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EXPLICIT SOLUTIONS TO HAMILTON-JACOBI EQUATIONS UNDER MILD CONTINUITY AND CONVEXITY ASSUMPTIONS

JEAN-PAUL PENOT AND MICHEL VOLLE

Abstract. The purpose of this paper is to extend the range of applicability of the Hopf and Lax-Oleinik explicit formulas for solving the Hamilton-Jacobi equation. The continuity assumptions are very weak and the convexity assumptions are rather mild; in particular, we do not assume that the data are finite-valued, so that equations derived from attainability problems can be considered. Only elementary facts from convex analysis, variational convergence and nonsmooth analysis are required.

Dedicated to Roger Témam on the occasion of his sixtieth birthday

1. Introduction

Given a normed vector space X with dual space X^* and functions $g : X \to \mathbb{R}\cup\{\infty\}$, $H : X^* \to \mathbb{R}\cup\{\infty\}$, the Hamilton-Jacobi equation is

$$\frac{\partial u}{\partial t}(x,t) + H(Du(x,t)) = 0 \qquad (1)$$

$$u(x,0) = g(x) \qquad (2)$$

where $u : X \times \mathbb{R}_+ \to \mathbb{R}\cup\{+\infty\}$ is the unknown function, and Du (resp. $\frac{\partial u}{\partial t}$) denotes the derivative of u with respect to its first (resp. second) variable. This equation has been extensively studied during the last decades (see [9]-[27], [31]-[33], [39]-[44], [52], [57], [58]-[61]...). In particular, existence and uniqueness questions have given rise to the key notion of viscosity solution and to striking results ([21], [20], [18], [10], [12]...)[1]. Since several books exist on the subject ([9], [7], [27], [43], [57]...), we refer the reader to these monographs for more information and without tempting to draw general views on such a large subject in this introduction.

It is the purpose of the present paper to show that the range of two known explicit formulas for the solution of (1)-(2) can be extended to general semicontinuity assumptions and mild convexity assumptions. In particular, we accept Hamiltonians and initial value functions which may take the value ∞ or are discontinuous. For existence results, our methods are elementary and involve not much more than simple definitions (which are recalled below) in nonsmooth analysis and in variational

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[1]See also the recent references given at the end of the paper.

convergence and basic facts in convex analysis (given in [24], for instance). Comparison and uniqueness results involve deeper techniques centered around various forms of the mean value theorem.

We make a strong use of duality techniques. Such methods have been in use in this field since the works of Legendre, Hamilton, and Hopf and more recently in [1], [11]-[14], [44], [60], [61]; nonetheless, we hope that our systematic use may shed a new light on such equations which have been studied by many different means ([9]-[27], [56], [45]...).

Throughout, for a function f on X, we denote by f^* its convex conjugate:

$$f^*(p) := \sup_{x \in X} \left(\langle p, x \rangle - f(x) \right) \qquad \text{for } p \in X^*.$$

Here $\langle \cdot, \cdot \rangle$ stands for the canonical pairing between X and X^*. We also write $p.x$ instead of $\langle p, x \rangle$. When we take the conjugate of a function on X^*, it is with respect with this duality, so that it is a function on X and not on X^{**}. The notion of solution we use here is close to the notion of viscosity solution; in fact, it coincides with it in finite dimensional spaces. It involves the Hadamard (or contingent) subdifferential of a function. Recall that the lower (or contingent or lower Hadamard) derivative of a function f on a normed space Z at $z \in \operatorname{dom} f := f^{-1}(\mathbb{R})$ is given by

$$df(z, w) := \liminf_{(s,u) \to (0_+, w)} \frac{1}{s} \left(f(z + su) - f(z) \right) \qquad \forall w \in Z.$$

The *Hadamard subdifferential* (or contingent subdifferential) of f at z is the set $\partial f(z)$ of $z^* \in Z^*$ such that $z^*(\cdot) \leq df(z, \cdot)$. It seems that these objects have been first used for Hamilton-Jacobi equations in [39], [30]-[32], in the context of optimal control theory; in finite dimensions and in spaces with smooth bump functions, they do not differ from more classical notions. In fact, for any normed vector space Z, $\partial f(z)$ contains the *viscosity subdifferential* of f at z which is the set of derivatives at z of smooth functions φ such that $f - \varphi$ attains its minimum at z; in spaces with a smooth bump function, in particular in any Hilbert space, $\partial f(z)$ coincides with the viscosity subdifferential of f at z. The set $\partial f(z)$ always contains the *Fréchet subdifferential* $\partial^- f(z)$ which is the set of $z^* \in Z^*$ such that

$$f(z + w) \geq f(z) + \langle z^*, w \rangle - \varepsilon(w) \|w\| \qquad \qquad \forall w \in Z$$

for some $\varepsilon(\cdot)$ with limit 0 at 0 and it always contains the Fenchel-Moreau subdifferential $\partial^c f(z)$ of convex analysis given as the set of $z^* \in Z^*$ such that $z^*(\cdot) \leq f(\cdot) - f(z) + \langle z^*, z \rangle$. When f is convex, it coincides with $\partial^c f(z)$. Let us note that the larger the subdifferential one uses is, the stronger the existence results obtained with this subdifferential are. On the other hand, the best uniqueness results are those which are formulated with the smallest subdifferentials. For this reason, following the classical line of ([18], [20], [15], [22]) one can use the viscosity subdifferential for uniqueness results and the Hadamard subdifferential for existence results; in one instance of our existence results, we turn to the Fréchet subdifferential in order to admit weaker assumptions. The reader which is just interested in the finite dimensional case will not bother about the distinctions between the subdifferentials we use.

Given a subdifferential ∂ one says that u is a *supersolution* (resp. *subsolution*) to (1) if for each $(x,t) \in X \times \mathbb{P}$ with $\mathbb{P} := (0, +\infty)$ and each $(p,q) \in \partial u(x,t)$ (resp. $(p,q) \in -\partial(-u)(x,t)$) one has $q + H(p) \geq 0$ (resp. $q + H(p) \leq 0$). We use the term *lower solution* if for each $(x,t) \in X \times \mathbb{P}$ and each $(p,q) \in \partial u(x,t)$ one has $q + H(p) \leq 0$. We are tempted to consider functions which are at the same time supersolutions and lower solutions as appropriate solutions. This viewpoint is close to the notion of l.s.c. solution in the sense of Barron and Jensen ([12], see also [7]) and is supported by the uniqueness results of [12] and of our last section.

The initial condition (2) is considered separetely in section 4. In such a way, one is able to distinguish assumptions required for (1) from assumptions used for (2). We also hope that the separate study of the Hopf and the Lax formula will help in the appropriate choice in the partially convex cases we consider. A hint is as follows: since the Hopf formula involves g^*, hence gives the same solution when a function g_1 satisfying $g^{**} \leq g_1 \leq g$ is substituted to g, it is natural to use it when $g = g^{**}$; on the other hand, when $H = H^{**}$, the use of the Lax formula is commendable.

2. The Hopf-Lax formula

The following result does not require any assumption but

$$\text{(3)} \qquad \mathrm{dom} g \neq \emptyset, \qquad \mathrm{dom} g^* \cap \mathrm{dom} H \neq \emptyset,$$

which is made in all this section and is satisfied if H is everywhere finite and if g is proper and bounded below by a continuous affine function. It involves the Hopf (or Hopf-Lax) formula:

$$\text{(4)} \qquad v(x,t) := (g^* + tH)^*(x) := \sup_{p \in \mathrm{dom} g^* \cap \mathrm{dom} H} \left(\langle p, x \rangle - g^*(p) - tH(p) \right),$$

for $(x,t) \in X \times \mathbb{R}_+$, where, for $t = 0$, the product $tH(p)$ is interpreted as 0 if $H(p) < \infty$, and $+\infty$ if $H(p) = +\infty$. We extend u (and any other function on $X \times \mathbb{R}_+$) by $+\infty$ on $X \times \mathbb{R}_-$. This formula shows that v is a closed convex function of (x,t), as $-\infty < v(\cdot, 0) \leq g^{**} \leq g$. Note that without assumption (3) the function v is identically $-\infty$ on $X \times \mathbb{R}$, so that the following statement is trivial. In the classical case, g is supposed to be Lipschitz and convex and H is supposed to be finite and continuous ([33], [6], [44], [26], [27]...) or boundedness and uniform continuity assumptions are made. Of course, since in the preceding formula g is involved through g^*, there is no loss of generality in assuming that g is convex and lower semicontinuous (l.s.c.) and one cannot expect to satisfy an initial data which does not have such a property. However, regularity properties such as uniform continuity or Lipschitz properties are not needed here, nor is convexity and finiteness of H. Let us illustrate these observations by some examples. The fact that neither g nor H is supposed to be finite allows to take indicator functions; recall that, for a subset S of X^*, the *indicator function* ι_S of S is defined by $\iota_S(p) = 0$ for $p \in S$, $+\infty$ else.

Example 2.1. Let H be an arbitrary function with nonempty domain and let $g = \langle p, \cdot \rangle$ with $p \in \operatorname{dom} H$. Then $v(x,t) = \langle p, x \rangle - tH(p)$ is a linear solution to (1)-(2) in the classical sense.

Example 2.2. More generally, let S be a subset of X^* and let g be given by $g(x) := \sup_{p \in S} \langle p, x \rangle$. Then $v(x,t) = (tH_C)^*(x)$, where C is the closed convex hull of S and $H_C := H + \iota_C$. In particular, if $H = \iota_D + \gamma$, with $D \subset X^*$, $\gamma \in \mathbb{R}$, one has $v(x,t) = \sup_{C \cap D} \langle p, x \rangle - \gamma t$; here $v(x,t) = -\infty$ when $C \cap D = \emptyset$. Again, this shows the usefulness of assumption (3). Moreover, we see that some compatibility assumption has to be imposed in order to satisfy the initial condition (2).

Example 2.3. Let $g = \iota_{\{a\}}$, where a is a given point in X and let H be an arbitrary function on X^*. Then $v(x,t) = (tH)^*(x-a)$ for $(x,t) \in X \times \mathbb{R}_+$

Example 2.4. Let $g = \min(\|\cdot\|, 1)$ and let $H = c\|\cdot\|$. Then $v(x,t) = 0$.

Example 2.5. Let $g = c\|\cdot\|$, where $c > 0$, and let $H(\cdot) = \min(\|\cdot\|, 1)$. Note that g and H are Lipschitzian, but H is nonconvex. Then, for $c \leq 1$ one has $v(x,t) = c(\|x\| - t)_+$, where $r_+ = \max(r, 0)$ for $r \in \mathbb{R}$ and for $c > 1$ one has $v(x,t) = (c\|x\| - t)_+$.

Example 2.6. Let $g = c\|\cdot\|$, where $c > 0$, and let H be given by $H(p) = -\sqrt{1 - \|p\|^2}$ for $p \in B^*$, the closed unit ball of X^*, $H(p) = +\infty$ for $p \in X^* \backslash B^*$. Then $g^* = \iota_{cB^*}$ and for $b := \min(c, 1)$ one has

$$\begin{aligned} v(x,t) &= b\|x\| + t(1-b^2)^{1/2} \quad \text{for } 0 \leq t \leq b^{-1}(1-b^2)^{1/2}\|x\|, \\ v(x,t) &= (\|x\|^2 + t^2)^{1/2} \quad \text{for } t \geq b^{-1}(1-b^2)^{1/2}\|x\|. \end{aligned}$$

Proposition 2.1. *Under assumption (3), the function v given by*

$$v(x,t) := (g^* + tH)^*(x)$$

is a supersolution to (1). Moreover, when X is complete, there exists a dense subset D of dom v *such that $\partial v(x,t) \neq \emptyset$ for each $(x,t) \in D$.*

Let us note that $\partial v(x,t)$ is nonempty whenever X is complete and (x,t) belongs to int(dom v). The proof below is inspired by [44]; a more direct proof is given in [51].

Proof. Let $t \in \mathbb{P}$, $x \in X$ and $(p,q) \in \partial v(x,t) = \partial^c v(x,t)$ since v is convex, so that for any $s \in]0,t[$, $w \in X$ one has, with $r := (g^* + tH)^*(x)$,

$$(g^* + sH)^*(w) \geq r + \langle p, w - x \rangle + q(s-t).$$

Taking the conjugates of both sides considered as functions of w and adding $(t-s)H$ we get

$$(g^* + sH)^{**} + (t-s)H \leq -r + \langle p, x \rangle - q(s-t) + \iota_{\{p\}} + (t-s)H,$$

where $\iota_{\{p\}}$ is the indicator function of $\{p\}$. Taking again the conjugates of both sides, using the relation

$$r = ((g^* + sH)^{**} + (t-s)H)^*(x)$$

deduced from the semi-group property satisfied by v (see [44]), and taking the values of both sides at x we obtain

$$r \geq r - \langle p, x\rangle + q(s-t) + \left(\iota_{\{p\}} + (t-s)H\right)^* (x)$$

or

$$0 \geq -\langle p, x\rangle - q(t-s) + (\langle p, x\rangle - (t-s)H(p)).$$

Since $t > s$, it follows that $q + H(p) \geq 0$.

The last assertion is a consequence of the classical Brøndsted-Rockafellar Theorem (see [24] for instance). □

The following lemma, which will help to give conditions ensuring that v is a subsolution, has an independent interest.

Lemma 2.2. *Let $t \in \mathbb{P}$, $x \in X$ be such that the conjugate in the definition of v is exact at x, i.e. there exists $\overline{p} \in \mathrm{dom} g^* \cap \mathrm{dom} H$ such that $\overline{p}$ is a maximizer of $\langle \cdot, x\rangle - g^* - tH$. Then $(\overline{p}, -H(\overline{p})) \in \partial v(x,t)$ so that*

$$\inf\{q + H(p) : (p,q) \in \partial v(x,t)\} = 0.$$

Moreover, for any $(p,q) \in \partial^+ v(x,t) := -\partial(-v)(x,t)$ one has $q + H(p) = 0$ and v is Gâteaux-differentiable at (x,t).

Proof. The assumption ensures that $v(x,t) = \langle \overline{p}, x\rangle - g^*(\overline{p}) - tH(\overline{p})$, so that, for any $(w,s) \in X \times \mathbb{P}$, we have

$$\begin{aligned} v(w,s) - v(x,t) &\geq (\langle \overline{p}, w\rangle - g^*(\overline{p}) - sH(\overline{p})) - (\langle \overline{p}, x\rangle - g^*(\overline{p}) - tH(\overline{p})) \\ &\geq \langle \overline{p}, w - x\rangle - (s-t)H(\overline{p}). \end{aligned}$$

Since v is convex, and since $X \times \mathbb{P}$ is a neighborhood of (x,t), this inequality is enough to ensure that $(\overline{p}, -H(\overline{p})) \in \partial v(x,t)$.

Since v is convex, whenever $(p,q) \in -\partial(-v)(x,t)$ the function v is Gâteaux-differentiable at (x,t) and one has $(p,q) = (\overline{p}, -H(\overline{p}))$. Thus $p = \overline{p}$, $q = -H(\overline{p}) = -H(p)$. □

Corollary 2.3. *Suppose that (condition (3) holds and) for some $(x,t) \in X \times \mathbb{P}$ and some $\overline{p} \in \partial v(\cdot,t)(x)$ one has $H(\overline{p}) = H^{**}(\overline{p})$. Then $(\overline{p}, \overline{q})) \in \partial v(x,t)$ if and only if one has $\overline{q} + H(\overline{p}) = 0$. If moreover $\partial^+ v(x,t)$ is nonempty then v is Gâteaux differentiable at (x,t).*

Proof. Since $H(\overline{p}) = H^{**}(\overline{p})$, and since condition (3) holds, the functions H^{**} and $g^* + tH^{**}$ are closed proper convex. The assumption $\overline{p} \in \partial v(\cdot,t)(x)$ is equivalent to $v(x,t) = \langle \overline{p}, x\rangle - (g^* + tH)^{**}(\overline{p})$. Since, for any function f, the greatest closed convex function majorized by f is f^{**}, we have

$$(g^* + tH)(\overline{p}) \geq (g^* + tH)^{**}(\overline{p}) \geq g^*(\overline{p}) + tH^{**}(\overline{p})$$

and as $H(\overline{p}) = H^{**}(\overline{p})$, we have equality in these relations and the value of each side is finite and equal to $\langle \overline{p}, x\rangle - v(x,t)$, so that $\overline{p} \in \mathrm{dom}\, g^* \cap \mathrm{dom}\, H$ and, by the preceding proof, we get $(\overline{p}, -H(\overline{p})) \in \partial v(x,t)$. Let $\overline{q}$ be such that $(\overline{p}, \overline{q}) \in \partial v(x,t)$. Since for any $(w,s) \in X \times \mathbb{P}$ we have

$$\overline{p}.x + \overline{q}t - v(x,t) \geq -v(w,s) + \langle \overline{p}, w \rangle + \overline{q}s,$$

taking the supremum over $w \in X$, we get

$$\langle \overline{p}, x \rangle + \overline{q}t - v(x,t) \geq v^*(\cdot, s)(\overline{p}) + \overline{q}s = (g^* + sH)^{**}(\overline{p}) + \overline{q}s = g^*(\overline{p}) + sH(\overline{p}) + \overline{q}s.$$

Since s can be arbitrarily large, we obtain $\overline{q} + H(\overline{p}) \leq 0$, hence $\overline{q} + H(\overline{p}) = 0$ by Proposition 2.1.

The last conclusion follows from the preceding lemma. □

In order to deal with subsolutions, let us make the following two assumptions in which l.s.c. means lower semicontinuous:

(A1) g is bounded above on bounded subsets of X;
(A2) H is weak* l.s.c. and there exist $a \in \mathbb{R}_+$, $b \in \mathbb{R}$ such that $H \geq b - a \|\cdot\|$.

Let us note that assumption (A1) is satisfied whenever X is finite dimensional, g is convex (or convex up to a power of the norm, in particular, when g is paraconvex or semiconvex) and finite on X and assumption (A2) is satisfied whenever H is weak* l.s.c. proper convex (or, more generally, when H is bounded below by a continuous affine function). However, neither (A1) nor (A2) involve an explicit convexity assumption. In particular, we note that certain integral functionals are lower semicontinuous without being convex ([16], [28], [29], [36], [37], [47]).

Theorem 2.4. *Suppose assumptions (A1), (A2) are satisfied. Then for each* $(x,t) \in X \times \mathbb{P}$ *and each* $(p,q) \in \partial^+ v(x,t) := -\partial(-v)(x,t)$ *one has* $q + H(p) = 0$ *and* v *is Gâteaux-differentiable at* (x,t). *In particular* v *is a subsolution to* (1).

Proof. Let $a \in \mathbb{R}_+$, $b \in \mathbb{R}$ be such that $H(\cdot) \geq b - a \|\cdot\|$. Then, for each $(x,t) \in X \times \mathbb{P}$ and each $p \in X^*$, taking $r := \|x\| + ta + 1$, $m := \sup\{g(w) : \|w\| \leq r\}$, we have

$$\begin{aligned} g^*(p) + tH(p) &\geq \sup_{w \in B(0,r)} (\langle p, w \rangle - g(w)) + tb - ta \|p\| \\ &\geq \sup_{w \in B(0,r)} \langle p, w \rangle - m + tb - ta \|p\| \\ &\geq (r - ta) \|p\| - m + tb, \end{aligned}$$

hence

$$\begin{aligned} -\langle p, x \rangle + g^*(p) + tH(p) &\geq -\|x\| \|p\| + (r - ta) \|p\| - m + tb \\ &\geq \|p\| - m + tb. \end{aligned}$$

Therefore the function $p \mapsto -\langle p, x \rangle + g^*(p) + tH(p)$ is weak* l.s.c. and coercive. Thus it attains its infimum on X^*. Then, the preceding lemma yields the conclusion. □

Let us remark that the preceding proof shows that the following assumption

(A′1) g is bounded above on a ball centered at 0 in X with radius $r > 0$

and (A2) ensure that for each $(x,t) \in X \times \mathbb{P}$ with $\|x\| + ta < r$ and each $(p,q) \in \partial^+ v(x,t) := -\partial(-v)(x,t)$ one has $q + H(p) = 0$. On the other hand, it also shows that assumption (A1) can be replaced by the assumption that g^* is hyper-coercive in the sense that $\lim_{\|p\|\to\infty} \|p\|^{-1} g^*(p) = +\infty$.

3. The Lax-Oleinik solution

In the present section we consider the Lax-Oleinik formula defined with the help of the infimal convolution operator $\Box$: for $(x,t) \in X \times \mathbb{P}$

$$u(x,t) := (g\Box h_t)(x) := \inf_{w\in X} [g(x-w) + h_t(w)], \tag{5}$$

where $h_t := (tH)^* = th(t^{-1}\cdot)$ for $t \in \mathbb{P}$, $h := H^*$ (considered as a function on X). The value of $u(\cdot,0)$ has no importance for the present section. For consistence, we set $u(\cdot,0) := g\Box h_0$, where $(0H)^* \leq h_0 \leq \iota_{\{0\}}$, with $0H := \iota_{\mathrm{dom}H}$; note that the choice $h_0 = \iota_{\{0\}}$ yields $u(\cdot,0) = g$. In the sequel, we assume that g is proper and H is bounded below by a (weak*) continuous affine functional, or equivalently,

$$\mathrm{dom} g \neq \emptyset, \quad \mathrm{dom} H^* \neq \emptyset. \tag{6}$$

In the classical case, H is supposed to be convex (or concave) and g is supposed to be Lipschitz ([33], [43], [26], [6]...) or H is supposed to be Lipschitz, convex and g is supposed to be continuous ([57]).

Example 3.1. Let H be an arbitrary function with nonempty domain and let $g = \langle p, \cdot\rangle$ with $p \in \mathrm{dom}\, H^{**}$.Then $u(x,t) = \langle p, x\rangle - tH^{**}(p)$ is a linear supersolution to (1)-(2).

Example 3.2. Let g be given by $g(x) := \sup_{p\in C}\langle p, x\rangle$ for a closed convex subset C of X^* and let $H := \iota_{\{d\}} + \gamma$ for some $d \in X^*$, $\gamma \in \mathbb{R}$. Then $u(x,t) = \langle d, x\rangle - \gamma t - \iota_C(d)$.

Example 3.3. Let $g = \iota_{\{a\}}$, with $a \in X$ and let H be an arbitrary function on X^*. Then $u(x,t) = (tH)^*(x-a)$.

Example 3.4. Let $g = \min(\|\cdot\|, 1)$ and let $H = c\|\cdot\|$. Then $u(x,t) = \min((\|x\| - ct)_+, 1)$.

Example 3.5. Let $g = c\|\cdot\|$ and let $H(\cdot) = \min(\|\cdot\|, 1)$. Then $u(x,t) = c\|x\|$.

Example 3.6. Let $g = c\|\cdot\|$, where $c > 0$, and let H be given by $H(p) = -\sqrt{1 - \|p\|^2}$ for $p \in B^*$, the closed unit ball of X^*, $H(p) = +\infty$ for $p \in X^*\backslash B^*$. Then $u = v$, where v is as in Example 2.6.

We observe that since $g^* + tH^{**} \leq g^* + tH$ for each $t \in \mathbb{R}_+$, we have

$$u(\cdot,t) = g\Box h_t \geq (g\Box h_t)^{**} = (g^* + tH^{**})^* \geq (g^* + tH)^* = v(\cdot,t). \tag{7}$$

In fact, for $t > 0$ one has

$$(u(\cdot,t))^{**} = (g\Box h_t)^{**} = (g^* + tH^{**})^* \geq (g^* + tH)^* = v(\cdot,t),$$

with equality when $H = H^{**}$ and for $t = 0$ we have $u(\cdot,0) \leq g\Box\iota_{\{0\}} = g$,

$$u(\cdot,0) \geq (g\Box\iota^*_{\mathrm{dom}H})^{**} = (g^* + \iota^{**}_{\mathrm{dom}H})^* \geq (g^* + \iota_{\mathrm{dom}H})^* = v(\cdot,0).$$

We also note the relation

$$h_r \square h_s = h_{r+s} \tag{8}$$

for any $r, s > 0$: since $h := H^*$ is convex, for $x \in X$, $t := r + s$ we have

$$\begin{aligned} h_t(x) &= th(t^{-1}x) = rh(r^{-1}rt^{-1}x) + sh(s^{-1}st^{-1}x) \\ &= h_r(rt^{-1}x) + h_s(st^{-1}x) \geq (h_r \square h_s)(x) \\ &\geq t \inf_{w \in X} \left(t^{-1}rh(r^{-1}w) + t^{-1}sh(s^{-1}(x-w))\right) \\ &\geq t \inf_{w \in X} h(t^{-1}w + t^{-1}(x-w)) = h_t(x). \end{aligned}$$

In the following lemma we use the (possibly non convex) *asymptotic function* of g given by

$$g_\infty(z) := \liminf_{(w,t) \to (z,\infty)} t^{-1}g(tw) \qquad \text{for } z \in X.$$

Here and in the sequel the convergence is taken in the weak topology (or the weak* topology when the function is defined on a dual space).

Proposition 3.1. *If the inf-convolution in the definition of u is exact at $(x,t) \in X \times \mathbb{P}$, then for each $(p,q) \in \partial u(x,t)$ one has $q + H^{**}(p) \geq 0$ hence $q + H(p) \geq 0$.*

The function u is l.s.c. and exactness of the inf-convolution occurs at any $(x,t) \in X \times \mathbb{P}$ when one of the following conditions is satisfied:
(a) X is reflexive, g is weakly l.s.c., bounded below by $b - c\,\|\cdot\|$ for some $b, c \in \mathbb{R}$, and H is bounded above on some ball with center 0 and radius $r > c$;
(b) X is reflexive, g is weakly l.s.c., bounded below, and H is bounded above around 0;
(c) X is finite dimensional, g is l.s.c., and $h_\infty(v) > -g_\infty(-v)$ for any unit vector v in X.

In fact, in conditions (a) and (b) one could suppose X is a dual space instead of a reflexive space, g is weak* l.s.c and H is the conjugate of a weak* closed proper convex function on X.

Proof. Let us suppose that the inf-convolution in the definition of u is exact at (x,t) : there exists some $z \in X$ such that

$$u(x,t) = g(x-z) + th(t^{-1}z).$$

Then, for any $s \in]0,t[$, we have

$$\begin{aligned} u(x - st^{-1}z, t-s) &= (h_{t-s} \square g)(x - st^{-1}z) \\ &\leq h_{t-s}(z - st^{-1}z) + g(x-z) \\ &\leq (t-s)h(t^{-1}z) + g(x-z) \\ &\leq u(x,t) - sh(t^{-1}z). \end{aligned}$$

It follows that

$$\limsup_{s \to 0_+} \frac{1}{s}\left(u(x - st^{-1}z, t-s) - u(x,t)\right) \leq -h(t^{-1}z)$$

and for each $(p,q) \in \partial u(x,t)$ we get

$$\langle p, -t^{-1}z\rangle + q(-1) \le -h(t^{-1}z).$$

Therefore

$$q + H^{**}(p) = q + \sup_{w\in X} (\langle p, w\rangle - h(w)) \ge q + \langle p, t^{-1}z\rangle - h(t^{-1}z) \ge 0.$$

In case (a), the second assertion is a consequence of the fact that, under its assumptions, the functions $k_x : w \mapsto g(x-w) + th(t^{-1}w)$ are weakly l.s.c. and coercive, uniformly in (x,t) for x in a bounded set and t in a compact subset of $\mathbb{P}$. In fact, if g is bounded below by $b - c\,\|\cdot\|$ for some $b, c \in \mathbb{R}$ and $H(p) \le m$ for each $p \in B(0,r)$ with $r > c$, then

$$th(t^{-1}w) \ge t \sup_{p\in B(0,r)} \big(\langle p, t^{-1}w\rangle - m\big) = r\,\|w\| - mt$$

while $g(x-w) \ge b - c_+\,\|w\| - c_+\,\|x\|$ with $r > c_+$, so that

$$k_x(w) \ge (r - c_+)\,\|w\| + b - c_+\,\|x\| - mt.$$

Assertion (b) is clearly a consequence of assertion (a).

On the other hand, when X is finite dimensional and $h_\infty(v) > -g_\infty(-v)$ for any unit vector v in X, k_x is also coercive, uniformly for (x,t) in any compact subset B of $X \times \mathbb{P}$. Otherwise, we can find $c > 0$ and sequences (w_n), (x_n, t_n) in X and B respectively with $(r_n) := (\|w_n\|) \to \infty$ such that $g(x_n - w_n) + t_n h(t_n^{-1} w_n) \le c$. Without loss of generality we may assume $w_n = r_n v_n$, where (v_n) converges to a unit vector v and (x_n, t_n) converges to some $(x,t) \in X \times \mathbb{P}$. Then we get

$$h_\infty(v) \le \liminf_n r_n^{-1} t_n h(r_n t_n^{-1} v_n) \le \limsup_n -r_n^{-1} g(x_n - r_n v_n) \le -g_\infty(-v),$$

a contradiction. The lower semicontinuity of u follows easily; see also [62], [63]. □

Example 3.7 Let $X = \mathbb{R}$, $g(x) = (1 - |x|)_+$, $H(p) = \iota_{[1,2]}(|p|)$. Then neither g nor H are convex and $u(x,t) = g(x)$. For $x = 1$, $t > 0$ one has $(p, 0) \in \partial u(x,t)$ for any $p \in [-1, 0]$ and $q + H(p) > 0$ for $p \in]-1, 0]$, $q = 0$. Here assumption (c) is satisfied.

Under a smoothness assumption, the preceding result can be reinforced (compare with [52]).

Lemma 3.2. *Suppose $H^{**} = H$ and $h := H^*$ is Gâteaux-differentiable. If the inf-convolution in the definition of u is exact at $(x,t) \in X \times \mathbb{P}$, then for each $(p,q) \in \partial u(x,t)$ one has $q + H(p) = 0$. Moreover, $\partial u(x,t)$, if nonempty, is a singleton, so that if moreover g is convex and u is continuous at (x,t), then u is Gâteaux-differentiable at (x,t).*

Proof. Let $F, G : X \times \mathbb{R} \to \overline{\mathbb{R}}$ be given by $G(x,t) = g(x) + \iota_{\{0\}}(t)$, $F(x,t) = h_t(x) = th(x/t)$ for $(x,t) \in X \times \mathbb{P}$, $F(x,0) = \iota^*_{\operatorname{dom} H}(x)$, $F(x,t) = +\infty$ for $(x,t) \in X\times\,]-\infty, 0[$. Clearly, for the choice $h_0 = \iota^*_{\operatorname{dom} H}$ one has

$$u = F \boxtimes G,$$

where $\boxtimes$ denotes the inf-convolution with respect to the variable (x,t). Moreover, the inf-convolution $F \boxtimes G$ is exact at $(x,t) \in X\times \mathbb{P}$ iff $g\Box h_t$ is exact at x :

$$(F \boxtimes G)(x,t) = F(x-y,t-s) + G(y,s) \Leftrightarrow (g\Box h_t)(x) = h_t(x-y) + g(y),\ s = 0.$$

Now, for any $(p,q) \in \partial u(x,t)$ one easily shows that $(p,q) \in \partial F(x-y,t)$. Since h is (Gâteaux) differentiable, this relation means that

$$p = h'(\frac{x-y}{t}),\ \ q = h(\frac{x-y}{t}) - h'(\frac{x-y}{t})(\frac{x-y}{t}). \tag{9}$$

Since h is convex and $H^{**} = H$, these equalities imply that

$$H(p) = h^*(p) = p.(\frac{x-y}{t}) - h(\frac{x-y}{t}) = -q.$$

Since (p,q) is uniquely determined by relation (9), the last conclusion follows. $\square$

Example 3.8. Suppose $X = \mathbb{R}$ and let H be given by $H(p) = p\ln p - p$ for $p > 0$, $H(0) = 0$, $H(p) = +\infty$ for $p < 0$. Then $h(x) = \exp x$ and by the two preceding results, for any l.s.c. function g satisfying $g_\infty(1) > \infty$, $g_\infty(-1) > 0$ the Lax-Oleinik solution u is a supersolution and a lower solution, i.e. a bilateral solution in the sense of [7].

Example 3.9. Suppose X is finite dimensional, the norm of X is differentiable off 0 and let H be given by $H(p) = \|p\|\ln\|p\| - \|p\| + 1$ for $p \neq 0$, $H(0) = 1$. Then $h(x) = \exp\|x\| - 1$ and the same conclusion holds, for any l.s.c. function g satisfying $g_\infty(v) > -\infty$ for any unit vector $v \in X$.

Example 3.10. Let $X = \mathbb{R}$, $g(x) = (1-|x|)_+$, $H(p) = \iota_{[-2,2]}(p)$. Then $H = H^{**}$ and $u(x,t) = g(x)$. For any $(x,\ t) \in X \times \mathbb{P}$ and any $(p,q) \in \partial u(x,t)$ one has $q + H(p) = 0$. This fact also follows from the next theorem.

Theorem 3.3. *For any $(x,t) \in X \times \mathbb{P}$ and for any $(p,q) \in \partial u(x,t)$, where u is given by (5), one has $q + H^{**}(p) \leq 0$; thus, if H is a closed proper convex function then u is a lower solution: for any $(x,t) \in X\times \mathbb{P}$ and for any $(p,q) \in \partial u(x,t)$ one has $q + H(p) \leq 0$. If moreover the inf-convolution in the definition of u is exact at (x,t) then $q + H(p) = 0$.*

Proof. Let $(x,t) \in X \times \mathbb{P}$ and let $(p,q) \in \partial u(x,t)$. For any $s > 0$ and $w \in W$, by relation (8) and the associativity of the infimal convolution we have

$$\begin{aligned} u(x+sw,t+s) &= ((h_t\Box g)\Box h_s)(x+sw) \\ &\leq u(x,t) + sh(w). \end{aligned}$$

It follows that

$$\limsup_{s\to 0_+} \frac{1}{s}\left(u(x+sw,t+s) - u(x,t)\right) \leq h(w).$$

Thus

$$\langle p,w\rangle + q1 \leq h(w)$$

and we get

$$q + H^{**}(p) = q + \sup_{w\in X}\left(\langle p,w\rangle - h(w)\right) \leq 0.$$

The last assertion is then a consequence of the preceding lemma. □

Remark. The preceding proof shows that the relation $q + H(p) \leq 0$ is valid whenever $(p, q) \in \partial^r u(x, t)$, where $\partial^r u(x, t)$ is the set of $(p, q) \in X^* \times \mathbb{R}$ such that

$$\forall (w, r) \in X \times \mathbb{R} \qquad \langle p, w \rangle + qr \leq \limsup_{s \to 0_+} s^{-1} \left(u(x + sw, t + sr) - u(x, t) \right).$$

This set is larger than $\partial u(x, t)$.

Variants of the preceding results can be given; in Proposition 3.1 one can use the Fréchet subdifferential and coercivity conditions. Another variant does not suppose the infimal convolution is exact in (5) but it uses the Fenchel-Moreau subdifferential ∂^c of convex analysis, a notion which is more restrictive than the Hadamard subdifferential. We will observe in the next corollary (in which the last assertion is due to C. Zalinescu) that when g is convex, the function u is convex, so that $\partial^c u = \partial u$.

Proposition 3.4. *For any $(x, t) \in X \times \mathbb{P}$ and for any $(p, q) \in \partial^c u(x, t)$, one has $q + H(p) \geq 0$ and in fact $q + H^{**}(p) = 0$.*

Proof. By the definition of $\partial^c u(x, t)$, for any $(w, s) \in X \times \mathbb{P}$ we have

$$(g \square h_s)(w) = u(w, s) \geq u(x, t) + \langle p, w - x \rangle + q(s - t).$$

Thus, for any $w, z \in X$, $s > 0$ we have

$$g(w - z) + h_s(z) \geq u(x, t) + \langle p, w - z \rangle + \langle p, z \rangle - \langle p, x \rangle + q(s - t),$$

hence, rearranging terms, taking suprema on $w' := w - z$ and then on z, we get

$$0 \geq u(x, t) + g^*(p) + h_s^*(p) - \langle p, x \rangle + q(s - t).$$

Since $h_s^*(p) = sH^{**}(p)$, since $g^*(p)$ is finite by assumption (6) and since s can be arbitrarily large, we get $q + H^{**}(p) \leq 0$. Since s can be arbitrarily close to 0, since $H^{**}(p) > -\infty$ by (6) and since by (7) we have $u(x, t) \geq (g^* + tH)^*(x)$, we get

$$\begin{aligned} qt &\geq u(x, t) + g^*(p) - \langle p, x \rangle \\ &\geq \inf_{w \in X} \left(g(x - w) + g^*(p) - \langle p, x \rangle + tH^*(w/t) \right) \\ &\geq \inf_{w \in X} \left(-\langle p, w \rangle + tH^*(w/t) \right) = -tH^{**}(p), \end{aligned}$$

hence $q \geq -H^{**}(p) \geq H(p)$. □

Corollary 3.5. *Suppose g is convex. Then, u is convex and for any $(x, t) \in X \times \mathbb{P}$, for any $(p, q) \in \partial u(x, t)$ one has $q + H(p) \geq 0$. If moreover H is a closed proper convex function on X^* then for any $(x, t) \in X \times \mathbb{P}$ and for any $(p, q) \in \partial u(x, t)$ one has $q + H(p) = 0$.*

Proof. When g is convex, u is convex as the performance function of the function $(w, x, t) \mapsto g(x - w) + th(t^{-1} w)$ in view of the relations

$$r't'h(\frac{x'}{t'}) + r''t''h(\frac{x''}{t''}) \geq th(\frac{1}{t}(r'x' + r''x''))$$

valid for $r', r'' \geq 0$, $r' + r'' = 1$, $t := r't' + r''t''$. It follows that $\partial u(x,t) = \partial^c u(x,t)$.$\Box$

Theorem 3.6. *For any $(x,t) \in X \times \mathbb{P}$ and for any $(p,q) \in \partial^+ u(x,t) := -\partial(-u)(x,t)$ one has $q + H^{**}(p) \leq 0$. In particular, if H is a closed proper convex function on X^*, u is a subsolution. If moreover g is convex, then $q + H(p) = 0$ and u is Gâteaux-differentiable at (x,t) when $\partial^+ u(x,t)$ is nonempty.*

Proof. As $H = h^*$, given $(x,t) \in X \times \mathbb{P}$, $(p,q) \in -\partial(-u)(x,t)$ it suffices to prove that $\langle p, w\rangle - h(w) \leq -q$ for each $w \in X$. For each $s \in]0, t[$, relation (8) yields

$$\begin{aligned} u(x,t) &= ((g \Box h_{t-s}) \Box h_s)(x) \\ &\leq (g \Box h_{t-s})(x - sw) + sh(w), \end{aligned}$$

so that

$$s^{-1}(u(x - sw, t - s) - u(x,t)) \geq -h(w).$$

Taking limits we obtain

$$\langle p, -w\rangle + q(-1) \geq \liminf_{s \to 0_+} s^{-1}(u(x - sw, t - s) - u(x,t)) \geq -h(w),$$

as expected. $\Box$

Remark. The preceding proof shows that the inequality $q + H^{**}(p) \leq 0$ is valid whenever $(p,q) \in -\partial^r(-u)(x,t)$. Moreover, when h is continuous, it suffices that for each $(w,r) \in X \times \mathbb{R}$ one has

$$\langle p, w\rangle + qr \geq \liminf_{(s,z) \to (0_+, w)} s^{-1}(u(x + sz, t + rs) - u(x,t)).$$

4. Initial conditions

In order to check the initial conditions we recall some basic definitions of epi-convergence (see [2] and the recent monograph [53] for instance). Given a family $(f_t)_{t>0}$ of functions on X parametrized by $\mathbb{P} :=]0, +\infty[$, we define its (weak) epi-limit inferior by

$$\left(e_w - \liminf_{t \to 0_+} f_t\right)(x) = \sup_{W \in \mathcal{N}(x)} \liminf_{t \to 0_+} \inf_{w \in W} f_t(w) = \liminf_{(w,t) \to (x, 0_+)} f_t(w),$$

where $\mathcal{N}(x)$ denotes the family of (weak) neighborhoods of x, and its epi-limit superior by

$$\left(e - \limsup_{t \to 0_+} f_t\right)(x) := \sup_{\varepsilon > 0} \limsup_{t \to 0_+} \inf_{w \in B(x,\varepsilon)} f_t(w),$$

where $B(x,\varepsilon)$ is the closed ball with center x and radius ε. The family is said to (Mosco) epi-converge to a function g on X if $e_w - \liminf_{t \to 0_+} f_t = g = e - \limsup_{t \to 0_+} f_t$. This notion can be given a simple interpretation in terms of set-convergence and is of great importance when dealing with duality questions. To the knowledge of the authors it has not yet been used for Hamilton-Jacobi equations but in [10], [12] (for the limit inferior in the concept of lower semicontinuous solution), [49] (under the form of level-convergence). Here we stress the relationship with classical variational convergences (see [2], [53] for example).

Proposition 4.1. *(a) One always has* $v(\cdot,0) \leq e_w - \liminf_{t\to 0_+} v(\cdot,t)$, $v(\cdot,0) \leq g$.
(b) With assumption (6) one has

$$v(\cdot,0) = e - \lim_{t\to 0_+} v(\cdot,t) \leq e_w - \liminf_{t\to 0_+} u(\cdot,t) \leq e - \limsup_{t\to 0_+} u(\cdot,t) \leq g.$$

(c) If $\mathrm{dom} g^* \subset \mathrm{dom} H$ *then* $v(\cdot,0) = g^{**}$, *hence* $v(\cdot,0) = g$ *when* $g = g^{**}$.
(d) If assumption (6) holds, if $\mathrm{dom} g^* \subset \mathrm{dom} H$ *and* $g = g^{**}$, *then the functions* $v(\cdot,t)$ *and* $u(\cdot,t)$ *epi-converge to* g *as* $t \to 0_+$. *If furthermore* H *is bounded below, then* $v(\cdot,t)$ *and* $u(\cdot,t)$ *pointwise converge to* g *as* $t \to 0_+$.

Proof. (a) The inequality $v(\cdot,0) \leq e_w - \liminf_{t\to 0_+} v(\cdot,t)$ is a consequence of the weak lower semicontinuity of v. We have $v(\cdot,0) = (g^* + \iota_{\mathrm{dom}H})^*$, where $\iota_{\mathrm{dom}H}$ denotes the indicator function of the domain of H. Since $g^* + \iota_{\mathrm{dom}H} \geq g^*$, we get $v(\cdot,0) \leq g^{**} \leq g$.

(b) When there exist $a \in X$, $b \in \mathbb{R}$ such that $H(\cdot) \geq \langle \cdot, a\rangle - b$, we have $H^*(a) \leq b$, hence, for each $w \in X$,

$$v(w,t) \leq u(w,t) = (g \Box h_t)(w) \leq g(w - ta) + tH^*(a) \leq g(w - ta) + tb. \tag{10}$$

Thus, for any $x \in X$ and any sequence $(t_n) \to 0_+$, taking $(w_n) := (x + t_n a)$ which converges to x, we see that $(e - \limsup u(\cdot,t_n))(x) \leq \limsup_n u(w_n,t_n) \leq \lim_n (g(x) + t_n b) = g(x)$. Similarly, as $tH \geq \iota_{\mathrm{dom}H} + \langle \cdot, ta\rangle - tb$ for $t > 0$, we get

$$\begin{aligned}
v(w,t) &\leq (g^* + \iota_{\mathrm{dom}H})^*(w - ta) + tb,\\
(e - \limsup v(\cdot,t_n))(x) &\leq \limsup_n v(w_n,t_n) \leq \lim_n ((g^* + \iota_{\mathrm{dom}H})^*(x) + t_n b)\\
&= v(x,0).
\end{aligned}$$

(c) If $\mathrm{dom} g^* \subset \mathrm{dom} H$ we have $g^* + \iota_{\mathrm{dom}H} = g^*$, hence

$$v(\cdot,0) = (g^* + \iota_{\mathrm{dom}H})^* = g^{**}.$$

(d) Gathering the preceding inequalities we obtain (when $g = g^{**}$)

$$\begin{aligned}
v(\cdot,0) &= e - \limsup_{t\to 0_+} v(\cdot,t) \leq e - \limsup_{t\to 0_+} u(\cdot,t) \leq g = g^{**} = v(\cdot,0)\\
v(\cdot,0) &\leq e - \lim_{t\to 0_+} v(\cdot,t) \leq e_w - \liminf_{t\to 0_+} u(\cdot,t) \leq e - \limsup_{t\to 0_+} u(\cdot,t) \leq v(\cdot,0).
\end{aligned}$$

Thus $v(\cdot,t)$ and $u(\cdot,t)$ epi-converge to g.

When H is bounded below, taking $a = 0$ in (10), so that $H^*(0) \leq b < +\infty$, it follows that for each $x \in X$

$$\limsup_{t\to 0_+} v(x,t) \leq \limsup_{t\to 0_+} u(x,t) \leq \limsup_{t\to 0_+} (g(x) + tb) \leq g(x)$$

and since

$$g(x) \leq e - \liminf_{t\to 0_+} v(\cdot,t)(x) \leq \liminf_{t\to 0_+} v(x,t) \leq \liminf_{t\to 0_+} u(x,t)$$

$v(\cdot,t)$ and $v(\cdot,t)$ pointwise converge to g. □

Corollary 4.2. *Suppose $g = g^{**}$, $\operatorname{dom} g^* \subset \operatorname{dom} H$ and (6) holds. Then v is a supersolution to (1)-(2) in the epi-limit sense. If the conjugation giving v is exact then*

$$\inf\{q + H(p) : (p,q) \in \partial u(x,t)\} = 0 \qquad \forall (x,t) \in X \times \mathbb{P}.$$

If $H = H^{**}$ then v is also a lower solution.

Example 4.1 Let $X = \mathbb{R}$, $g(x) = 0$ for $x \in \mathbb{R}_-$, $g(x) = +\infty$ for $x > 0$, $H(p) = (|p| - 1)^2$. Then $g^*(p) = \iota_{\mathbb{R}_+}(p)$ and $\operatorname{dom} g^* = \mathbb{R}_+ \subset \operatorname{dom} H$. Then $v(x,t) = x + \frac{1}{4t}x^2$ for $x \geq -2t$, $v(x,t) = -t$ for $x < -2t$. We note that H is nonconvex and v is differentiable a.e. with (1) satisfied a.e.

Example 4.2 Let $g(x) = c\,\|x\|$, $H(p) = a\,\|p\|$ with $c > 0$, $a > 0$. Then $g^* = \iota_{cB^*}$ and $\operatorname{dom} g^* \subset \operatorname{dom} H$. Then $v(x,t) = u(x,t) = c\,(\|x\| - ta)_+$.

Example 4.3 Let $g(x) = c\,\|x\|$, $H(p) = \frac{1}{2}\,\|p\|^2$. Then $g^* = \iota_{cB^*}$ and $\operatorname{dom} g^* \subset \operatorname{dom} H$. Then $v(x,t) = u(x,t) = \frac{1}{2t}\,\|x\|^2$ for $\|x\| \leq ct$, $v(x,t) = u(x,t) = c\,\|x\| - \frac{c^2}{2}t$ for $\|x\| > ct$.

In the following statement, the convexity assumption on g is dropped, but only u is considered.

Proposition 4.3. *Suppose either*

(a) If $g = g^{**}$, $\operatorname{dom} g^* \subset \operatorname{dom} H$ or

(b) X is finite dimensional, g is l.s.c., $g \Box h_\infty \geq g$ and

$$h_\infty(z) > -g_\infty(-z) \qquad \text{for each } z \in X \backslash \{0\}. \tag{11}$$

Then $e_w - \liminf_{t \to 0_+} u(\cdot,t) \geq g$. If moreover H is bounded below by a continuous affine function (resp. is bounded below), then the function $u(\cdot,t)$ epi-converges (resp. pointwise converges) to g as $t \to 0_+$.

Proof. In case (a) we have

$$g = v(\cdot,0) \leq e_w - \liminf_{t \to 0_+} v(\cdot,t) \leq e_w - \liminf_{t \to 0_+} u(\cdot,t)$$

by the preceding proposition.

(b) Suppose on the contrary that $e_w - \liminf_{t \to 0} u(\cdot,t)(x) < g(x)$ for some $x \in X$; then there exist some $r < g(x)$ and sequences $(t_n) \to 0_+$, $(x_n) \to x$ such that $(u(x_n,t_n)) \to r$. The definition of an infimal convolution ensures that we can find a sequence (w_n) such that

$$g(x_n - w_n) + t_n h(t_n^{-1} w_n) \to r \quad \text{as } n \to \infty.$$

Taking subsequences, we may assume that either (w_n) has a limit w or that $(s_n) := (\|w_n\|) \to \infty$ and $(z_n) := (s_n^{-1} w_n)$ has a limit z with norm 1. In the first case, we get

$$r \geq g(x - w) + h_\infty(w) \geq (g \Box h_\infty)(x) \geq g(x) > r,$$

a contradiction. In the second case, we get

$$-g_\infty(-z) \geq -\liminf_n s_n^{-1} g(s_n(s_n^{-1}x_n - z_n)) \geq \liminf_n s_n^{-1} t_n h(s_n t_n^{-1} z_n)) \geq h_\infty(z),$$

a contradiction with our assumption. Therefore $e - \liminf_{t \to 0} u(\cdot,t) \geq g$.

The second assertion follows as in the preceding proof. □

Remark. A similar result holds when X is an infinite dimensional reflexive Banach space and assumption (11) is replaced with the stronger assumption

$$\liminf_{\|x\|\to\infty} h(x)/\|x\| > -\liminf_{\|x\|\to\infty} g(x)/\|x\|.$$

Corollary 4.4. *Suppose X is finite dimensional, g is l.s.c., $\operatorname{dom} H^*$ is nonempty and condition (11) holds. Then u is a supersolution to (1)-(2) in the epi-limit sense. If moreover $H = H^{**}$ then u is also a lower solution.*

5. Coincidence of the Hopf-Lax and of the Lax-Oleinik solutions

Without convexity assumptions, the Hopf solution and the Lax solution may differ drastically, as Examples 2.5 and 3.5 show. Let us first make clear the fact that, under convexity assumptions, the Hopf solution and the Lax solution are close. Throughout we assume that condition (3) is satisfied. This assumption discards the case $H = 0$, g a non continuous linear form, for which the conclusion of the following proposition is not satisfied.

Proposition 5.1. *Suppose condition (3) holds, g is convex and H is a closed proper convex function. Then the Hopf solution v coincides with the l.s.c. hull $\overline{u}$ of the Lax solution u on $X \times \mathbb{P}$.*

Proof. As observed above, when $H = H^{**}$ and g is convex, for each $t \in \mathbb{P}$, $u(\cdot,t)$ being convex on X, one has $v(\cdot,t) = u(\cdot,t)^{**} = \overline{u(\cdot,t)}$, the lower semicontinuous hull of $u(\cdot,t)$ because v does not take the value $-\infty$. Thus, for each $(x,t) \in X \times \mathbb{P}$ one has, by Proposition 4.1,

$$v(x,t) = \liminf_{x'\to x} u(x',t) \geq \liminf_{(x',t')\to(x,t)} u(x',t') \geq \liminf_{(x',t')\to(x,t)} v(x',t') \geq v(x,t),$$

v being l.s.c., and equality holds. □

Remark. If $\mathrm{dom}g^* \subset \mathrm{dom}H$ then $\overline{u} = v$ on $X \times \mathbb{R}$. On $X \times \mathbb{R}_-$ this is trivial. For $t = 0$, by Proposition 4.1 we have

$$g^{**} = v(\cdot,0) = e - \lim_{t\to 0_+} v(\cdot,t) \leq e_w - \liminf_{t\to 0_+} u(\cdot,t) \leq g,$$

Since $g^{**} = \overline{g}$, and $v(\cdot,0) \leq u(\cdot,0)$ by our choice of $u(\cdot,0)$, the relations $v(\cdot,0) = e_w - \liminf_{t\to 0_+} u(\cdot,t) = \overline{u}(\cdot,0) = \overline{g}$ ensue. This fact also follows from [35] Theorem 2.1 (which can be extended to the framework of the preceding proposition and does not assume that $\mathrm{dom}g^* \subset \mathrm{dom}H$), showing that v is the biconjugate of u with respect to the two variables (x,t).

Let us draw some consequences of the coincidence of v and u at some point $(x,t) \in X \times \mathbb{P}$.

Proposition 5.2. *Suppose that for some $(x,t) \in X \times \mathbb{P}$ one has $v(x,t) = u(x,t)$. Then, for each $(p,q) \in \partial^+ u(x,t)$ one has $(p,q) \in \partial^+ v(x,t)$ and $q + H(p) \geq 0$. Moreover for each $(p,q) \in \partial v(x,t)$ (in particular for each $(p,q) \in \partial^+ v(x,t)$) one has $(p,q) \in \partial u(x,t)$ and if $H(p) = H^{**}(p)$, then $q + H(p) = 0$.*

Proof. Since $u \geq v$, the inclusion $\partial^+ u(x,t) \subset \partial^+ v(x,t)$ holds whenever $v(x,t) = u(x,t)$. Since v is convex, for any $(p,q) \in \partial^+ u(x,t) \subset \partial^+ v(x,t)$, we have $(p,q) \in \partial v(x,t)$, hence $q + H(p) \geq 0$ by Proposition 2.1.

Let $(p,q) \in \partial v(x,t) = \partial^c v(x,t) \subset \partial^c u(x,t) \subset \partial u(x,t)$. If $H(p) = H^{**}(p)$ Theorem 3.3 yields $q + H(p) \leq 0$ while Proposition 2.1 ensures that $q + H(p) \geq 0$. $\square$

Now, let us present criteria ensuring that v and u coincide.

Proposition 5.3. *Suppose g is convex and H is a closed proper convex function. Then for each $(x,t) \in X \times \mathbb{P}$ such that $\partial u(\cdot,t)(x) \neq \emptyset$ (or a fortiori $\partial u(x,t) \neq \emptyset$) one has $v(x,t) = u(x,t)$, $\partial v(\cdot,t)(x) = \partial u(\cdot,t)(x)$ and $\partial v(x,t) = \partial u(x,t)$.*

Proof. Under the assumptions on g and H the function u is convex and $v(\cdot,t) = u(\cdot,t)^{**}$. Since for each $(x,t) \in X \times \mathbb{P}$ and each $p \in \partial u(\cdot,t)(x) = \partial^c u(\cdot,t)(x)$ the function $u(\cdot,t)$ is lower semicontinuous at x, one has $u(\cdot,t)(x) = u(\cdot,t)^{**}(x) = v(\cdot,t)(x)$ and $\partial u(\cdot,t)(x) = \partial^c u(\cdot,t)(x) = \partial^c u(\cdot,t)^{**}(x) = \partial^c v(\cdot,t)(x) = \partial v(\cdot,t)(x)$. The last assertion is proved similarly. $\square$

Theorem 5.4. *Suppose X is reflexive, the cone $Z := \mathbb{R}_+(\mathrm{dom} g^* - \mathrm{dom} H)$ is closed and symmetric and g and H are closed proper convex functions. Then for each $(x,t) \in X \times \mathbb{P}$ one has $v(x,t) = u(x,t)$. Moreover the infimal convolution in the definition of u is exact wherever it is finite.*

Proof. This follows from a general result of Attouch and Brézis ([3]) since in that case one has $(g^* + tH)^* = g^{**} \square (tH)^* = g \square h_t$. For other criteria in this line see [5].

6. Uniqueness results

Uniqueness results are a prominent feature in the viscosity approach to Hamilton-Jacobi equations ([20]-[21]). Another notable uniqueness result due to Barron and Jensen concerns l.s.c. solutions or unilateral solutions ([12], see also [9], [7]); however the technique used in [12] seems to be limited to the finite dimensional case. Infinite dimensional results have been the object of much interest during the last few years ([15], [22]...), in the stationary case as in the evolutionary case. Here, instead of sum rules, we use mean value results; see also [22] in which uniform continuity assumptions are involved.

In our first result we use a mean value theorem for viscosity subdifferentials or Hadamard subdifferentials (see [4], [48] and its references). In fact, it is valid for a large class of subdifferentials provided $Z := X \times \mathbb{R}$ satisfies some regularity condition close to the trustworthiness condition of Ioffe [36], [37], [38], called *reliability* in [48]. This condition requires that for any l.s.c. function f on Z, for any Lipschitzian convex function g on Z and for any $z \in \mathrm{dom}\, f$ at which $f+g$ attains its infimum and for any $\varepsilon > 0$ there exist $u, v \in B(z,\varepsilon)$ such that $|f(u) - f(z)| < \varepsilon$ and $0 \in \partial f(u) + \partial g(v) + \varepsilon B^*$, where B^* is the unit ball of Z^*. This condition is satisfied when Z is an Asplund space and ∂ is the Fréchet subdifferential or when Z has a smooth enough bump function and ∂ is the viscosity subdifferential or the Hadamard subdifferential.

Here we do not give a formal definition of what can be called a subdifferential, but we assume ∂ satisfies usual properties such as coincidence with the Fenchel-Moreau subdifferential in the convex case and the rule $\partial(f+\ell+c)(z) = \partial f(z) + \ell$ for any $c \in \mathbb{R}$, $\ell \in Z^*$, $z \in Z$ (see [38], [48] for instance).

Lemma 6.1. ([48]) Suppose Z is reliable for a subdifferential ∂. Let $f : Z \to \mathbb{R} \cup \{\infty\}$ be a l.s.c. function finite at $z_0 \in Z$. Then, for any $\overline{z} \in Z$ there exist $z \in [z_0, \overline{z}[$ and sequences $(z_n) \to z$, (z_n^*) in Z^* such that $z_n^* \in \partial f(z_n)$ for each $n \in \mathbb{N}$ and

$$\liminf_n \langle z_n^*, \overline{z} - z_0 \rangle \geq f(\overline{z}) - f(z_0).$$

Theorem 6.2. *Suppose $X \times \mathbb{R}$ is reliable for a subdifferential ∂. Let $w : X \times \mathbb{P} \to \mathbb{R}$ be a lower solution to (1) which is l.s.c. and such that for each $x \in X$ one has $\liminf_{(z,t)\to(x,0_+)} w(z,t) \leq g(x)$. Then $w \leq u$, the Lax solution.*

A similar result holds if w is a subsolution.

Proof. Let us observe that for any $s, t \in \mathbb{P}$, $x, x', y \in X$ we have

$$w(x', t+s) \leq w(x'-ty, s) + tH^*(y). \tag{12}$$

This follows from the mean value inequality

$$w(x', t+s) - w(x'-ty, s) \leq \liminf_n (p_n.ty + q_n t)$$

for some $(p_n, q_n) \in \partial w(z_n)$ where (z_n) is a sequence converging to some $z \in [(x'-ty, s), (x', t+s)]$ and from the inequalities $q_n \leq -H(p_n)$, $p_n.y - H(p_n) \leq H^*(y)$.

Taking the limit inferior when $(x', s) \to (x, 0_+)$ in (12) and using the assumption about the initial condition, we get

$$w(x,t) \leq g(x-ty) + tH^*(y).$$

Taking the infimum on y we get $w(x,t) \leq u(x,t)$. □

Corollary 6.3. *Suppose, with condition (3) and the assumptions of the preceding theorem, that $H = H^{**}$ and for each $t \in \mathbb{P}$ the function $w(\cdot, t)$ is convex and proper. Then $w \leq v$, the Hopf solution.*

Proof. Since $H = H^{**}$, and since condition (3) holds, for each $t \in \mathbb{P}$ one has $v(\cdot, t) = u(\cdot, t)^{**}$, the greatest closed convex function majorized by $u(\cdot, t)$. Therefore $w(\cdot, t) \leq v(\cdot, t)$. □

The following corollary is given in [35] under the additional assumption that $\operatorname{dom} H^*$ is open.

Corollary 6.4. *Suppose $X \times \mathbb{R}$ is reliable (for the Hadamard subdifferential), g and H are closed proper convex functions satisfying (3). Then the Hopf solution is the greatest lower solution w of (1) which is l.s.c. and such that*

$$\liminf_{(z,t)\to(x,0)} w(z,t) \leq g(x).$$

Proof. As observed in Proposition 5.1, when g and H are closed proper convex functions and (3) holds, the Hopf solution v is the l.s.c. hull of the Lax solution u. Since $w \leq u$, and since w is l.s.c., we also have $w \leq v$. We know from Proposition 4.1 that $\liminf_{(z,t)\to(x,0)} v(z,t) \leq g(x)$. □

The next results will use multi-directional mean value inequalities, in the line of [34], [35]. The simplest such inequality is similar to [17] Theorem 2.3 p. 114; its proof is obtained by adding the use of the lop-sided Moreau minimax theorem ([46]). Here we say that a function f on a normed space Z is *tangentially convex* if for any $z \in \operatorname{dom} f$ the Hadamard lower derivative $f'(z,\cdot)$ is convex and $f'(z,0) = 0$. This class contains usual marginal functions and convex composite functions satisfying a classical qualification condition.

Lemma 6.5. *Let Z be a reflexive Banach space and let $f : Z \to \mathbb{R} \cup \{+\infty\}$ be a weakly l.s.c. function which is tangentially convex. Given $z_0 \in Z$ and a bounded closed convex subset Y of Z there exist $z \in [z_0, Y[:= \{(1-t)z_0 + ty : y \in Y,\ t \in [0,1[\}$ and $z^* \in \partial f(z)$ (the Hadamard subdifferential) such that*

$$\min_Y f - f(z_0) \leq z^*.(y - z_0) \quad \forall y \in Y.$$

Theorem 6.6. *Suppose X is a reflexive Banach space, H is u.s.c. at each point of $\operatorname{dom} g^*$, is bounded above on bounded subsets and such that*

$$\limsup_{\|p\|\to\infty} H(p)/\|p\| < +\infty.$$

Let $w : X \times \mathbb{R}_+ \to \mathbb{R}$ be a weakly l.s.c., tangentially convex, Hadamard supersolution to (1) such that $w(\cdot,0) \geq g$ (or even $\geq g^{**}$). Then $w \geq v$, the Hopf solution.

Let us observe that when X is reflexive the assumption $\limsup_{\|p\|\to\infty} H(p)/\|p\| < +\infty$ is a consequence of the relation $(-H)_\infty(p) > -\infty$ for each $p \in X^*$ (the asymptotic function being taken in a sequential way, with respect to the weak topology); conversely, this assumption implies the relation $(-H)_\infty(p) > -\infty$ for each $p \in X^* \backslash \{0\}$.

Proof. It suffices to prove that for each $\overline{p} \in \operatorname{dom} g^* \cap \operatorname{dom} H$ and for any $(x,t) \in X \times \mathbb{P}$, one has

$$f(x,t) := w(x,t) - \overline{p}.x + g^*(\overline{p}) + tH(\overline{p}) \geq 0. \tag{13}$$

Suppose on the contrary that (with some fixed $\overline{p} \in \operatorname{dom} g^* \cap \operatorname{dom} H$) there is some $(\overline{x}, \overline{t}) \in X \times \mathbb{P}$ such that $f(\overline{x}, \overline{t}) < 0$. Let $\alpha \in]0, -f(\overline{x}, \overline{t})[$. Since

$$w(\cdot,0) \geq (g \geq)\ g^{**} \geq (g^* + \iota_{\operatorname{dom} H})^* \geq \overline{p}(\cdot) - g^*(\overline{p}) - 0.H(\overline{p})$$

one has $\inf_{x\in X} f(x,0) \geq 0$. Thus, Lemma 6.5 ensures that, for each $r > 0$, there exist $(x_r, t_r) \in [(\overline{x}, \overline{t}), B(\overline{x}, r) \times \{0\}[$ and $(p_r, q_r) \in \partial f(x_r, t_r)$ such that, for each $x \in B(\overline{x}, r)$,

$$\alpha \leq p_r.(x - \overline{x}) - q_r \overline{t}.$$

Taking the infimum over $x \in B(\overline{x}, r)$ it follows that

$$\alpha \leq -r\,\|p_r\| - q_r \overline{t}.$$

The inclusion $(p_r, q_r) \in \partial f(x_r, t_r)$ being equivalent to the relation $(p_r + \overline{p}, q_r - H(\overline{p})) \in \partial w(x_r, t_r)$, so that $q_r - H(\overline{p}) + H(p_r + \overline{p}) \geq 0$ (as $t_r > 0$), we get

$$\alpha + r \|p_r\| \leq -q_r \overline{t} \leq \overline{t} \left(H(p_r + \overline{p}) - H(\overline{p}) \right). \tag{14}$$

Let $\rho > 0$ be such that $H(\overline{p} + p) - H(\overline{p}) < \alpha / \overline{t}$ for $p \in B(0, \rho)$. The preceding inequalities ensure that $\|p_r\| \geq \rho$. Since H is bounded above on bounded sets, and

$$H(p_r + \overline{p}) \geq H(\overline{p}) + \overline{t}^{-1} (\alpha + r\rho)$$

we must have $\|p_r\| \to \infty$ as $r \to \infty$. Then inequality (14) yields a contradiction with our assumption $\limsup_{\|p\| \to \infty} H(p) / \|p\| < +\infty$. □

A more involved mean value inequality will give a variant of the preceding result improving [35] Theorem 3.3 which assumes that X is a Hilbert space, H is closed proper convex and globally Lipschitzian (this last assumption being equivalent to our growth condition under the convexity assumption on H).

Lemma 6.7. *([38]) Let Z be a reliable Banach space for a subdifferential ∂ and let $f : Z \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function. Given $\varepsilon > 0$, $z_0 \in Z$ and a bounded closed convex subset Y of Z, there exist $z \in \mathrm{co}(z_0, Y) + B(0, \varepsilon)$ and $z^* \in \partial f(z)$ such that*

$$\sup_{\delta > 0} \min_{y' \in Y + B(0,\delta)} f(y') - f(z_0) \leq z^*.(y - z_0) \quad \forall y \in Y.$$

Theorem 6.8. *Suppose $X \times \mathbb{R}$ is reliable for a subdifferential ∂, H is u.s.c. on $\mathrm{dom}\, g^*$, H is bounded above on bounded sets and such that $\limsup_{\|p\| \to \infty} H(p) / \|p\| < +\infty$. Let $w : X \times \mathbb{R}_+ \to \mathbb{R}$ be a l.s.c. supersolution to (1) such that $w(\cdot, 0) \geq g$ (or even $\geq g^{**}$) and such that for each bounded subset M of X one has*

$$\liminf_{t \to 0_+} \inf_{y \in M} \left(w(y, t) - w(y, 0) \right) \geq 0.$$

Then $w \geq v$, the Hopf solution.

Note that the semicontinuity assumption $\liminf_{t \to 0_+} \inf_{y \in Y} \left(w(y, t) - w(y, 0) \right) \geq 0$ is satisfied if X is reflexive and if w is weakly lower semicontinuous at each point of $X \times \{0\}$ and $w(\cdot, 0)$ is weakly continuous.

Proof. Again we prove (13) by supposing on the contrary that for some $\overline{p} \in \mathrm{dom}\, g^*$ there is some $(\overline{x}, \overline{t}) \in X \times \mathbb{P}$ such that $f(\overline{x}, \overline{t}) < 0$, where $f(x, t) := w(x, t) - \overline{p}.x + g^*(\overline{p}) + tH(\overline{p})$ as above. Taking again $\alpha \in]0, -f(\overline{x}, \overline{t})[$, noting that $f(\cdot, 0) \geq 0$ and using our uniform lower semicontinuity assumption on balls for w, hence for f, for each $r > 0$ we can find $s_r \in]0, \overline{t}/2[$ such that $f(x, s) \geq \alpha + f(\overline{x}, \overline{t})$ for each $(x, s) \in B(\overline{x}, r + 1) \times [0, 2s_r]$. Then, taking $Y := B(\overline{x}, r) \times \{s_r\}$, $\varepsilon \in]0, s_r[$, $\varepsilon < 1$, Lemma 6.7 yields some $(x_r, t_r) \in \mathrm{co}((\overline{x}, \overline{t}), B(\overline{x}, r) \times \{s_r\}) + B(0, \varepsilon)$ and $(p_r, q_r) \in \partial f(x_r, t_r)$ such that for each $x \in B(\overline{x}, r)$

$$\alpha \leq p_r.(x - \overline{x}) - q_r(\overline{t} - s_r).$$

Our choice of ε ensures that $t_r > 0$. Thus, we can finish the proof as above, replacing $\overline{t}$ by $\overline{t} - s_r$. □

In the preceding result and in the next corollary, taking for ∂ the Fréchet subdifferential, or the Hadamard subdifferential, the reliability assumption is satisfied.

If X has a Lipschitzian C^1 bump function, the reliability assumption is satisfied for the viscosity subdifferential.

Corollary 6.9. *Suppose X is reflexive, $X \times \mathbb{R}$ is reliable for a subdifferential ∂ and H is a closed proper convex function satisfying condition (3). Suppose H is u.s.c. on $\operatorname{dom} g^*$ and satisfies a linear growth condition: $H(\cdot) \leq b + c\|\cdot\|$ for some $b, c \in \mathbb{R}$. Let w be a weakly l.s.c. function on $X \times \mathbb{R}_+$ which is convex and proper in its first variable, satisfies $w(x,0) = \liminf_{(z,t)\to(x,0)} w(z,t) = g(x)$ for each $x \in X$ and is a supersolution and a lower solution to (1). Then $w = v$, the Hopf solution.*

Proof. Under our assumptions, we have $u \geq w \geq v$. Since for each $t > 0$ the function $w(\cdot, t)$ is convex and l.s.c. and proper, we get $(u(\cdot,t))^{**} \geq w(\cdot,t)$; since $H = H^{**}$ we have $(u(\cdot,t))^{**} = v(\cdot,t)$. It follows that $(u(\cdot,t))^{**} = w(\cdot,t) = v(\cdot,t)$. $\square$

Our last statement is close to classical results.

Corollary 6.10. *Suppose X is reflexive. Suppose g is a closed proper convex function, H is convex and such that $\limsup_{\|p\|\to\infty} H(p)/\|p\| < +\infty$. Then $u = v$ is the unique weakly l.s.c. supersolution and lower solution w to (1) which satisfies $w(\cdot,0) = g$.*

Proof. The convexity and growth assumptions on H imply that H is continuous and bounded above on bounded sets. Since X is reflexive, it can be endowed with a norm which is Hadamard (and even Fréchet) differentiable off 0 ([23] p. 286), so that X is reliable for the Hadamard subdifferential. Since H is continuous, one has $\mathbb{R}_+(\operatorname{dom} g^* - \operatorname{dom} H) = X$ and $u = v$ by Theorem 5.4. Then the result follows from Theorems 6.2 and 6.6. $\square$

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Jean-Paul PENOT
Université de Pau, Faculté des Sciences, Laboratoire de Mathématiques Appliquées, UPRES A 5033 Av. de l'Université 64000 PAU, France

Michel VOLLE
Université d'Avignon, Faculté des Sciences, Département de Mathématiques 33 rue Louis Pasteur 84000 AVIGNON, France