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APPROXIMATE CONVEX FUNCTIONS

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Abstract. The purpose of this paper is to study a class of generalized convex functions defined on a Banach space, called approximate convex functions which are stable under finite sums and finite suprema, and for which most of the known subdifferentials such as the Clarke, the Mordukhovich and the Ioffe approximate subdifferential coincide and share several properties of the Fenchel-Moreau-Rockafeller convex subdifferential.

1. Introduction

The class of Lipschitz convex functions on a Banach space possesses the following important properties:

(a) it is stable under finite sums and finite suprema;

(b) the optimality condition $0 \in \partial f(x)$, *where* ∂f *stands for the classical subdifferential of convex functions, is sufficient for* x *to be a local minimum of the functions* $y \to f(y) + \epsilon\|y - x\|$ *for every* $\epsilon > 0$;

(c) equality holds for the sum rule: $\partial(f_1 + f_2)(x) = \partial f_1(x) + \partial f_2(x)$; *and*

(d) an integration property holds: If $\partial f_1(x) = \partial f_2(x)$ *for all* $x \in X$, *then* $f_1 - f_2$ *is a constant.*

It is a challenging problem to know, as pointed out by Ioffe [4], whether the above class can be extended to nonconvex functions so that it still verifies properties a)-d) with a suitably choosen subdifferential and contains all continuously differentiable functions. Of course, if such an extended class exists, it cannot contain all Lipschitz functions because for these functions, most of known subdifferentials do not satisfy equality in the sum rule. A smaller class, consisting of Lipschitz and primal lower nice functions [Poliquin 14], [Thibault-Zagrodny 18] on a Hilbert space also verifies properties (a)-(d), but it does not contain all continuously differentiable functions. The purpose of the present paper is to introduce a new class consisting of generalized convex functions on a Banach space, called approximate convex functions, which meets the above requirements. The main feature of this class of functions is twofold:

(1) it includes convex functions, as well as, continuously differentiable functions;

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(2) the subdifferential in the sense of Clarke and Mordukhovich coincides with the Ioffe geometric subdifferential, as well as with the Ioffe approximate subdifferential when the functions are Lipschitz.

The so-called ϵ-convex functions introduced by Jofré-Luc-Théra [7] serve as the main tool to define approximate convex functions and to derive their properties. The paper is organized as follows. In Section 2, we study $\epsilon-$convex functions, their $\epsilon-$conjugate functions and $\epsilon-$subdifferentials. Much theory about convex functions can be extended to $\epsilon-$convex functions, including Fenchel-Moreau's duality theorem. Section 3 deals with approximate convex functions and their basic properties such as continuity, directional derivability etc.. It is shown that for approximate convex functions properties a), b) and d) are satisfied, while property c) is true under the Attouch-Brézis qualification assumption [2].

In the last section we answer a question raised by A. Ioffe about the existence of a class of functions which satisfies properties a)-d) and contains Lipschitz convex functions as well as continuously differentiable functions.

2. $\epsilon-$CONVEX FUNCTIONS

Let X be a real Banach space with topological dual X^*. Throughout the paper, $B(x, \delta)$ denotes the closed ball in X with center at x and radius $\delta > 0$, and B^* the closed unit ball of X^*. Let f be a function from X to $\mathbb{R}\cup\{+\infty\}$. As usual, we denote by $\operatorname{dom} f = \{x \in X : f(x) < +\infty\}$ and $\operatorname{epi} f := \{(x, \alpha) \in \operatorname{dom} f \times \mathbb{R} : f(x) \le \alpha\}$ the *effective domain* and *the epigraph* of f, respectively. The function f is *proper* if it has a nonempty domain.

Recall ([7],[10]) that the function f is $\epsilon-$*convex* with $\epsilon > 0$ if it satisfies the following inequality for every x, $y \in X$, and $\lambda \in (0, 1)$:

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y) + \epsilon\lambda(1-\lambda)\|x-y\|. \tag{2.1}$$

It was shown in [7], [10] that $\epsilon-$convex functions have several interesting properties and are useful for approximate calculus. In this section, by developing $\epsilon-$conjugate functions and $\epsilon-$subdifferential, we obtain more characterizations about convex functions.

2.1. $\epsilon-$conjugate functions

Let f be an $\epsilon-$convex function from X to $\mathbb{R} \cup \{+\infty\}$. Let $y \in X$ be fixed. We define the $\epsilon-$*conjugate* function $f_y^*(\epsilon, .) : X^* \to \mathbb{R} \cup \{+\infty\}$ of f at y by

$$f_y^*(\epsilon, \xi) := \sup_{x\in X}\{\langle \xi, x\rangle - f(x) - \epsilon\|x-y\|\}. \tag{2.2}$$

Obviously, $f_y^*(\epsilon, .)$ is a convex function. Its Fenchel-Legendre conjugate is denoted by $f_y^{**}(\epsilon, .x)$ and given by

$$f_y^{**}(\epsilon, x) := \sup_{\xi\in X^*}\{\langle \xi, x\rangle - f_y^*(\epsilon, \xi)\}. \tag{2.3}$$

As we shall see, $\epsilon-$conjugate functions of $\epsilon-$convex functions have many properties similar to conjugate functions of convex functions. For other generalizations of conjugate functions the interested reader is refered to [20] and the references therein.

For our aim, recall first that the Clarke directional derivative of f at $x \in \operatorname{dom} f$ is given by

$$f^{\uparrow}(x,v) := \sup_{\delta>0} \limsup_{y \xrightarrow{f} \to x} \inf_{u \in B(v,\delta)} \frac{f(y+tu)-f(y)}{t},$$

while the Clarke subdifferential of f at $x \in \operatorname{dom} f$ is defined by

$$\partial^C f(x) := \big\{ x^* \in X^* |\ \langle x^*, v\rangle \leq f^{\uparrow}(x,v)\ \forall v \in X \big\},$$

where as usual, $y \xrightarrow{f} \to x$ means $y \to x$ and $f(y) \to f(x)$. If $x \notin \operatorname{dom} f$, we set $\partial^C f(x) = \emptyset$.

Recall also [10] that the ϵ-subdifferential of f at x is defined by

$$\partial^\epsilon f(x) := \big\{ x^* \in X^* :\ \langle x^*, v\rangle \leq f(x+v) - f(x) + \epsilon\|v\|\ , \forall v \in X \big\}.$$

We shall need the following estimation established in [10] for the ϵ-subdifferential of f:

$$\partial^C f(x) \subseteq \partial^\epsilon f(x). \tag{2.4}$$

As usual, the infimal convolution $h \Box g$ of two convex functions h and g is defined by

$$(h \Box g)(x) = \inf\{h(y) + g(z) |\ y + z = x\}$$

and the convention $+(\infty) - (+\infty) = 0$ is adopted.

Proposition 2.1 *Assume that f is ϵ-convex and lower semicontinuous. Then the following assertions hold:*

i) As a function of y, the ϵ-conjugate function $f_y^*(\epsilon, \xi)$ is Lipschitz with a Lipschitz constant equal to ϵ;

ii) As a function of ϵ, the ϵ-conjugate function $f_y^*(\epsilon, \xi)$ is decreasing;

iii) As a function of ξ, the ϵ-conjugate function $f_y^*(\epsilon, \xi)$ is convex and lower semicontinuous, and it is proper if f is proper;

iv) $\xi \in \partial^\epsilon f(x) \quad \Longleftrightarrow \quad f(x) + f_x^*(\epsilon, \xi) = \langle \xi, x\rangle$;

v) For every $x \in X$, one has $f_y^{**}(\epsilon, x) \leq f(x) + \epsilon\|x - y\|$.

Proof. To prove i), let $\xi \in X^*$ be fixed. Let $y, y' \in X$. By the definition

$$f_y^*(\epsilon,\xi) := \sup_{x\in X}\{\langle \xi, x\rangle - f(x) - \epsilon\|x-y\|\} \leq \sup_{x\in X}\{\langle \xi, x\rangle - f(x) - \epsilon\|x-y'\| + \epsilon\|y'-y\|\}.$$

Consequently, $f_y^*(\epsilon,\xi) \leq f_{y'}^*(\epsilon,\xi) + \epsilon\|y-y'\|$. Interchanging the roles of y and y', one obtains $f_y^*(\epsilon,\xi) \geq f_{y'}^*(\epsilon,\xi) - \epsilon\|y-y'\|$ and i) follows.

Assertion ii) is derived from the definition.

For assertion iii), observe that for every fixed $x, y \in X$, the function $\xi \to \langle \xi, x\rangle - f(x) - \epsilon\|x-y\|$ is affine on X^*; therefore it is convex and continuous. Hence $f_y^*(\epsilon, .)$ is convex and lower semicontinuous on X^*. Now assume that f is proper. We show that $f^*(\epsilon, .)$ is proper. Indeed, there is $x \in \operatorname{dom} f$ such that $\partial^C f(x)$ is nonempty because f is proper lower semicontinuous. Let $\xi \in \partial^C f(x) \subseteq \partial^\epsilon f(x)$ (by (2.4)).

One has $\langle \xi, z-x\rangle \le f(z) - f(x) + \epsilon\|z-x\|$ for all $z \in X$. Use the inequality $\|z-x\| \le \|z-y\| + \|y-x\|$ to deduce

$$\langle \xi, z\rangle - f(z) - \epsilon\|z-y\| \le \langle \xi, x\rangle - f(x) + \epsilon\|x-y\|$$

for all $z \in X$. Consequently, $f^*(\epsilon, \xi) \le \langle \xi, x\rangle - f(x) + \epsilon\|x-y\|$ and hence $\xi \in \operatorname{dom} f_y^*(\epsilon, .)$, which shows that $f^*(\epsilon, .)$ is proper.

For assertion iv), it suffices to observe the following chain of equivalences:

$$\begin{aligned} \xi \in \partial^\epsilon f(x) &\iff \langle \xi, y-x\rangle \le f(y) - f(x) + \epsilon\|y-x\| \quad \forall y \in X \\ &\iff \langle \xi, y\rangle - f(y) - \epsilon\|y-x\| \le \langle \xi, x\rangle - f(x) \quad \forall y \in X \\ &\iff f_x^*(\epsilon, \xi) = \langle \xi, x\rangle - f(x). \end{aligned}$$

For the last assertion, let $x, y \in X$. By definition, for all $\xi \in X^*$ we have,

$$f_y^*(\epsilon, \xi) \ge \langle \xi, x\rangle - f(x) - \epsilon\|x-y\|.$$

Equivalently, for all $\xi \in X^*$:

$$\langle \xi, x\rangle - f_y^*(\epsilon, \xi) \le f(x) + \epsilon\|x-y\|.$$

Therefore, by (2.3), we obtain $f_y^{**}(\epsilon, x) \le f(x) + \epsilon\|x-y\|$.The proof is complete. ∎

Proposition 2.2 *Let f and g be lower semicontinuous functions from X to $\mathbb{R} \cup \{+\infty\}$. Assume that f is ϵ_1-convex and g is ϵ_2-convex. Then for every $y \in X$ and $\xi \in X^*$, one has*

$$(f+g)_y^*(\epsilon_1+\epsilon_2, \xi) \le (f_y^*(\epsilon_1, .)\Box g_y^*(\epsilon_2, .))(\xi). \tag{2.5}$$

Equality holds if in addition, $y \in \operatorname{Int}(\operatorname{dom} f)$ and y is a local minimum point of the function $f + g - \langle \xi, .\rangle$ on X.

Proof. For the first part, let $\xi_1, \xi_2 \in X^*$ and $x \in X$. One obtains

$$f_y^*(\epsilon_1, \xi_1) \ge \langle \xi_1, x\rangle - f(x) - \epsilon_1\|x-y\|$$

and

$$g_y^*(\epsilon_2, \xi_1) \ge \langle \xi_1, x\rangle - g(x) - \epsilon_2\|x-y\|.$$

Therefore,

$$f_y^*(\epsilon_1, \xi_1) + g_y^*(\epsilon_2, \xi_2) \ge (f+g)_y(\epsilon_1+\epsilon_2, \xi_1+\xi_2)$$

and (2.5) holds. Under the additional condition, one has $0 \in \partial^C(f + g - \langle \xi, .\rangle)(y) \subseteq \partial^C f(y) + \partial^C g(y) - \xi$. Therefore, there are $\xi_1 \in \partial^C f(y)$ and $\xi_2 \in \partial^C g(y)$ such that $\xi = \xi_1 + \xi_2$. Due to (iv) of Proposition 2.1, we obtain

$$f_y^*(\epsilon_1, \xi_1) = \langle \xi_1, y\rangle - f(y) \qquad g_y^*(\epsilon_2, \xi_2) = \langle \xi_2, y\rangle - g(y)$$

and

$$(f+g)_y^*(\epsilon_1+\epsilon_2, \xi) = \langle \xi, y\rangle - f(y) - g(y).$$

Hence $(f_y^*(\epsilon_1, .)\Box g_y^*(\epsilon_2, .))(\xi) \le (f+g)_y^*(\epsilon_1+\epsilon_2, \xi)$ and the proof is complete. ∎

As the example below shows, inequality (2.5) is strict in general. Take f and $g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = -|x|$ and $g(x) = 2|x|$. Observe that f is 2-convex and g is convex. One has $(f+g)_1^*(2, 0) = -1$ while $(f_1^*(2, .)\Box g_1^*(0, .))(0) = 0$.

2.2. The Fenchel-Moreau extended duality theorem

We shall see in this subsection that the Fenchel-Moreau duality theorem can be extended to express the relation between an $\epsilon-$convex function and its second conjugate $f_y^{**}(\epsilon,.)$. For this purpose we need the following approximate mean value theorem proved by Zagrodny [21] (see also [9] for a generalized version):

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function and let a, b be two distinct points of $\operatorname{dom} f$. *Then there exist sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{x_n^*\}_{n\in\mathbb{N}}$ such that $x_n \to c \in [a,b)$; $x_n^* \in \partial^C f(x_n)$ and*

$$(i)\ \liminf_{n\to\infty}\langle x_n^*, b - x_n\rangle \geq \frac{f(b)-f(a)}{\|b-a\|}\|b-c\|\,;$$

$$(ii)\ \liminf_{n\to\infty}\langle x_n^*, b-a\rangle \geq f(b)-f(a).$$

It follows from this theorem that for a proper lower semicontinuous function f, the domain of $\partial^C f := \{x \in X |\ \partial^C f(x) \neq \emptyset\}$ is graphically dense in $\operatorname{dom} f$. Hence if f is a proper lower semicontinuous $\epsilon-$convex function, then $\partial^\epsilon f(x) \neq \emptyset$ on a graphically dense subset of $\operatorname{dom} f$.

Theorem 2.3 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous $\epsilon-$convex function. Then for all $x, y \in X$, we have*

$$|f(x) - f_y^{**}(\epsilon, x)| \leq \epsilon\|x-y\|. \tag{2.6}$$

As a result, $f_y^*(\epsilon,.)$ and $f_y^{**}(\epsilon,.)$ are proper lower semicontinuous convex function. Moreover, $f_y^{**}(\epsilon, x) > -\infty$ for all $x, y \in X$.

Proof. Let $x, y \in X$. By virtue of Proposition 2.1, $f_y^{**}(\epsilon, x) \leq f(x) + \epsilon\|x-y\|$. In order to prove (2.6), it suffices to show that

$$f_y^{**}(\epsilon, x) \geq f(x) - \epsilon\|x-y\|. \tag{2.7}$$

Let us consider the two following cases:

Case 1. $\partial^C f(x) \neq \emptyset$. Take some $\xi \in \partial^C f(x)$. By (2.4), $\xi \in \partial^\epsilon f(x)$. One has

$$\langle \xi, z - x\rangle \leq f(z) - f(x) + \epsilon\|z-x\| \quad \text{for all } z \in X.$$

Use the inequality $\|z-x\| \leq \|z-y\| + \|y-x\|$ to deduce

$$\langle \xi, z\rangle - f(z) - \epsilon\|z-y\| \leq \langle \xi, x\rangle - f(x) + \epsilon\|x-y\| \quad \text{for all } z \in X.$$

Consequently, $\langle \xi, x\rangle - f^*(\epsilon, \xi) \geq f(x) - \epsilon\|x-y\|$ and we derive (2.7).

Case 2. $\partial^C f(x) = \emptyset$. It must hapen one of the following two situations:

Either **2.1:** there exists a sequence $\{c_n\}_{n\in\mathbb{N}}$ converging to x and satisfying $f(c_n) < f(x)$ for all $n \in \mathbb{N}$ or **2.2:** there is a positive number γ such that $B(x, 2\gamma) \cap \operatorname{dom} f = \emptyset$.

In Case 2.1, for every n, we define the function f_n by

$$f_n(z) := \begin{cases} f_n(z) := f(z) & \text{if } z \neq x \\ f(x) & \text{if } f(x) \text{ is finite} \\ f(c_n) + 1 & \text{if } f(x) = +\infty. \end{cases}$$

Applying Zagrodny's M.V.T to the function f_n on $[c_n, x]$, select $y_n \in B([c_n, x], \frac{1}{n}) = \{z \in X|\ d_{[c_n,x]}(z) \leq \frac{1}{n}\}$; $y_n \neq x$ and $y_n^* \in \partial^C f_n(y_n)$ such that $\langle y_n^*, x - y_n\rangle > 0$ ($d_A(z)$ stands for the distance from z to the set A). Since $c_n \to x$, then $y_n \to x$. Note that $\partial^C f_n(z) = \partial^C f(z)$ for all $z \neq x$. Therefore, $\partial^C f_n(y_n) = \partial^C f(y_n)$.

According to Case 1, we have

$$\langle y_n^*, y_n\rangle - f_y^*(\epsilon, y_n^*) \geq f(y_n) - \epsilon\|y_n - y\|.$$

Hence $\langle y_n^*, x\rangle - f_y^*(\epsilon, y_n^*) \geq f(y_n) - \epsilon\|y_n - y\|$. This yields

$$f_y^{**}(\epsilon, x) \geq f(y_n) - \epsilon\|y_n - y\|.$$

Since f is lower semicontinuous , just take the limit as n tends to ∞ to obtain (2.7).

In Case 2.2, pick $a \in \mathrm{dom} f$ and for $n \in \mathbb{N}$, define the function g_n by

$$g_n(z) := \begin{cases} f(z) & \text{if } z \neq x \\ n & \text{otherwise.} \end{cases}$$

Since f is lower semicontinuous , then f is bounded from below on some neighbourhood V of the segment $[a, x]$, i. e., there exists $\alpha \in \mathbb{R}$ such that $f(z) \geq \alpha$ for all $z \in V$.

Now, apply again the mean value theorem to g_n on $[a, x]$. There exist sequences $\{x_n^m\}_{m\in\mathbb{N}}$; $\{x_n^{*m}\}_{m\in\mathbb{N}}$ such that $\lim_{m\to\infty} x_n^m = c_n \in [a, x)$; $x_n^{*m} \in \partial^C g_n(x_n^m)$ and

$$\liminf_{m\to\infty} \langle x_n^{*m}, x - x_n^m\rangle \geq \frac{g_n(x) - g_n(a)}{\|x - a\|}\|c_n - x\|. \tag{2.8}$$

When n is large enough, say $n \geq n_0$, one has $g_n(x) = n > f(a) = g_n(a)$. Since $\partial^C g_n(x_n^m)$ is nonempty, we must have $c_n \notin B(x, \gamma)$, that is, $\|c_n - x\| > \gamma$. For $n \geq n_0$, according to inequality (2.8), there is some index m_n such that $x_n^{m_n} \in V$ and

$$\langle x_n^{*m_n}, x - x_n^{m_n}\rangle > \frac{n - f(a)}{\|x - a\|}\gamma.$$

Equivalently, we have

$$\langle x^{*m_n}, x\rangle > \langle x_n^{*m_n}, x_n^{m_n}\rangle + \frac{n - f(a)}{\|x - a\|}\gamma.$$

On the other hand, use Case 1 to obtain

$$\langle x_n^{*m_n}, x_n^{m_n}\rangle - f_y^*(\epsilon, x_n^{*m_n}) \geq f(x_n^{m_n}) - \epsilon\|x_n^{m_n} - y\|.$$

Combining the above inequalities we obtain

$$\langle x_n^{*m_n}, x\rangle - f_y^*(\epsilon, x_n^{*m_n}) \geq \frac{n - f(a)}{\|x - a\|}\gamma + f(x_n^{m_n}) - \epsilon\|x_n^{m_n} - y\|.$$

Consequently,

$$f_y^{**}(\epsilon, x) \geq \frac{n - f(a)}{\|x - a\|}\gamma + f(x_n^{m_n}) - \epsilon \|x_n^{m_n} - y\|.$$

Finally, note that $f(x_n^{m_n}) \geq \alpha$ for all $n \geq n_0$ and that the sequence $\{\|x_n^{m_n} - y\|\}$ is bounded. Taking the limit as $n \to \infty$, in the above inequality we obtain $f_y^{**}(\epsilon, x) = +\infty$, and (2.7) holds. The proof is complete. ∎

Note that when f is convex, by setting $\epsilon = 0$, Theorem 2.3 subsumes the classical Fenchel-Moreau duality theorem.

Corollary 2.6 *Let $f : X \to 1\mathrm{R} \cup \{+\infty\}$ be a proper lower semicontinuous $\epsilon-$convex function. The following assertions are equivalent:*

(i) $\xi \in \partial^\epsilon f(x)$;

(ii) $x \in \partial f_x^*(\epsilon, .)(\xi)$;

(iii) $\xi \in \partial f_x^{**}(\epsilon, .)(x)$.

Proof. First, the implication (ii) ⇒ (iii) is known because $f_x^*(\epsilon, .)$ and $f_x^{**}(\epsilon, .)$ are convex functions. For the implication (i) ⇒ (ii), let $\xi \in \partial^\epsilon f(x)$. Due to Proposition 2.1, we have

$$\langle \xi, x\rangle - f_x^*(\epsilon, \xi) = f(x).$$

On the other hand, it follows from (2.2) that $f(x) \geq \langle \xi', x\rangle - f_x^*(\epsilon, \xi')$ for all $\xi' \in X^*$. The above relations imply

$$\langle x, \xi' - \xi\rangle \leq f_x^*(\epsilon, \xi') - f_x^*(\epsilon, \xi) \text{ for all } \xi' \in X^*,$$

which shows that $x \in \partial f_x^*(\epsilon, .)(\xi)$.

For the implication (iii) ⇒ (i), let $\xi \in \partial f_x^{**}(\epsilon, .)(x)$. One has $\langle \xi, y - x\rangle \leq f_x^{**}(\epsilon, y) - f_x^{**}(\epsilon, x)$ for all $y \in X$. According to Theorem 2.3, $f_x^{**}(\epsilon, x) = f(x)$ and $f_x^{**}(y) \leq f(y) + \epsilon\|x - y\|$. Hence

$$\langle \xi, y - x\rangle \leq f(y) - f(x) + \epsilon\|x - y\| \text{ for all } y \in X,$$

which shows that $\xi \in \partial^\epsilon f(x)$. ∎

3. Approximate convex functions

Let $f : X \to 1\mathrm{R} \cup \{+\infty\}$ be a lower semicontinuous function. For every $\delta > 0$, we define the function f_δ by $f_\delta(x) = f(x)$ if $x \in B(x_0, \delta)$ and $+\infty$ otherwise.

We say that the function f is *approximate convex* at $x_0 \in X$ if for each $\epsilon > 0$, there exists $\delta > 0$ such that f_δ is $\epsilon-$convex, and f is approximate convex on a nonempty set $C \subseteq X$ if it is approximate convex at every $x \in C$. When $C = X$, we say simply that f is approximate convex.

In this section we shall concentrate our efforts to the study of the class of approximate convex functions.

3.1 Basic properties

It follows immediately from the definition that convex functions are approximate convex, and the converse is not true. Below we shall give some more sufficient conditions for a function to be approximate convex. Let us recall [8] that a function

$f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be $\gamma-$ *paraconvex* with $\gamma \in \mathbb{R}$, if there is a constant $\kappa > 0$ such that for all $x, y \in X$ and $\lambda \in (0, 1)$:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) + \kappa\lambda(1-\lambda)\|x - y\|^\gamma.$$

Recall also that a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is strictly differentiable at $x_0 \in X$ if there exists $Df(x_0) \in X^*$ such that

$$\lim_{x,y \to x_0} \frac{f(y) - f(x) - Df(x_0)(x - y)}{\|x - y\|} = 0.$$

Proposition 3.1 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$. Each of the following conditions is sufficient for f to be approximate convex at $x_0 \in X$:*

i) f is $\gamma-$paraconvex with $\gamma > 1$;

ii) f is strictly differentiable at x_0;

iii) $f = f_1 + f_2$, or $f = \max\{f_1, f_2\}$ where f_1 and f_2 are approximate convex at x_0;

iv) $f = g \circ A$ where A is a continuous affine mapping from X to a Banach space Y and g is a function from Y to $\mathbb{R} \cup \{+\infty\}$ which is approximate convex at $Ax_0 \in Y$.

Proof. It is obvious that each of conditions i), iii), iv) implies that f is approximate convex at x_0. Actually, under condition i) f is approximate convex at any point of X. Now we show that f is approximate convex at x_0 if ii) is verified. By the strict differentiablility of f at x_0, for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(y) - Df(x_0)(x - y)| \leq \frac{\epsilon}{2}\|x - y\| \quad \forall x, y \in B(x_0, \delta).$$

Therefore, for every $x, y \in B(x_0, \delta)$ and $\lambda \in (0, 1)$ one has

$$|f(\lambda x + (1-\lambda)y) - f(x) - (1-\lambda)Df(x_0)(y - x)| \leq \frac{\epsilon}{2}(1-\lambda)\|x - y\|$$

and

$$|f(\lambda x + (1-\lambda)y) - f(y) - \lambda Df(x_0)(x - y)| \leq \frac{\epsilon}{2}\lambda\|x - y\|.$$

Consequently, for every $x, y \in B(x_0, \delta)$ and $\lambda \in (0, 1)$ one has

$$f(\lambda x + (1-\lambda)y) \leq f(x) + (1-\lambda)Df(x_0)(y - x) + \frac{\epsilon}{2}(1-\lambda)\|x - y\|$$

and

$$f(\lambda x + (1-\lambda)y) \leq f(y) + \lambda Df(x_0)(x - y) + \frac{\epsilon}{2}\lambda\|x - y\|.$$

Multiplying the above inequalities by λ and $(1-\lambda)$ respectively and summing them up yields

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) + \lambda\epsilon(1-\lambda)\|x - y\|,$$

which shows that f is approximate convex at x_0. ∎

The next proposition establishes a Lipschitz property of approximate convex functions. Its proof follows the lines of the convex case ([13],[15]).

Proposition 3.2 *Suppose that $f : X \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous function. If f is approximate convex at $x_0 \in \mathrm{Int}(\mathrm{dom} f)$, then f is locally Lipschitz at x_0.*

Proof. Since f is approximate convex at x_0, there exist $\epsilon > 0$ and $\delta > 0$ such that $B(x_0, \delta) \subset \operatorname{dom} f$ and equality (2.1) is satisfied for all $x, y \in B(x_0, \delta)$ and $\lambda \in (0, 1)$. First, we want to show that f is locally bounded at x_0. Let $U_n := \{x \in B(x_0, \delta) | \, f(x) \leq n\}$, $n = 1, 2, ...$ Then, $B(x_0, \delta) = \bigcup_{n \in \mathbb{N}} U_n$ and all the U_n are closed. Thanks to the Baire category theorem, there is some index n_0 such that the interior of U_{n_0} denoted by Int U_{n_0} is nonempty. Pick $z_0 \in \text{Int } U_{n_0}$ and $\alpha > 1$ such that $y_0 := z_0 + \alpha(x_0 - z_0) \in \text{Int } U_{n_0}$ and select some nonnegative number $\gamma < \delta$ such that for all $x \in B(x_0, \gamma)$ one has $z := y_0 + \alpha(x - y_0) \in \text{Int } U_{n_0}$. We have

$$
\begin{aligned}
f(x) &= f(\alpha^{-1} z + (1 - \alpha^{-1}) y_0) \\
&\leq \alpha^{-1} f(z) + (1 - \alpha^{-1}) f(y_0) + \epsilon \alpha^{-1} (1 - \alpha^{-1}) \|y_0 - z\| \\
&\leq \alpha^{-1} n_0 + (1 - \alpha^{-1}) f(y_0) + \epsilon \alpha^{-1} (1 - \alpha^{-1}) 2\delta.
\end{aligned}
$$

Thus, f is bounded from above, say, by M, on $B(x_0, \gamma)$. To show that it is locally bounded from below, note that for all $x \in B(x_0, \gamma)$, obviously, $2x_0 - x \in B(x_0, \gamma)$ and consequently

$$f(x_0) \leq 1/2 f(x) + 1/2 f(2x_0 - x) + \epsilon/2 \|x - x_0\|.$$

Therefore, $f(x) \geq 2f(x_0) - M - 2\epsilon\gamma$ for all $x \in B(x_0, \gamma)$ and f is bounded on $B(x_0, \gamma)$. Hence we may assume that $|f(x)| \leq M$ for all $x \in B(x_0, \gamma)$. Now, for any $x, y \in B(x_0, \gamma/2)$, then $z := x + (\gamma/2\eta)(x - y) \in B(x_0, \gamma)$ with $\eta := \|x - y\|$. Hence,

$$f(x) = f\Big(\frac{2\eta}{\gamma + 2\eta} z + \frac{\gamma}{\gamma + 2\eta} y\Big) \leq \frac{2\eta}{\gamma + 2\eta} f(z) + \frac{\gamma}{\gamma + 2\eta} f(y) + \frac{2\epsilon\eta\gamma}{(\gamma + 2\eta)^2} \|z - y\|.$$

It follows that

$$f(x) - f(y) \leq \frac{2\eta}{\gamma}(f(z) - f(x)) + \epsilon\gamma \|x - y\| \leq \Big(\frac{4M}{\gamma} + \epsilon\gamma\Big) \|x - y\|.$$

Interchanging the roles of x and y, we obtain the required result. ∎

The following corollary will be used in the sequel.

Corollary 3.3 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function that is approximate convex on the segment $[a, b]$ with $a \neq b$ and $[a, b] \subset \operatorname{dom} f$. Then the restriction of f on this segment is continuous.*

Proof. Let us consider the function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by $\varphi(t) = f(ta + (1 - t)b)$ if $t \in [0, 1]$; $\varphi(t) = f(a)$ if $t \geq 1$ and $\varphi(t) = f(b)$ if $t \leq 0$. We wish to show that φ is continuous on $[0, 1]$. Note that φ is approximate convex at all $t \in (0, 1)$ and by virtue of Proposition 3.2, φ is continuous on $(0, 1)$. We need only to show that $\lim_{t \to 0^+} \varphi(t) = \varphi(0)$ (similarly, $\lim_{t \to 1^-} \varphi(t) = \varphi(1)$). For each $\epsilon > 0$, thanks to approximate convexity of f at b take a real $\delta > 0$ such that

$$\varphi(\lambda t_1 + (1 - \lambda) t_2) \leq \lambda \varphi(t_1) + (1 - \lambda) \varphi(t_2) + \epsilon \lambda (1 - \lambda) |t_1 - t_2|$$

for all $t_1, t_2 \in [0, \delta]$ and $\lambda \in (0, 1)$. Let $\{t_n\}_{n \in \mathbb{N}}$ be a sequence converging to 0, $t_n > 0$. For n is large, one has $t_n < \delta$ and therefore

$$\varphi(t_n) \leq t_n/\delta \varphi(\delta) + (1 - t_n/\delta) \varphi(0) + \epsilon t_n (1 - t_n) \delta.$$

Passing to the limit as $n \to \infty$, we obtain $\limsup_{n\to\infty} \varphi(t_n) = \varphi(0)$ and consequently, (since φ is lower semicontinuous) $\lim_{n\to\infty} \varphi(t_n) = \varphi(0)$. The proof is complete. ∎

We present a characterization of approximate convexity via convex functions.

Theorem 3.4 *Let X be a Banach space and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Then, f is approximate convex at $x_0 \in X$ if and only if for each $\epsilon > 0$, there exists $\delta > 0$ such that for any $y \in B(x_0, \delta)$ we can find a lower semicontinuous convex function $g_y(.) : X \to \mathbb{R} \cup \{+\infty\}$ satisfying the following inequality for every $x \in B(x_0, \delta)$.*

$$|f(x) - g_y(x)| \le \epsilon \|x - y\|.$$

Proof. For the part "if", assume that f is approximate convex at x_0. For each $\epsilon > 0$, take $\delta > 0$ such that the function f_δ is $\epsilon-$convex. Fix $y \in B(x_0, \delta)$ and define $g_y(x) := f^{**}_{\delta y}(\epsilon, x)$, where $f^{**}_{\delta y}(\epsilon, .)$ is the $\epsilon-$second conjugate function of f_δ. The result follows from Theorem 2.3.

For the part "only if", by the assumption, for each $\epsilon > 0$, there exists $\delta > 0$ such that for every $z \in B(x_0, \delta)$, we can find a convex function g_z satisfying

$$|f(x) - g_z(x)| \le \frac{\epsilon}{2}\|x - z\| \text{ for all } x \in B(x_0, \delta).$$

Let $x, y \in B(x_0, \delta)$; $\lambda \in (0, 1)$. Since $\lambda x + (1 - \lambda)y \in B(x_0, \delta)$ we have

$$|f(x) - g_{\lambda x+(1-\lambda)y}(x)| \le \frac{\epsilon}{2}(1 - \lambda)\|x - y\|$$

and

$$|f(y) - g_{\lambda x+(1-\lambda)y}(y)| \le \frac{\epsilon}{2}\lambda\|x - y\|.$$

It follows that

$$\lambda f(x) + \frac{\epsilon}{2}\lambda(1 - \lambda)\|x - y\| \ge \lambda g_{\lambda x+(1-\lambda)y}(x)$$

and

$$(1 - \lambda)f(y) + \frac{\epsilon}{2}\lambda(1 - \lambda)\|x - y\| \ge (1 - \lambda)g_{\lambda x+(1-\lambda)y}(y).$$

On adding the above two inequalities and noticing that $g_{\lambda x+(1-\lambda)y}(.)$ is convex and $g_{\lambda x+(1-\lambda)y}(\lambda x + (1 - \lambda)y) = f(\lambda x + (1 - \lambda)y)$, we obtain the required inequality:

$$f(\lambda x + (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda)f(y) + \epsilon\lambda(1 - \lambda)\|x - y\|.$$

The proof is complete. ∎

Corollary 3.5 *If $f : X \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and approximate convex at $x_0 \in \operatorname{dom} f$, then for every $v \in X$, the directional derivative*

$$f'(x_0, v) := \lim_{t\downarrow 0} \frac{f(x_0 + tv) - f(x_0)}{t}$$

of f exists and is sublinear on X.

Proof. Since f is approximate convex at x_0, making use of Theorem 3.4, for every $\epsilon > 0$, there exist $\delta > 0$ and a lower semicontinuous convex function $g_{x_0}(.)$ such that

$$|f(x) - g_{x_0}(x)| \le \epsilon\|x - x_0\| \text{ for all } x \in B(x_0, \delta).$$

Equivalently,

$$f(x) - \epsilon\|x - x_0\| \leq g_{x_0}(x) \leq f(x) + \epsilon\|x - x_0\| \text{ for all } x \in B(x_0, \delta).$$

Fix $v \in X$ and take $t > 0$ small enough to have $t\|v\| < \delta$. Then,

$$\frac{f(x_0 + tv) - f(x_0)}{t} - \epsilon\|v\| \leq \frac{g_{x_0}(x_0 + tv) - g_{x_0}(x_0)}{t} \leq \frac{f(x_0 + tv) - f(x_0)}{t} + \epsilon\|v\|$$

(note that $g_{x_0}(x_0) = f(x_0)$). Since g_{x_0} is convex, it is a basic fact in Convex Analysis that

$$g'_{x_0}(x_0, v) = \lim_{t \downarrow 0} \frac{g_{x_0}(x_0 + tv) - g_{x_0}(x_0)}{t}.$$

Taking the limit as $t \downarrow 0$ in the preceding inequalities, we obtain

$$\limsup_{t \downarrow 0} \frac{f(x_0 + tv) - f(x_0)}{t} \leq g'_{x_0}(x_0, v) + \epsilon\|v\| \tag{3.1}$$

and

$$\liminf_{t \downarrow 0} \frac{f(x_0 + tv) - f(x_0)}{t} \geq g'_{x_0}(x_0, v) - \epsilon\|v\|. \tag{3.2}$$

Consequently, for any $\epsilon > 0$,

$$\limsup_{t \downarrow 0} \frac{f(x_0 + tv) - f(x_0)}{t} \leq \liminf_{t \downarrow 0} \frac{f(x_0 + tv) - f(x_0)}{t} + 2\epsilon\|v\|.$$

This means that

$$f'(x_0, v) = \lim_{t \downarrow 0} \frac{f(x_0 + tv) - f(x_0)}{t}$$

exists. Obviously, $f'(x_0, .)$ is positively homogeneous. We shall use (3.1) and (3.2) to prove its subadditivity. Fix $v_1, v_2 \in X$, use (3.1) and write

$$f'(x_0, v_1 + v_2) \leq g'_{x_0}(x_0, v_1 + v_2) + \epsilon\|v_1 + v_2\|.$$

By (3.2), then $f'(x_0, v_1) \geq g'_{x_0}(x_0, v_1) - \epsilon\|v_1\|$ and $f'(x_0, v_2) \geq g'_{x_0}(x_0, v_2) - \epsilon\|v_2\|$. On the other hand, since g_{x_0} is convex, it is well-known that $g'_{x_0}(x_0, .)$ is sublinear. Hence, for any $\epsilon > 0$,

$$f'(x_0, v_1 + v_2) \leq f'(x_0, v_1) + f'(x_0, v_2) + \epsilon(\|v_1 + v_2\| + \|v_1\| + \|v_2\|).$$

Consequently, $f'(x_0, v_1 + v_2) \leq f'(x_0, v_1) + f'(x_0, v_2)$ and the proof is complete. ∎

3.2. Subdifferential of approximate convex functions

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function and let ϵ be a fixed nonnegative real. Recall that the Fréchet $\epsilon-$subdifferential of f at $x \in \operatorname{dom} f$ is defined by

$$\partial^F_\epsilon f(x) := \Big\{x^* \in X^* | \liminf_{\|h\| \to 0} \frac{f(x + h) - f(x) - \langle x^*, h\rangle}{\|h\|} \geq -\epsilon\Big\}.$$

When $\epsilon = 0$, we set $\partial^F f(x) := \partial^F_0 f(x)$. The limiting Fréchet $\epsilon-$subdifferential is defined by

$$\hat{\partial}_\epsilon f(x) := \text{seq} - \limsup_{y \xrightarrow{f} x} \partial^F_\epsilon f(y)$$

where, " seq-limsup"denotes the sequential Painlevé-Kuratowski upper limit of sets, i.e.,

$$\text{seq} - \limsup_{y \xrightarrow{f} x} \partial_\epsilon^F f(y) = \Big\{ x^* \in X^* |\ \exists x_n \xrightarrow{f} x,\ x_n^* \xrightarrow{w^*} x^* \text{ with } x_n^* \in \partial_\epsilon^F f(x_n) \Big\}$$

with "$\xrightarrow{w^*}$" denoting the weak* convergence in the dual space X^*.

The Mordukhovich subdifferential of f at $x \in \text{dom } f$ is the set ([11], [12]):

$$\partial^M f(x) := \text{seq} - \limsup_{y \xrightarrow{f} x, \epsilon \downarrow 0} \partial_\epsilon^F f(y).$$

We agree that $\partial^M f(x) = \partial_\epsilon^F f(x) = \hat{\partial}\epsilon f(x) = \emptyset$ if $x \notin \text{dom} f$. We also need to recall the definition of the ϵ-*approximate* subdifferential introduced by A. Ioffe in [4],[5]. Denote by $\mathcal{F}(X)$ the collection of all finite dimensional subspaces of X, then

$$\partial^A f(x) := \bigcap_{L \in \mathcal{F}(X)} \limsup_{y \xrightarrow{f} x, \epsilon \downarrow 0} \partial_\epsilon^- f_{y+L}(y),$$

where,

$$f_{y+L}(x) = \begin{cases} f(x) & \text{if} \quad x \in y + L \\ +\infty & \text{otherwise} \end{cases}$$

and

$$\partial_\epsilon^- f(y) := \Big\{ x^* \in X^* |\ \langle x^*, v \rangle \le \liminf_{u \to v, t \downarrow 0} \frac{f(y + tu) - f(y)}{t} + \epsilon \|v\| \quad \forall v \in X \Big\}.$$

"limsup" is used here to express the "topological Painlevé-Kuratowski limit", namely, for a multivalued mapping $F : X \rightrightarrows X^*$, then $x^* \in \limsup_{y \to x} F(y)$ if for each weak$^*-$neighbourhood W of the origin of X^* and for each neighbourhood V of x, there exists $y \in V$ such that $(w + x^*) \cap F(y) \neq \emptyset$.

To define the Ioffe geometric subdifferential (denoted by $\partial^G f(.)$), we recall ([6]) that the $G-$*normal cone* to $C \subseteq X$ at $x \in C$ is the set

$$N^G(C, x) := \text{cl}^* \big(\bigcup_{\lambda > 0} \lambda \partial^A d_C(x) \big),$$

where $d_C(x)$ denotes the distance from x to C and cl* means the weak*closure. Now, $\partial^G f(x)$ is given by

$$\partial^G f(x) := \big\{ x^* \in X^* |\ (x^*, -1) \in N^G(\text{epi} f, (x, f(x)) \big\}.$$

It is well-known that if X is finite dimensional, then $\partial^G f(x)$, $\partial^A f(x)$ and $\partial^M f(x)$ coincide. The next proposition gives an important feature of approximate convex functions. We are keeping the notation

$$\partial f(x_0) := \big\{ x^* \in X^* |\ \langle x^*, v \rangle \le f'(x_0, v) \text{ for all } v \in X \big\},$$

which agrees with the subdifferential in the sense of convex analysis when f is convex.

Theorem 3.6 *Let X be a Banach space. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Assume that f is approximate convex at $x_0 \in \text{dom} f$. Then we have* *(i)* $\partial^C f(x_0) = \partial^M f(x_0) = \partial^F f(x_0) = \partial f(x_0) = \partial^G f(x_0)$;

(ii) $\partial f(x_0) \subseteq \partial^A f(x_0)$. *Equality holds if in addition, f is Lipschitz at x_0.*

Proof. Clearly we have:

$$\partial^F f(x_0) \subseteq \partial^C f(x_0), \partial^F f(x_0) \subseteq \partial^M f(x_0) \text{ and } \partial^F f(x_0) \subseteq \partial f(x_0).$$

We need to show that $\partial^C f(x_0) \subseteq \partial^F f(x_0)$; $\partial^M f(x_0) \subseteq \partial^F f(x_0)$ and $\partial f(x_0) \subseteq \partial^F f(x_0)$. Since by assumption, f is approximate convexity at x_0, for each $\epsilon > 0$, there is $\delta > 0$ such that the function

$$f_\delta(x) := \begin{cases} f(x) & \text{if } x \in B(x_0, \delta), \\ +\infty & \text{otherwise} \end{cases}$$

is $\epsilon-$convex. Let $x^* \in \partial^C f(x_0)$. One has

$$\begin{aligned} &\partial^C f(x_0) = \partial^C f_\delta(x_0) \\ &= \big\{ x^* \in X^* | \ \langle x^*, h \rangle \leq f(x_0 + h) - f(x_0) + \epsilon \|h\| \quad \forall h \in B(0, \delta) \ \big\}. \end{aligned}$$

It follows that $x^* \in \partial^F f(x_0)$ and the inclusion $\partial^C f(x_0) \subseteq \partial^F f(x_0)$ is established.

As to equality $\partial^G f(x_0) = \partial^F f(x_0)$, observe that

$$\partial^F f(x_0) \subseteq \partial^G f(x_0) \subseteq \partial^C f(x_0).$$

Since as already shown $\partial^F f(x_0) = \partial^C f(x_0)$, we obtain $\partial^G f(x_0) = \partial^F f(x_0) = \partial^C f(x_0)$.

For the inclusion $\partial^M f(x_0) \subseteq \partial^F f(x_0)$, let $x^* \in \partial^M f(x_0)$. There exist sequences $\{\epsilon_n\} \downarrow 0$, $\{x_n\} \to x$, $\{x_n^*\} \stackrel{w^*}{\to} x^*$ with $x_n^* \in \partial^F_{\epsilon_n} f(x_n)$. Pick a sequence of nonnegative numbers $\gamma_n \downarrow 0$. By definition, for each n, we can find a number $\eta_n > 0$ such that

$$\text{(3.3)} \qquad \langle x_n^*, h \rangle \leq f(x_n + h) - f(x_n) + (\epsilon_n + \gamma_n)\|h\| \text{ for all } h \in B(x_n, \eta_n).$$

As above, for any $\epsilon > 0$, take $\delta > 0$ such that for all $x, y \in B(x_0, \delta)$ and $\lambda \in (0, 1)$, inequality (2.1) is satisfied, i. e.,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + 1 - \lambda) f(y) + \epsilon \lambda (1 - \lambda)\|x - y\|.$$

We may assume that $x_n \in B(x_0, \delta)$ for all $n \geq n_0$. For any $y \in B(x_0, \delta)$, choose $t \in (0, 1)$ such that $t\|y - x_n\| < \eta_n$. From (3.3) and (2.1) one deduces that

$$\begin{aligned} &\langle x_n^*, t(y - x_n) \rangle \\ &\qquad \leq f(x_n + t(y - x_n)) - f(x_n) + t(\epsilon_n + \gamma_n)\|y - x_n\| \\ &\qquad \leq (1 - t) f(x_n) + t f(y) - f(x_n) + t(\epsilon(1 - t) + \epsilon_n + \gamma_n)\|y - x_n\|. \end{aligned}$$

Therefore

$$\langle x_n^*, y - x_n \rangle \leq f(y) - f(x_n) + (\epsilon + \epsilon_n + \gamma_n)\|y - x_n\|.$$

Passing to limit as n tends to ∞, we obtain

$$\langle x^*, y - x_0 \rangle \leq f(y) - f(x_0) + \epsilon \|y - x_0\|.$$

This shows that $x^* \in \partial^F f(x_0)$ and the inclusion $\partial^M f(x_0) \subseteq \partial^F f(x_0)$ holds.

Finally, Let $x^* \in \partial f(x_0)$. Again by the approximate convexity of f at x_0, for any $y \in B(x_0, \delta)$ fixed, and $t \in (0,1)$, one has

$$\frac{f(x_0 + t(y - x_0)) - f(x_0)}{t} \leq f(y) - f(x_0) + \epsilon(1-t)\|y - x_0\|.$$

Taking the limit as $t \downarrow 0$ we obtain $f'(x_0, y - x_0) \leq f(y) - f(x_0) + \epsilon\|y - x_0\|$ and consequently $x^* \in \partial^F f(x_0)$. Hence, $\partial f(x_0) = \partial^F f(x_0)$.

Let us now prove assertion (ii). Clearly, $\partial f(x_0) = \partial^F f(x_0) \subseteq \partial^A f(x_0)$. Suppose now that f is Lipschitz around x_0. There are $\kappa > 0$ and $\delta_0 > 0$ such that $|f(x) - f(y)| \leq \kappa\|x - y\|$ for all $x, y \in B(x_0, \delta_0)$. Let $x^* \in \partial^A f(x_0)$. Take $v \in X, \epsilon, \gamma > 0$, define $W := \{x^* \in X^* | \, |\langle x^*, v\rangle| \leq \gamma\}$ and take $L \in \mathcal{F}(X)$ such that $v \in L$. Set $V := B(x_0, \eta)$ with $\eta > 0$. By definition, there exist $y \in V$ and $y^* \in \partial_\epsilon^- f_{y+L}(y)$ such that

$$|\langle y^* - x^*, v\rangle| \leq \gamma. \tag{3.4}$$

Since f is approximate convex at x_0, there is $\delta > 0$ with $\delta < \delta_0$ such that f_δ is $\epsilon-$convex. For any $\eta > 0$ and $t > 0$ small enough to have $\eta + t\|v\| < \delta$, one has $y + tv \in B(x_0, \delta)$. For all $s \in (0, t)$, by using the decomposition $y + sv = (y + tv)s/t + y(t - s)/t$ one obtains:

$$f(y + sv) \leq \frac{s}{t} f(y + tv) + \frac{t - s}{t} f(y) + \frac{\epsilon s(t - s)}{t^2} t\|v\|.$$

Equivalently

$$\frac{f(y + sv) - f(y)}{s} \leq \frac{f(y + tv) - f(y)}{t} + \epsilon(1 - \frac{s}{t})\|v\|.$$

Letting $s \downarrow 0$ in the latter inequality yields

$$\langle y^*, v\rangle \leq \frac{f(y + tv) - f(y)}{t} + 2\epsilon\|v\|.$$

On the other hand, since f is Lipschitz with a Lipschitz constant κ and $y \in B(x_0, \eta)$, one has

$$f(y + tv) - f(y) \leq f(x_0 + tv) - f(x_0) + 2\kappa\eta.$$

Combining (3.4) with the latter inequalities we obtain

$$\langle x^*, v\rangle \leq \frac{f(x_0 + tv) - f(x_0)}{t} + 2\epsilon\|v\| + \frac{2\kappa\eta}{t} + \gamma.$$

By letting $\eta \downarrow 0$ and $\gamma \downarrow 0$ in this inequality we have

$$\langle x^*, v\rangle \leq \frac{f(x_0 + tv) - f(x_0)}{t} + 2\epsilon\|v\|$$

and then by letting further $t \downarrow 0$ and $\epsilon \downarrow 0$, we obtain $\langle x^*, v\rangle \leq f'(x_0, v)$. This shows that $x^* \in \partial f(x_0)$ and completes the proof. ∎

Corollary 3.7 *Assume that* $f : X \to \mathbb{R} \cup \{+\infty\}$ *is lower semicontinuous and approximate convex at* $z \in X$*. Then the condition* $0 \in \partial^C f(z)$ *implies that* z *is a local minimum of the functions* $f(.) + \epsilon\|. - z\|$ *for every* $\epsilon > 0$.

Proof. By Theorem 3.6, $\partial^C f(z) = \partial^F(z)$. The condition $0 \in \partial^F f(z)$ means that for each $\epsilon > 0$, one has $f(z+h) + \epsilon\|h\| \geq f(z)$ for h sufficiently close to 0. This shows that z is a local minimum of the function $f(.) + \epsilon\|. - z\|$. ∎

Let us now prove a sum rule for the subdifferential of approximate convex functions (see [1], [17], [19] for convex functions).

Theorem 3.8 *Let X be a Banach space. Let f_1 and $f_2 : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous functions. Suppose that* $\operatorname{dom} f_1$ *and* $\operatorname{dom} f_2$ *are convex sets and that f_1 and f_2 are approximate convex at $x_0 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$. Then one has*

$$\partial(f_1 + f_2)(x_0) \supseteq \partial f_1(x_0) + \partial f_2(x_0).$$

Equality holds provided the Attouch-Brézis condition holds, i.e.

$$\bigcup_{\lambda>0} \lambda(\operatorname{dom} f_1 - \operatorname{dom} f_2) \text{ is a closed subspace of } X.$$

Proof. The inclusion $\partial(f_1 + f_2)(x_0) \supseteq \partial f_1(x_0) + \partial f_2(x_0)$ follows immediately from the definition. Now assume that the Attouch-Brézis condition holds. First we show that

$$\partial(f_1 + f_2)(x_0) \subseteq \operatorname{cl}(\partial f_1(x_0) + \partial f_2(x_0)),$$

where "cl" denotes the closure in the norm topology. Let $\epsilon > 0$ be arbitrarily fixed. Since f_1 and f_2 are approximate convex functions, by Theorem3.4, there exist $\delta > 0$ and lower semicontinuous convex functions $g^1_{x_0}(.)$ and $g^2_{x_0}(.)$ such that for all $x \in B(x_0, \delta)$ the following inequalities are satisfied

$$|f_i(x) - g^i_{x_0}(x)| \leq \epsilon\|x - x_0\|, \quad i = 1, 2.$$

This implies that

$$\operatorname{dom} f_i \cap B(x_0,\delta) = \operatorname{dom} g^i_{x_0} \cap B(x_0,\delta) \text{ and } \partial g^i_{x_0}(x_0) \subseteq \partial f_i(x_0) + \epsilon B^* \text{ for } i = 1, 2.$$

By setting $x = x_0$ and by estimating $f_1 + f_2$ by the above inequality for $x \in B(x_0, \delta)$ we obtain:

$$\partial(f_1 + f_2)(x_0) \subseteq \partial(g^1_{x_0} + g^2_{x_0})(x_0) + 2\epsilon B^* \tag{3.5}$$

Note further that

$$\bigcup_{\lambda>0} \lambda\big(\operatorname{dom} f_1 - \operatorname{dom} f_2\big) = \bigcup_{\lambda>0} \lambda\Big(\operatorname{dom} f_1 \cap B(x_0,\delta) - \operatorname{dom} f_2 \cap B(x_0,\delta)\Big).$$

Indeed, the inclusion "$\supseteq$" is obvious. The converse inclusion follows from the fact that $\operatorname{dom} f_1 \subseteq \cup_{\lambda>0}\lambda(\operatorname{dom} f_1 \cap B(x_0,\delta))$ because $\operatorname{dom} f_1$ is convex. We derive from the Attouch-Brézis condition that

$$\bigcup_{\lambda>0} \lambda\Big(\operatorname{dom} g^1_{x_0} \cap B(x_0,\delta) - \operatorname{dom} g^2_{x_0} \cap B(x_0,\delta)\Big)$$

is a closed subspace of X. Hence, the sum rule valid for convex functions (see [2]), yields:

$$\partial(g^1_{x_0} + g^2_{x_0})(x_0) = \partial g^1_{x_0}(x_0) + \partial g^2_{x_0}(x_0).$$

Combining this formula with (3.5) yields:

$$\partial(f_1+f_2)(x_0) \subseteq \partial f_1(x_0) + \partial f_2(x_0) + 4\epsilon B^*.$$

As $\epsilon > 0$ is arbitrary, we conclude

$$\partial(f_1+f_2)(x_0) \subseteq \mathrm{cl}(\partial f_1(x_0) + \partial f_2(x_0)).$$

Let us now show that $\partial f_1(x_0) + \partial f_2(x_0)$ is norm-closed. Let $\{x^*_\alpha\}$ be a net of elements of $\partial f_1(x_0) + \partial f_2(x_0)$ norm-converging to x^*. We want to show that $x^* \in \partial f_1(x_0) + \partial f_2(x_0)$. Without any loss of generality, we may assume that the net $\{x^*_\alpha\}$ is norm-bounded. Let $x^*_\alpha = y^*_\alpha + z^*_\alpha$ with $y^*_\alpha \in \partial f_1(x_0)$ and $z^*_\alpha \in \partial f(x_0)$. Denote by $L := \bigcup_{\lambda>0} \lambda\Big(\mathrm{dom} f_1 - \mathrm{dom} f_2\Big)$ and take $v := \lambda(x_1 - x_2) \in L$ with $x_1 \in \mathrm{dom} f_1$, $x_2 \in \mathrm{dom} f_2$. Assume that f_1, f_2 are $\epsilon-$convex on $B(x_0,\delta)$ for some $\epsilon, \delta > 0$. Choose $t > 0$ small enough to have $x_0 + t(x_1 - x_0)$ and $x_0 + t(x_2 - x_0) \in B(x_0, \delta)$. One has

$$\begin{aligned}
\langle y^*_\alpha, v\rangle &= \frac{\lambda}{t}[\langle y^*_\alpha, t(x_1 - x_0)\rangle - \langle y^*_\alpha, t(x_2 - x_0)\rangle]\\
&= \frac{\lambda}{t}[\langle y^*_\alpha, t(x_1 - x_0)\rangle - \langle x^*_\alpha - z^*_\alpha, t(x_2 - x_0)\rangle]\\
&\le \frac{\lambda}{t}[f(x_0 + t(x_1 - x_0)) - f(x_0) + \epsilon t\|x_1 - x_0\| + t\|x^*_\alpha\|\|x_2 - x_0\| +\\
&+ f_2(x_0 + t(x_2 - x_0)) - f_2(x_0) + \epsilon t\|x_2 - x_0\|\|].
\end{aligned}$$

Since $\{x^*_\alpha\}$ is norm-bounded, the above inequality implies that the net $\{y^*_\alpha\}$ is weak$^*-$bounded on L and consequently $\{y^*_\alpha\}$ is norm-bounded on L. Since L is a closed subspace of X, then L is a Banach space itself and therefore the net of the restrictions $\{y^*_{\alpha,L}\}$ of $\{y^*_\alpha\}$ to L has a weak$^*-$convergent subsequence in the topological dual L^* of L. We can assume that $y^*_{\alpha,L} \overset{w^*}{\to} y^*_L \in L^*$. By the Hahn-Banach theorem, there exists an extension $y^* \in X^*$ of y^*_L. In order to complete the proof, we want to show that $y^* \in \partial f_1(x_0)$ and $x^* - y^* \in \partial f_2(x_0)$. Let $\epsilon > 0$. There is $\delta > 0$ such that f is ϵ-convex on $B(x_0, \delta)$. Since $y^*_\alpha \in \partial f_1(x_0)$, for all $x \in B(x_0, \delta)$, one has

$$\langle y^*_\alpha, x - x_0\rangle \le f_1(x) - f_1(x_0) + \epsilon\|x - x_0\|\,.$$

Let $x \in B(x_0, \delta)$. If $x \in \mathrm{dom} f_1$ then $x - x_0 \in L$, hence $\langle y^*_\alpha, x - x_0\rangle \to \langle y^*, x - x_0\rangle$ and consequently

$$\langle y^*, x - x_0\rangle \le f_1(x) - f_1(x_0) + \epsilon\|x - x_0\|.$$

Obviously, this inequality also holds if $x \notin \mathrm{dom} f_1$. Hence, $y^* \in \partial^F f_1(x_0) = \partial f_1(x_0)$. Similarly, $x^* - y^* \in \partial^F f_2(x_0) = \partial f_2(x_0)$ and the proof is complete. ∎

As a direct consequence of Theorem 3.8 we have:

Corollary 3.9 *Assume that f_1 and f_2 are approximate convex at $x_0 \in \mathrm{dom}\, f_1 \cap$ $\mathrm{Int}(\mathrm{dom} f_2)$ (or, equivalently f_2 is Lipschitz around x_0). Then we have:*

$$\partial(f_1+f_2)(x_0) = \partial f_1(x_0) + \partial f_2(x_0).$$

Proof. The proof follows immediately from Theorem 3.8. ∎

Corollary 3.10 *Suppose that $f : X \to \mathbb{R} \cup \{+\infty\}$ is proper lower semicontinuous and approximate convex at $x_0 \in X$. Then*

$$\hat{\partial}_\epsilon f(x_0) = \partial^F_\epsilon f(x_0) = \partial f(x_0) + \epsilon B^*.$$

Proof. By virtue of Corollary 3.9, one has

$$\partial^F_\epsilon f(x_0) := \partial^F(f + \epsilon\|. - x_0\|)(x_0) = \partial f(x_0) + \epsilon B^*.$$

The proof of equality $\hat{\partial}_\epsilon f(x_0) = \partial^F_\epsilon f(x_0)$ is similar to equality $\partial^M f(x_0) = \partial^F f(x_0)$ in Proposition 3.6. ∎

This corollary is a generalization of Proposition 2.3 and Proposition 2.8 of [7] established in the case when f is convex or continuously differentiable.

3.3. Integration of subdifferentials of approximate convex functions

In the sequel, f is supposed to be a lower semicontinuous function from a Banach space X to the extended real line $\mathbb{R} \cup \{+\infty\}$ and ∂f is any subdifferential which satisfies the following conditions:

(i) $\partial f(x) = \partial^F f(x)$ if f is approximate convex at x, or equivalently

$$\partial f(x) = \left\{x^* \in X^* | \ \langle x^*, v\rangle \le f'(x, v) \quad \text{for all } v \in X\ \right\};$$

(ii) $0 \in \partial f(x)$ if x is a local minimum of f;

(iii) If g is a Lispchitz convex function or a Lipschitz concave function on X, then

$$\partial(f + g)(x) \subseteq \partial f(x) + \partial g(x) \quad \textit{for} \text{ every } \ x \in \text{dom}\, f \cap \text{dom}\, g;$$

(iv) If f and g coincide on a neighbourhood of x, then $\partial f(x) = \partial g(x)$.

It was shown in [9, 18], that for any subdifferential which satisfies conditions (i)-(iv), the Zagrodny Mean Value Theorem (M. V. T) is valid. Clarke's subdifferential, Mordukhovich's subdifferential , G-subdifferential... verify conditions (i)-(iv). The following property of approximate convex functions will be needed.

Lemma 3.11 *Let $g : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function that is approximate convex at $x_0 \in$dom$\,g$ and let $v \in X$. Assume that $g'(x_0, v) > -\infty$. Then for every $\alpha > 0$, there exists $\eta > 0$ such that for every $s, t \in \mathbb{R}$ with $0 < s < t < \eta$, the following inequality is satisfied:*

$$\frac{g(x_0 + tv) - g(x_0 + sv)}{t - s} \le \frac{g(x_0 + tv) - g(x_0)}{t} + \alpha\|v\|.$$

Proof. Since g is approximate convex at x_0, for $\alpha > 0$ there is $\delta_0 > 0$ such that for all $x, y \in B(x_0, \delta_0)$; $\lambda \in (0, 1)$, (assume that $B(x_0, \delta_0) \subseteq U$) one has

$$g(\lambda x + (1 - \lambda)y) \le \lambda g(x) + (1 - \lambda)g(y) + \frac{\alpha}{4}\lambda(1 - \lambda)\|x - y\|. \tag{3.6}$$

Observe that the result is obvious if there is $\delta > 0$ such that $g(x_0 + tv) = +\infty$ for all $t \in (0, \delta)$. Assume now there is $\delta_1 > 0$ with $\delta_1 < \delta_0$ such that $x_0 + \delta_1 v \in$dom$g$. Pick

$\delta > 0$ such that $\delta\|v\| < \delta_0$ and $\delta < \delta_1$. For $0 < s < t < \delta/2$, using the representation $x_0 + sv = \frac{s}{t}(x_0 + tv) + \frac{t-s}{t}x_0$ and according to (3.6), one has

$$g(x_0 + sv) \le \frac{s}{t}g(x_0 + tv) + \frac{t-s}{t}g(x_0) + \frac{\alpha s(t-s)}{4t}\|v\|.$$

Consequently

$$\frac{g(x_0 + tv) - g(x_0 + sv)}{t-s} \ge \frac{g(x_0 + sv) - g(x_0)}{s} - \frac{\alpha}{4}\|v\|.$$

Passing to the limit as $s \downarrow 0$, one deduces

$$\frac{g(x_0 + tv) - g(x_0)}{t} \ge g'(x_0, v) - \frac{\alpha}{4}\|v\| \quad \text{for all } t \text{ with } 0 < t < \delta/2. \tag{3.7}$$

On the other hand, using the representation

$$x_0 + (t+s)v = \frac{s}{t}(x_0 + 2tv) + \frac{t-s}{t}(x_0 + tv),$$

we derive from (3.6) that

$$\frac{g(x_0 + (t+s)v) - g(x_0 + tv)}{s} \le \frac{g(x_0 + 2tv) - g(x_0 + (t+s)v)}{t-s} + \frac{\alpha}{4}\|v\|.$$

Passing to the limit as $s \downarrow 0$, we obtain

$$g'(x_0 + tv, v) \le \frac{g(x_0 + 2tv) - g(x_0 + tv)}{t} + \frac{\alpha}{4}\|v\|.$$

Hence

$$\limsup_{t\downarrow 0} g'(x_0 + tv, v) \le \limsup_{t\downarrow 0}\Big[2\frac{g(x_0 + 2tv) - g(x_0)}{2t} - -\frac{g(x_0 + tv) - g(x_0)}{t}\Big] + \frac{\alpha}{4}\|v\| = g'(x_0, v) + \frac{\alpha}{4}\|v\|.$$

Therefore, there exists $\eta \in (0, \delta/2)$ such that

$$g'(x_0 + tv, v) \le g'(x_0, v) + \frac{\alpha}{2}\|v\| \quad \text{for all } t \in (0, \eta). \tag{3.8}$$

Finally, use again the approximate convexity of f. For any $r > 0$, $s > 0$ such that $0 < s < t < t + r < \eta$, using the decomposition

$$x_0 + tv = \frac{t-s}{t+r-s}(x_0 + (t+r)v) + \frac{r}{t+r-s}(x_0 + sv)$$

and (3.6), we derive

$$\frac{g(x_0 + tv) - g(x_0 + sv)}{t-s} \le \frac{g(x_0 + (t+r)v) - g(x_0 + tv)}{r} + \frac{\alpha}{4}\|v\|.$$

Letting $r \downarrow 0$, we obtain

$$\frac{g(x_0 + tv) - g(x_0 + sv)}{t-s} \le g'(x_0 + tv, v) + \frac{\alpha}{4}\|v\|.$$

Combining this inequality with (3.7), (3.8) yield the result. ∎

The following theorem is an extension of Thibault & Zagrodny (Theorem 2.1 in [18]) to the case of approximate convex functions.

Theorem 3.12 *Let $U \subseteq X$ be a nonempty convex open subset of X. Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous functions and let $\epsilon \geq 0$. Assume that $U \cap \operatorname{dom} f \neq \emptyset, \operatorname{dom} g$ is convex and g is approximate convex on U and that the following condition is satisfied:*

$$(\ast) \qquad \partial f(x) \subseteq \partial g(x) + \epsilon B^* \quad \forall x \in X.$$

Then $U \cap \operatorname{dom} f = U \cap \operatorname{dom} g$ and for every $x \in U$; $y \in U \cap \operatorname{dom} g$, one has

$$g(x) - g(y) - \epsilon\|x - y\| \leq f(x) - f(y) \leq g(x) - g(y) + \epsilon\|x - y\|. \tag{3.9}$$

Proof. The proof we present here follows Thibault & Zagrodny [18]. We mention that the domain of ∂f is graphically dense in $\operatorname{dom} f$, hence $U \cap \operatorname{dom} \partial f \neq \emptyset$ because $U \cap \operatorname{dom} f \neq \emptyset$. First we prove the second inequality of (3.9) for $x \in U$ and $y_0 \in U \cap \operatorname{dom} \partial f$. Indeed, it holds trivially if $x = y_0$. According to the assumption $(\ast)$ the set $\partial g(y_0)$ is non-empty and consequently $y_0 \in U \cap \operatorname{dom} g$. Take now $x \neq y_0$ and set $v := x - y_0$. Obviously, $g^{'}(y_0, v) > -\infty$.

Claim 1. For $\alpha > 0$, there is $\eta > 0$ such that for every $t \in (0, \eta)$, the following inequality holds true:

$$f(y_0 + tv) - f(y_0) \leq g(y_0 + tv) - g(y_0) + (\epsilon + 5\alpha/4)t\|v\|. \tag{3.10}$$

Indeed, given $\alpha > 0$ and $\eta > 0$ be as in Lemma 3.11. Let $t \in (0, \eta)$. For every $n = 1, 2, ...,$ we define the function f_n by

$$f_n(x) := \begin{cases} f(x) & \text{if } x \neq y_0 + tv \\ f(y_0 + tv) & \text{if } x = y_0 + tv \text{ and } f(y_0 + tv) \text{ is finite} \\ n & \text{if } x = y_0 + tv \text{ and } f(y_0 + tv) = +\infty. \end{cases}$$

Note that at any $x \neq y_0 + tv$, $\partial f_n(x)$ and $\partial f(x)$ coincide. By virtue of the mean value theorem, for every n, we can find sequences $\{x_n^m\}_{m \in \mathbb{N}}$ converging to $c_n := y_0 + s_n v \in [y_0, x_0 + tv)$, and $x_n^{*m} \in \partial f(x_n^m)$ such that

$$\liminf_{m \to \infty} \langle x_n^{*m}, y_0 + tv - x_n^m \rangle \geq \frac{f_n(y_0 + tv) - f(y_0)}{t\|v\|}(t - s_n)\|v\|.$$

Since $x_n^{*m} \in \partial f(x_n^m) \subseteq \partial g(x_n^m) + \epsilon B^*$ and as $x_n^m \in B(y_0, \delta_0)$ for m sufficiently large, it follows from the approximate convexity of g that

$$\langle x_n^{*m}, y_0 + tv - x_n^m \rangle \leq g(y_0 + tv) - g(x_n^m) + (\epsilon + \alpha/4)\|y_0 + tv - x_n^m\|.$$

The above inequalities yield:

$$\frac{f_n(y_0 + tv) - f(y_0)}{t}(t - s_n) \leq g(y_0 + tv) - g(x_n^m) + (\epsilon + \alpha/4)\|y_0 + tv - x_n^m\|.$$

Passing to the limit as $m \to \infty$, we obtain

$$\frac{f_n(y_0 + tv) - f_n(y_0)}{t} \leq \frac{g(y_0 + tv) - g(y_0 + s_n v)}{t - s_n} + (\epsilon + \alpha/4)\|v\|.$$

This and Lemma 3.11 imply

$$f_n(y_0 + tv) - f_n(y_0) \leq g(y_0 + tv) - g(y_0) + (\epsilon + 5\alpha/4)t\|v\|.$$

Taking now the limit as $n \to \infty$ in this inequality we obtain (3.10) which completes the claim 1.

Claim 2. *The following inequality is true:*

$$f(x) - f(y_0) \leq g(x) - g(y_0) + \epsilon\|x - y_0\|. \tag{3.11}$$

If $g(x) = +\infty$, then the inequality above is obvious. Assume that $g(x) < +\infty$. Define $C := \{t \in [0,1] | \text{ (3.10) holds } \}$. Obviously, C is bounded and by Claim 1, it is nonempty. Since f is a lower semicontinuous function, further, since $g(y_0)$ and $g(x)$ are finite, according to Corollary 3.3, the restriction of g to $[y_0, x]$ is continuous and therefore C is also closed. Hence, it is a compact subset of $\mathbb{R}$ and consequently $\max C$ exists. Assume that $t_0 = \max C$. Actually, $t_0 = 1$. Indeed, if $t_0 < 1$, one has $g(y_0 + t_0 v) < +\infty$ because $g(x) < +\infty$. By Proposition 3.2, the restriction of g to $[y_0, x]$ is locally Lipchitzian at $y_0 + t_0 v$. Hence, $g^{'}(y_0 + t_0 v, v)$ is finite. Therefore, similarly to the proof of (3.10), by replacing y_0 by $y_0 + t_0 v$, we find a number $t_1 \in (t_0, 1)$ such that

$$f(y_0 + t_1 v) - f(y_0 + t_0 v) \leq g(y_0 + t_1 v) - g(y_0 + t_0 v) + (\epsilon + 5\alpha/4)(t_1 - t_0)\|v\|.$$

This inequality together with (3.10) imply that $t_1 \in C$, a contradiction. Thus $t_0 = 1$ and for this t_0 and for all $\alpha > 0$ we have

$$f(x) - f(y_0) \leq g(x) - g(y_0) + (\epsilon + 5\alpha/4)\|x - y_0\|.$$

Therefore (3.11) holds true.

Next we prove the second inequality of (3.9) for $x \in U$ and $y \in U \cap \operatorname{dom} f$. Since $\operatorname{dom} \partial f$ is graphically dense in $\operatorname{dom} f$, there exists a sequence $\{y_n\}_{n \in \mathbb{N}} \subset U \cap \operatorname{dom} \partial f$ such that $y_n \xrightarrow{f} y$. By (3.11) one has

$$f(x) - f(y_n) \leq g(x) - g(y_n) + \epsilon\|x - y_n\| \quad \forall x \in U.$$

This and the lower semicontinuouity of g imply

$$f(x) - f(y) \leq g(x) - g(y) + \epsilon\|x - y\| \quad \forall x \in U \tag{3.12}$$

as requested. The latter inequality also shows that $U \cap \operatorname{dom} f = U \cap \operatorname{dom} g$. Consequently, the second inequality of (3.9) is true for every $x \in U$ and $y \in U \cap \operatorname{dom} g$. Finally, interchanging the roles of x and y in inequality (3.12), we have

$$g(x) - g(y) \leq f(x) - f(y) + \epsilon\|x - y\|$$

for every $x \in \operatorname{dom} f$. Obviously, the above inequality holds trivially if $f(x) = +\infty$. Therefore, the first inequality in (3.9) holds for every $x \in U$ and $y \in U \cap \operatorname{dom} f$. The proof is complete. ∎

Corollary 3.13 *Let X be a Banach space and let f, g be lower semicontinuous functions from X to $\mathbb{R} \cup \{+\infty\}$. Assume that $\operatorname{dom} f \neq \emptyset, \operatorname{dom} g$ is convex and g is approximate convex on X and that $\partial f(x) \subseteq \partial g(x)$ for all $x \in X$. Then $f(x) - g(x)$ is a constant.*

Proof. Apply Theorem 3.12 for $U = X$ and $\epsilon = 0$. ∎

Note that inequality (3.9) is no longer true if $\operatorname{dom} g$ is not convex. This fact is due to the local nature of approximate convexity. Take for instance the functions f and g from $\mathbb{R}$ to $\mathbb{R} \cup +\{\infty\}$ and given by $f(1) = f(0) = g(0) = 0; g(1) = 1$ and $f(x) = g(x) = +\infty$ for all $x \neq 0; x \neq 1$. One has $\partial f(x) = \partial g(x)$ for all $x \in \mathbb{R}$. However, $f - g$ is not a constant.

4. Application

In this section we are going to apply the previously obtained properties of approximate convex functions to give an answer to the following question of [4]: Does there exist a class (L) of functions verifying properties b)-d) mentioned in the introduction such that

(a') It is stable under finite sums, finite infima and finite suprema;

e) It contains continuous convex functions and continuously differentiable functions.

Obviously, the answer to the above question is negative because the infimum of two continuously differentiable functions does not necessarily verify properties b) and c). For instance, the functions $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \min\{-x, x^2\}$ does not satisfy property (b) at $x = 0$; while f_1 and $f_2 : \mathbb{R} \to \mathbb{R}$ defined by $f_1(x) = \min\{x, -x\} = -|x|$ and $f_2(x) = |x|$ do not verify (c) at $x = 0$. This observation leads us to look for a class of functions that is stable under finite sums and finite suprema, and verifies properties b)-e). Let us denote by **LAC** the family of Lipschitz approximate convex functions on a Banach space X. According to Proposition 3.1, Corollary 3.7, 3.9 and 3.13, **LAC** is stable under finite sums and finite suprema and verifies properties b)-e). Now, in order to obtain a class of functions larger than Lipschitz functions we can proceed as follows. Let f be any proper lower semicontinuous, approximate convex function such that $\cup_{\lambda>0}\lambda(\operatorname{dom} f - \operatorname{dom} f)$ is closed subspace, for instance f is a convex function such that dom f is finite dimensional. Define $\mathbf{LAC_f}$ as the family consisting of f, of all functions from **LAC** together with their finite sums and finite suprema. Then $\mathbf{LAC_f}$ is a class of not necessarily Lipschitz functions that verifies properties a)-e). Note that by taking f and g such that $\partial(f+g) \neq \partial f + \partial g$, we obtain $\mathbf{LAC_f} \neq \mathbf{LAC_g}$. In other words, maximal classes of lower semicontinuous functions verifying properties a)-e) do exist (by Zorn's lemma) but the largest one does not exist.

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