$$

and the Lipschitz continuity of $\rho_{\mathcal{AB}}$ is proved.

(2) It follows from $|g(x)| \le r_2 h(x)$ that $|\rho_{\mathcal{AB}}(x)| = |g(x)/h(x)| \le r_2$ for all $x \in [-r_1, r_1]^n$. □

It is easily seen that every bounded Lipschitz function $\rho : [-r_1, r_1]^n \to \mathbb{R}$ can be extended to a bounded Lipschitz function $\tilde{\rho}$ on $\mathbb{R}^n$ without increasing its Lipschitz constant and bound. In fact, define $\tilde{\rho} : \mathbb{R}^n \to \mathbb{R}$ by

$$\tilde{\rho}(x) = \tilde{\rho}(x_1, \dots, x_n) = \rho(\varepsilon(x_1), \dots, \varepsilon(x_n)), \quad x \in \mathbb{R}^n$$

where

$$\varepsilon(s) = \begin{cases} r_1, & \text{if } s > r_1 \\ s, & \text{if } -r_1 \le s \le r_1 \\ -r_1, & \text{if } s < -r_1. \end{cases}$$

Let $(\mathcal{A},\mathcal{B}) \in \mathcal{F}_\delta$. Then it follows from Proposition 4 and the fact above that the extension $\tilde{\rho}_{\mathcal{AB}}$ of $\rho_{\mathcal{AB}}$ is Lipschitz continuous on $\mathbb{R}^n$ with the same Lipschitz constant of $\rho_{\mathcal{AB}}$ and satisfies $\sup_{x\in\mathbb{R}^n} |\tilde{\rho}_{\mathcal{AB}}(x)| \le r_2$. Therefore, by Proposition 3 the state equation (4.4) for the feedback law $\tilde{\rho}_{\mathcal{AB}}$ has a unique solution $x(t, x_0, \tilde{\rho}_{\mathcal{AB}})$ with the initial condition $x(0) = x_0$. Though the extension $\tilde{\rho}_{\mathcal{AB}}$ of $\rho_{\mathcal{AB}}$ is not unique in general, the solution $x(t, x_0, \tilde{\rho}_{\mathcal{AB}})$ is uniquely determined by $\rho_{\mathcal{AB}}$ using inequality (4.8) of Proposition 3. Consequently, in the following the extension $\tilde{\rho}_{\mathcal{AB}}$ is written as $\rho_{\mathcal{AB}}$ without confusion.

The performance index of this fuzzy feedback control system is evaluated with the following integral cost function:

$$J = \int_{B_r}\int_0^T w(x(t,\zeta,\rho_{\mathcal{AB}}), \rho_{\mathcal{AB}}(x(t,\zeta,\rho_{\mathcal{AB}})))dtd\zeta, \tag{4.13}$$

where $w : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a positive continuous function. The following theorem guarantees the existence of a fuzzy controller which minimizes and maximizes (4.13).

Theorem 5. *The mapping*

$$(\mathcal{A},\mathcal{B}) \in \mathcal{F}_\delta \mapsto \int_{B_r}\int_0^T w(x(t,\zeta,\rho_{\mathcal{AB}}), \rho_{\mathcal{AB}}(x(t,\zeta,\rho_{\mathcal{AB}})))dtd\zeta$$

has a minimum (*maximum*) value on the compact metric space $\mathcal{F}_\delta$ of all admissible fuzzy controllers defined by (4.9).

Proof. By Proposition 2, $\mathcal{F}_\delta$ is compact metric space with respect to the the product topology on $C[-r_1, r_1]^{mn} \times L^\infty[-r_2, r_2]^m$.

Assume that $(\mathcal{A}^k, \mathcal{B}^k) \to (\mathcal{A},\mathcal{B})$ in $\mathcal{F}_\delta$. Routine calculation gives an estimate

$$\begin{aligned}\sup_{x\in[-r_1,r_1]^n} |\rho_{\mathcal{A}^k\mathcal{B}^k}(x) - \rho_{\mathcal{AB}}(x)| \le{}& \frac{2mr_2}{\delta^2}\sum_{i=1}^m \left|\int_{-r_2}^{r_2} y\mu_{B_i^k}(y)dy - \int_{-r_2}^{r_2} y\mu_{B_i}(y)dy\right| \\ &+ \frac{mr_2^2}{\delta^2}\sum_{i=1}^m \left|\int_{-r_2}^{r_2} \mu_{B_i^k}(y)dy - \int_{-r_2}^{r_2} \mu_{B_i}(y)dy\right| \\ &+ \frac{4mr_2^3}{\delta^2}\sum_{i=1}^m \|\alpha_{\mathcal{A}_i^k} - \alpha_{\mathcal{A}_i}\|_\infty,\end{aligned}$$

where $\|\alpha_{\mathcal{A}_i^k} - \alpha_{\mathcal{A}_i}\|_\infty := \sup_{x\in[-r_1,r_1]^n} |\alpha_{\mathcal{A}_i^k}(x) - \alpha_{\mathcal{A}_i}(x)|$.

Fix $(t,\zeta) \in [0,T] \times B_r$. Then it follows from the estimate above that

$$\lim_{k\to\infty} \sup_{x\in[-r_1,r_1]^n} |\rho_{\mathcal{A}^k\mathcal{B}^k}(x) - \rho_{\mathcal{AB}}(x)| = 0. \tag{4.14}$$

Hence, by (b) of Proposition 3, we have

$$\lim_{k\to\infty} \|x(t,\zeta,\rho_{\mathcal{A}^k\mathcal{B}^k}) - x(t,\zeta,\rho_{\mathcal{AB}})\| = 0. \tag{4.15}$$

Further, it follows from (4.14), (4.15) and (1) of Proposition 4 that

$$\lim_{k\to\infty} \rho_{\mathcal{A}^k\mathcal{B}^k}(x(t,\zeta,\rho_{\mathcal{A}^k\mathcal{B}^k})) = \rho_{\mathcal{AB}}(x(t,\zeta,\rho_{\mathcal{AB}})). \tag{4.16}$$

Since $w : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is positive and continuous, it follows from (4.15), (4.16) and the Lebesgue's dominated convergence theorem that the mapping

$$(\mathcal{A}, \mathcal{B}) \in \mathcal{F}_\delta \mapsto \int_{B_r} \int_0^T w(x(t,\zeta,\rho_{\mathcal{AB}}), \rho_{\mathcal{AB}}(x(t,\zeta,\rho_{\mathcal{AB}})))dtd\zeta$$

is continuous on the compact metric space $\mathcal{F}_\delta$. Thus it has a minimum (maximum) value on $\mathcal{F}_\delta$, and the proof is complete. □

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Takashi Mitsuishi
Department of Information Engineering, Faculty of Engineering, Shinshu University, 4-17-1 Wakasato, Nagano 380-8553, Japan
E-mail address: xmitsu@cs.shinshu-u.ac.jp

Jun Kawabe
Department of Mathematics, Faculty of Engineering, Shinshu University, 4-17-1 Wakasato, Nagano 380-8553, Japan
E-mail address: jkawabe@gipwc.shinshu-u.ac.jp

Katsumi Wasaki
Department of Information Engineering, Faculty of Engineering, Shinshu University, 4-17-1 Wakasato, Nagano 380-8553, Japan
E-mail address: wasaki@cs.shinshu-u.ac.jp

Yasunari Shidama
Department of Information Engineering, Faculty of Engineering, Shinshu University, 4-17-1 Wakasato, Nagano 380-8553, Japan
E-mail address: shidama@cs.shinshu-u.ac.jp