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THE PARATINGENT SPACE AND A CHARACTERIZATION OF C^1-MAPS DEFINED ON ARBITRARY SETS

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Abstract. We use the paratingent space (introduced by Glaeser, J. Analyse Math, 6 (1958), and strictly related to Bouligand's paratingent cone) to obtain a topological-geometrical characterization of the maps defined on arbitrary subsets of $\mathbf{R}^m$ that admit C^1-extensions. This result constitutes an improvement of Whitney's Extension Theorem and has many applications. Among them a new proof of the Inverse Function Theorem and a generalized version of it.

1. Introduction

The paratingent space $\Delta_p(A)$ of $A \subset \mathbb{R}^m$, essentially introduced by Glaeser [6], is the subspace of $\mathbb{R}^m$ defined by the property that the map $p \mapsto \Delta_p(A)$ is "the smallest" set-valued map with a closed graph whose values are linear subspaces of $\mathbb{R}^m$ and contain Bouligand's paratingent cone (Section 2). When considering C^1-submanifolds of $\mathbb{R}^m$ the paratingent space reduces to the usual tangent space.

The aim of this work is to show that the paratingent space can be used to characterize the maps defined on arbitrary subsets of $\mathbb{R}^n$ that admit C^1-extensions.

The main result of this work is the following. *Let $f : X \subset \mathbb{R}^n \to \mathbb{R}^m$ be continuous at $p \in X$. The map f admits a C^1-extension around p if and only if $\Delta_{(p,f(p))}(G_f)$ is a graph* (where G_f is the graph of f in $\mathbb{R}^n \times \mathbb{R}^m$ and saying that $\Delta_{(p,f(p))}(G_f)$ is a graph means that its projection into $\mathbb{R}^n$ is injective) (Statement and applications of this result are in Section 3 while its proof is in Section 5).

There are some important aspects to point out. First, it is surprising that from the fact that $\Delta_{(x,f(x))}(G_f)$ is a graph at p it follows that f is C^1 on a neighborhood of p.

Second, to prove this result we use Whitney's Extension Theorem but our result constitutes an improvement of Whitney's Theorem itself. In fact, Whitney's Theorem gives conditions for a map (defined on a closed set X) to have a C^1-extension but it requires an *a priori* knowledge of the differential of the extension at every point of X. On the contrary we don't ask such an information but, in case $\Delta_{(p,f(p))}(G_f)$ is a graph, we construct a differential for the extension and then we construct a C^1-extension of f. That is, we use only the information carried by f (see Section 5). To clarify this point it is important to observe that the paratingent space can be obtained from Bouligand's paratingent cone by applying to it, a finite number of times, two simple topological-geometrical constructions (see Glaeser [6] and Section 4.1). The possibility of such a construction emphasizes the properties of the paratingent space and makes, at least theoretically, explicitly testable the

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hypotheses of our main result: the idea is that we first construct $\Delta_{(p,f(p))}(G_f)$ by the mentioned topological-geometrical means and then we check if it is a graph; in such a case we can assert that f admits a C^1-extension around p.

The third aspect to point out is that our main result has many applications.

Strictly related to our main result is the fact that *if* $\dim(\Delta_p(A))=k$ *then there exists a* C^1*-submanifold of* $\mathbb{R}^m$ *of dimension* k *that contains a neighborhood of* p *in* A *and, moreover, there are not* C^1*-submanifolds of* $\mathbb{R}^m$ *of lower dimension satisfying this request* (Section 3.1). This result was partly proved by Glaeser [6]. In view of this property $\dim(\Delta_p(A))$ can be called the *differential (or* C^1*) dimension* of A at p.

The differential dimension can be used to prove the following characterization of the C^1-submanifolds of $\mathbb{R}^m$, based by no means on the concept of diffeomorphism. *A set* $M \subset \mathbb{R}^m$ *is a* C^1*-submanifold if and only if it is locally compact and the paratingent and contingent cones are at each point of* M *linear spaces of the same dimension* (Section 3.2). This result was previously proved in Tierno [9] but the proof we give here is new and deeper. Using this characterization we prove that under simple conditions on $A \subset \mathbb{R}^m$ the set $\mathrm{bd}B(A,\delta) = \{x \in \mathbb{R}^m : \ d(x,A) = \delta\}$ (that is, the sphere centered on A) is a C^1-submanifold of $\mathbb{R}^m$.

Consider now a C^1-map $f : X \subset \mathbb{R}^n \to Y \subset \mathbb{R}^m$. The properties of the paratingent space allow us to define a differential $df_p^{X,Y} : \Delta_p(X) \to \Delta_{f(p)}(Y)$ (see Section 3.3). With this extended definition of the differential we prove that f *is a local diffeomorphism at* p *if and only if* $df_p^{X,Y}$ *is injective.* This result is a generalization of the Inverse Function Theorem and in fact we prove that the Inverse Function Theorem can be easily deduced from it.

To conclude we prove that the paratingent space $\Delta_p(X)$ can be characterized as the space where the differential at p of every C^1-map, defined on a neighborhood of p and null on X, is zero. Surprisingly we find that this fact is equivalent to Whitney's Extension Theorem (see Section 5). This fact shows also that $\Delta_p(X)$ is the biggest subspace of $\mathbb{R}^m$ where we can define the differential of a C^1-map $f : X \subset \mathbb{R}^n \to \mathbb{R}^m$.

About the structure of the paper. After the introduction and the definitions we state our main result (Section 3). We then deduce many consequences. In Section 4 we give an explicit way to construct the paratingent space and we prove many properties of Bouligand's cones and of the paratingent space. Finally, in Section 5, we prove our main result and we clarify its relationship with Whitney's Extension Theorem.

Contents

2. Contingent cone, paratingent cone and paratingent space

2.1. **Some terms and notations.** We use the notation $H : X \rightrightarrows Y$ to mean that H is a set-valued map from X into Y (that is, it associates to every element $x \in X$ a subset of Y).

Let X and Y be metric spaces. A set-valued map $H : X \rightrightarrows Y$ is said to have a closed graph if from

$$x_k \in X,\ y_k \in H(x_k),\ (x_k, y_k) \to (x, y),$$

it follows that $y \in H(x)$. This means that the graph of H, i.e. the set $\{(x, y) \in X \times Y : \ y \in H(x)\}$, is closed in $X \times Y$.

The set-valued map H is said to be *lower semicontinuous* at $x \in X$ if for any $y \in H(x)$ and any sequence of elements $x_k \in X$ converging to x, there exists a sequence of elements $y_k \in H(x_k)$ converging to y. The set-valued map H is said to be lower semicontinuous if it is lower semicontinuous at every point.

The set-valued map H is said to be *continuous* if it is both with a closed graph and lower semicontinuous.

A set-valued map $H : X \rightrightarrows Y$ of a set X into a vector space Y is said to be *linear subspace-valued* or more simply *subspace-valued* if $H(x)$ is a linear subspace of Y for every $x \in X$.

Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ and $f : X \to Y$. With G_f we denote the graph of f and we consider it as a subset of $\mathbb{R}^n \times \mathbb{R}^m$. The map f is said to be of class C^1_X or, of class C^1 on X, if for every $x \in X$ there exist an open neighborhood U of x in $\mathbb{R}^n$ and a C^1 map $g : U \to \mathbb{R}^m$ that coincides with f on $X \cap U$. The map $f : X \to Y$ is said to be a diffeomorphism or, more precisely, a C^1-diffeomorphism, if it is C^1_X, invertible and has a C^1_Y inverse; in this case X and Y are said to be C^1-diffeomorphic.

An n-dimensional C^1-submanifold of $\mathbb{R}^m$ is a subset M of $\mathbb{R}^m$ that is locally C^1-diffeomorphic to $\mathbb{R}^n$ which means that for every $p \in M$ there exists a neighborhood of p in M that is C^1-diffeomorphic to $\mathbb{R}^n$.

For a differentiable map f defined on an open set we denote by df_p its differential at p.

We say that a set $X \subset \mathbb{R}^n \times \mathbb{R}^m$ is a graph if the projection of X into $\mathbb{R}^n$ is injective that is, if for every $x \in \mathbb{R}^n$ there exists at most one $y \in \mathbb{R}^m$ such that (x, y) belongs to X.

We use the standard notation $B(x, \rho) = \{y \in \mathbb{R}^n \ : \ d(x, y) < \rho\}$ and we denote the scalar product of $\mathbb{R}^n$ by $\langle \cdot\, , \cdot \rangle$. Finally, if $V \subset \mathbb{R}^m$ we write $\langle V \rangle$ to denote the linear subspace of $\mathbb{R}^m$ spanned by V.

2.2. **Contingent cone, paratingent cone and paratingent space.** We begin by briefly recalling the definitions of the contingent cone and the paratingent cone. For literature and results regarding these cones see [1, 5, 7, 8, 9] and the references therein.

Definition Let $A \subset \mathbb{R}^m$ and $p \in \bar{A}$.
• The set

$$C_p(A) = \Big\{ v \in \mathbb{R}^m : \text{ there exist } x_k \in A,\ x_k \to p \text{ and } \alpha_k \in [0, +\infty) \text{ such that } \alpha_k(x_k - p) \to v \Big\}$$

is called the *contingent (cone)* of A at p.
• The set

$$S_p(A) = \Big\{ v \in \mathbb{R}^m : \text{ there exist } x_k, y_k \in A,\ x_k, y_k \to p \text{ and } \alpha_k \in \mathbb{R} \text{ such that } \alpha_k(x_k - y_k) \to v \Big\}$$

is called the *paratingent (cone)* of A at p.
• The subspace of $\mathbb{R}^m$ spanned by the paratingent cone of A at p is denoted by $T_p(A)$. That is, we set $T_p(A) = \langle S_p(A) \rangle$. (We will not be concerned with the subspace spanned by the contingent).

The following properties of $C_p(A)$ and $S_p(A)$ can easily be proved:

- $C_p(A) \subset S_p(A)$, $C_p(\bar{A}) = C_p(A)$, $S_p(\bar{A}) = S_p(A)$.
- If U is a neighborhood of p in $\mathbb{R}^m$ then $C_p(A) = C_p(A \cap U)$ and $S_p(A) = S_p(A \cap U)$.
- If $A \subset B$ then $C_p(A) \subset C_p(B)$ and $S_p(A) \subset S_p(B)$.
- If $v \in S_p(A)$ then $-v \in S_p(A)$, that is, $S_p(A)$ is symmetrical.

The sets $C_p(A)$, $S_p(A)$ and $T_p(A)$ give rise to the set-valued maps $p \mapsto C_p(A)$, $p \mapsto S_p(A)$ and $p \mapsto T_p(A)$. For these maps we shall always consider as a domain the set $\bar{A}$, that is, when we refer to $p \mapsto C_p(A)$, $p \mapsto S_p(A)$ and $p \mapsto T_p(A)$ we think of

$$\begin{array}{ccc} \bar{A} & \overset{\longrightarrow}{\longrightarrow} & \mathbb{R}^m \\ p & \longmapsto & C_p(A),\ S_p(A),\ T_p(A). \end{array}$$

We have the following interesting result.

Proposition 2.1. *Let $A \subset \mathbb{R}^m$. The set-valued map $p \mapsto S_p(A)$ has a closed graph.*

The proof is not difficult and can be found in Aubin-Frankowska [1], Shi [8], Tierno [9].

Remark Even though $T_p(A)$ is spanned by $S_p(A)$, the set-valued map $p \mapsto T_p(A)$ is not, in general, a map with a closed graph (unless $A \subset \mathbb{R}$; in such a case $T_p(A) = S_p(A)$ for every $p \in \bar{A}$). As an example consider the function $f : [0,1] \to \mathbb{R}$ given by $f(0) = 0$, $f(\frac{t}{2n} + \frac{1-t}{2n\pm 1}) = \frac{t}{n^2}$ for $t \in [0,1]$ and $n \in \mathbf{N}$ and set $A = G_f \subset \mathbb{R}^2$. First we want to prove that $S_{(0,0)}(A) = \mathbb{R} \times 0$. Assume $a_k = (x_k, f(x_k))$, $b_k = (y_k, f(y_k))$,

$x_k, y_k \to 0$ ($x_k \neq y_k$) and $(a_k - b_k)/\|a_k - b_k\| \to (x, y) \in S_{(0,0)}(A)$. Assume $x_k, y_k < \frac{1}{2n}$ then it is easily seen that

$$\frac{f(x_k) - f(y_k)}{\|x_k - y_k\|} < \frac{1/n^2}{1/n} = \frac{1}{n},$$

and this implies $y = 0$. Now, the subspace spanned by $S_{(1/n, f(1/n))}(A)$ is $\mathbb{R}^2$ for every $n \in \mathbf{N}$ and this shows that the subspace spanned by the paratingent space is not, in this case, a map with a closed graph.

We shall study in more detail the properties of the contingent cone and the paratingent cone in Section 4; we will now take such properties for known and we will refer to the proofs in Section 4 when using them.

We now turn our attention to the definition of the paratingent space. The paratingent space will constitute our main object of study and it will lead us to many interesting results.

We have seen that $p \mapsto S_p(A)$ is a map with a closed graph, but observe that in general its values are not linear subspaces. On the contrary $p \mapsto T_p(A)$ is a subspace-valued map but it may fail to have a closed graph.

We now define a new tangent space, the "paratingent space" $\Delta_p(A)$, that gives rise to a set-valued map $p \mapsto \Delta_p(A)$ that puts together the properties of the maps $p \mapsto S_p(A)$ and $p \mapsto T_p(A)$. Such a map is, in a sense, "the smallest" subspace-valued map with a closed graph that "contains" the map $p \mapsto S_p(A)$. The exact definition of the paratingent space is as follows

Definition Let $A \subset \mathbb{R}^m$ and $p \in \bar{A}$. Set

$$\Psi(A) = \Big\{ H : \bar{A} \rightrightarrows \mathbb{R}^m : H \text{ is subspace-valued, has a closed graph and } H(x) \supset S_x(A) \text{ for } x \in \bar{A} \Big\}.$$

The set

$$\Delta_p(A) = \bigcap_{H \in \Psi(A)} H(p)$$

is called the *paratingent space* of A at p.

The paratingent space was introduced by Glaeser in [6, Chapter 2, § 5] for closed subsets of $\mathbb{R}^m$ with the name of "paratingent linearisé".

Elementary properties of $\Delta_p(A)$:

- $\Delta_p(A) = \Delta_p(\bar{A})$, $\Delta_p(A) \supset T_p(A)$,
  $A \subset B \Rightarrow \Delta_p(A) \subset \Delta_p(B)$,
  If U is a neighborhood of p in $\mathbb{R}^m$ then $\Delta_p(A \cap U) = \Delta_p(A)$.
- $\Delta_p(A)$ is a subspace of $\mathbb{R}^m$ (as an intersection of subspaces of $\mathbb{R}^m$).
- the set-valued map $p \mapsto \Delta_p(A)$ is a map with a closed graph (its graph being the intersection of closed sets)

As a consequence of these properties we get that the map $p \mapsto \Delta_p(A)$ is in $\Psi(A)$ and since, obviously, $\Delta_p(A) \subset H(p)$ for every $H \in \Psi(A)$, we may say that "$p \mapsto \Delta_p(A)$ is the smallest map in $\Psi(A)$".

We shall see that the paratingent space enjoys very strong properties. Moreover, in Section 4.1 we shall give an alternative "more constructive" definition of $\Delta_p(A)$ that will make the properties of this space even more interesting. The most important property of the paratingent space constitutes our main result and it is the object of the next section.

3. Statement of the main result and applications

In this section we state our main result. We then relate it to other results and point out some important applications. The proof of our main result will be given in Section 5. Our main result shows a strong relationship between the paratingent space and the maps of class C^1 defined on arbitrary sets. The result says that a continuous map $f : X \to Y$ is of class C^1 on a neighborhood of $p \in X$ if and only if the paratingent space of the graph of f at the point $(p, f(p))$ is itself a graph, that is, if it does not contain "vertical" vectors.

Precisely stated the result says

Theorem 3.1. *Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ and $p \in X$. A map $f : X \to Y$ is C^1 on a neighborhood of p in X (i.e. there exists a C^1 map defined on an open subset of $\mathbb{R}^n$ containing p, that coincides with f on a neighborhood of p in X) if and only if f is continuous at p and $\Delta_{(p,f(p))}(G_f)$ is a graph.*

(The proof of Theorem 3.1 will be given in Section 5).

Obviously the interesting part of the theorem is the "if-part".

Before showing some consequences and applications of this result we want to point out three remarkable aspects.

1) It is interesting to observe that we only require that $\Delta_{(x,f(x))}(G_f)$ nicely behaves (that is, it is a graph) at a single point p to obtain that the map f nicely behaves (that is, it is C^1) on a whole neighborhood of p.

2) Our theorem is strictly related to Whitney's C^1-Extension Theorem under various points of view (the statement of Whitney's Theorem is in Section 5). In fact we shall use Whitney's Theorem to prove our result but we shall also see that our result sharpens and completes Whitney's result. All this will be done in Section 5, however a first aspect can be immediately pointed out. Indeed Theorem 3.1 allows us to decide if a map admits a C^1-extension by checking only the properties of its graph (i.e. by checking only the properties of its values) and not relying upon the existence of other functions as Whitney's Extension Theorem does. In this sense Theorem 3.1 may be considered an improvement of Whitney's Theorem.

The conditions that ensure the existence of a "global" C^1-extension are the content of Corollary 3.2.

3) We shall see in Section 4.1 that the paratingent space admits a constructive definition. Therefore every testable property of the paratingent space of a set X can be considered as an explicitly testable property of X. In particular Theorem 3.1

gives an explicitly testable condition to know if a map defined on an arbitrary set admits a C^1-extension.

3.1. **Application 1: Characterizations of C^1-maps.** We begin by pointing out some results strictly related to Theorem 3.1.

First, the "global" version of Theorem 3.1.

Corollary 3.2. *Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ and let $f : X \to Y$ be continuous. If for every $x \in X$, $\Delta_{(x,f(x))}(G_f)$ is a graph, then there exist an open set U and a C^1-map $g : U \to \mathbb{R}^m$ such that $U \supset X$ and $g = f$ on X (that is, f is C^1_X).*

Moreover if X is closed, U can be chosen to be $\mathbb{R}^n$.

Proof. The proof can be easily obtained applying the partitions of unity; in order not to be too cumbersome we leave the details to the interested reader. □

We can substitute $\Delta_{(x,f(x))}(G_f)$ with $T_{(x,f(x))}(G_f)$ or with $S_{(x,f(x))}(G_f)$ in the statement of Corollary 3.2 provided that X satisfies certain conditions, as shown in the next result.

Theorem 3.3. *Let $X \subset \mathbb{R}^n$ be locally compact and let $f : X \to \mathbb{R}^m$ be continuous.*

1. *Assume $T_x(X) = S_x(X)$ for every $x \in X$, then f is C^1_X if and only if $T_{(x,f(x))}(G_f)$ is a graph for every $x \in X$.*
2. *Assume $n = 1$, that is $X \subset \mathbb{R}$, then f is C^1_X if and only if $S_{(x,f(x))}(G_f)$ is a graph for every $x \in X$, equivalently, if and only if*
$$\lim_{x,y \in X,\, x,y \to p} \frac{f(x) - f(y)}{x - y}$$
exists for every $p \in X$.
3. *Assume that X is contained in a 1-dimensional C^1-submanifold of $\mathbb{R}^n$, or that X is open, or that X is a C^1-submanifold of $\mathbb{R}^n$. In all these cases, f is C^1_X if and only if $S_{(x,f(x))}(G_f)$ is a graph for every $x \in X$, equivalently, if and only if*
$$\lim_{x,y \in X,\, x,y \to p,\, \frac{x-y}{\|x-y\|} \to v} \frac{f(x) - f(y)}{\|x - y\|}$$
exists for every $v \in S_p(X)$ with $\|v\| = 1$ and for every $p \in X$.

Proof. We begin our proof with the following remark.

Remark If $f : X \to \mathbb{R}^m$ is continuous it is not difficult to see that the fact that $S_{(p,f(p))}(G_f)$ is a graph is equivalent to say that
$$\lim_{x,y \in X,\, x,y \to p,\, \frac{x-y}{\|x-y\|} \to v} \frac{f(x) - f(y)}{\|x - y\|}$$
exists for every $v \in S_p(X)$ with $\|v\| = 1$.

1) Since $S_{(x,f(x))}(G_f) \subset T_{(x,f(x))}(G_f)$, $S_{(x,f(x))}(G_f)$ is a graph for every $x \in X$. If $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ denotes the projection onto the first factor we have that $\pi(S_{(x,f(x))}(G_f)) = S_x(X)$ and $\pi(T_{(x,f(x))}(G_f)) = T_x(X)$ (this property of tangent

cones is in Proposition 4.12). Therefore, considering that $T_x(X) = S_x(X)$, we obtain that $T_{(x,f(x))}(G_f) = S_{(x,f(x))}(G_f)$ for every $x \in X$. Using the fact that G_f is locally compact (it is homeomorphic to X) we deduce that $S_{(x,f(x))}(G_f) = T_{(x,f(x))}(G_f) = \Delta_{(x,f(x))}(G_f)$ for every $x \in X$. This implies that $\Delta_{(x,f(x))}(G_f)$ is a graph for every $x \in X$ and therefore, by Theorem 3.1, f is C^1_X.

2) First observe that $X \subset \mathbb{R}$ implies that $S_x(X) = T_x(X)$ for every $x \in X$. By considering the components of f we may assume $m = 1$, that is, that f takes its values on $\mathbb{R}$. Therefore $S_{(x,f(x))}(G_f) \subset \mathbb{R} \times \mathbb{R}$. Since $S_{(x,f(x))}(G_f)$ is symmetric (with respect to the origin) and since it is a graph, we have that either $S_{(x,f(x))}(G_f) = \{0\}$ or $S_{(x,f(x))}(G_f)$ is a 1-dimensional subspace of $\mathbb{R} \times \mathbb{R}$. In both cases we obtain $S_{(x,f(x))}(G_f) = T_{(x,f(x))}(G_f)$ and this, in view of point 1), concludes the proof.

3) If X is contained in a 1-dimensional C^1-submanifold of $\mathbb{R}^n$ the conclusion derives from point 2) and the proof is omitted.

To prove the result when X is open we need the following result due to Mirică [7, Prop. 3.11]: *Let $U \subset \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}^m$ be locally Lipschitz at $x \in U$. Then, if $(a + b, c) \in S_{(x,f(x))}(G_f)$, there exist (a, d), $(b, e) \in S_{(x,f(x))}(G_f)$ such that $d + e = c$.*

(For the definition of locally Lipschitz see Section 4) This property is particularly interesting if $S_{(x,f(x))}(G_f)$ is a graph. In such a case it follows that $S_{(x,f(x))}(G_f)$ is the graph of a linear map. We know (Proposition 4.2) that the fact that $S_{(x,f(x))}(G_f)$ is a graph implies that f is locally Lipschitz, therefore using the property above we conclude our proof.

The proof when X is a C^1-submanifold of $\mathbb{R}^n$ is left to the reader.

The statements concerning the existence of limits follow from our previous remark. □

Remark 1) Part 2 of the previous theorem was previously proved (for closed subsets of $\mathbb{R}$, which is equivalent) by Whitney [11].

2) The hypothesis of local compactness of X in Theorem 3.3 cannot be removed. Consider, for instance, the function $f : [0, 1] \to \mathbb{R}$ defined in the Remark after Prop. 2.2. Now restrict this function to the set $X = [0, 1] \setminus \{1/n : n \in \mathbf{N}\}$. All the hypotheses of Theorem 3.3 are satisfied except for the local compactness of X, but this new function does not have any C^1-extension in a neighborhood of 0 since all its extensions must coincide with f in a neighborhood of 0.

3) Part 3 of Theorem 3.3 can also be proved directly, that is, without the use of Theorem 3.1 (compare Tierno [9, Prop. 4]).

4) It is interesting to observe that under the hypotheses of Theorem 3.3-2,3 checking if f has a C^1-extension reduces (at least theoretically) to check the existence of a limit (even if, in general, a difficult one).

3.2. **Application 2: The differential dimension and a characterization of C^1-manifolds.** In this Section we introduce, by means of the paratingent space, the notion of differential dimension of a subset of $\mathbb{R}^m$, we then use it to give a geometrical characterization of the C^1-submanifolds of $\mathbb{R}^m$.

The next propositions motivate the definition.

Theorem 3.4. *Let $X \subset \mathbb{R}^m$ and $p \in X$. Let F be a subspace of $\mathbb{R}^m$ such that $\Delta_p(X) \cap F = \{0\}$. Then, in a neighborhood of p the set X may be regarded as the graph of a C^1-map from a subset of $F^\perp$ into F.*

Proof. Let $\pi : \mathbb{R}^m \to F^\perp$ and $\mu : \mathbb{R}^m \to F$ be the orthogonal projections onto $F^\perp$ and onto F. Let K be a compact neighborhood of p in $\bar{X}$ such that π is injective on K (the existence of such a neighborhood is a property shown in Proposition 4.3). Set $g = \pi_{|K} : K \to \pi(K)$ and $h = \mu \circ g^{-1}$. Since g is a homeomorphism, h is continuous. The set K may be regarded as the graph of h. Since we have

$$\Delta_{(\pi(p),h(\pi(p)))}(G_h) \cap F = \Delta_p(K) \cap F = \{0\},$$

we can invoke Theorem 3.1 to conclude that h is C^1 on a neighborhood of $\pi(p)$. □

Corollary 3.5. *Let $X \subset \mathbb{R}^m$ and $p \in X$. If $dim(\Delta_p(X)) = n$, then there exist C^1-submanifolds of $\mathbb{R}^m$ of dimension n that contain a neighborhood of p in X, and there are not C^1-submanifolds of $\mathbb{R}^m$ of dimension lower then n that contain a neighborhood of p in X .*

Proof. By Theorem 3.4, there is a neighborhood of p in X which may be regarded as the graph of a C^1-map f defined on a subset of $\Delta_p(X)$ and with values in $\Delta_p(X)^\perp$. Consider now a C^1-extension $\bar{f} : U \to \Delta_p(X)^\perp$ of f defined on an open subset U of $\Delta_p(X)$. The graph of $\bar{f}$ is an n-dimensional C^1-submanifold containing a neighborhood of p in X.

If V is a C^1-submanifold of $\mathbb{R}^m$ containing a neighborhood of p in X then $\Delta_p(X) \subset \Delta_p(V)$ and therefore $\dim(\Delta_p(X)) \leq \dim(\Delta_p(V))$. □

Loosely speaking Theorem 3.4 says that if $\dim(\Delta_p(X)) = n$, then there exists a neighborhood of p in X that can be smoothly flattened into $\mathbb{R}^n$, i.e. one can find a diffeomorphism of a neighborhood of p in X into $\mathbb{R}^k$ if and only if $k \geq \dim(\Delta_p(X))$.

This result was partly proved by Glaeser [6, Chapter 2, Theorem 1].

Definition Let $X \subset \mathbb{R}^m$ and $p \in X$. The integer $\dim(\Delta_p(X))$ is called the *differential dimension* (or *C^1-dimension*) of X at p and it is denoted by $\dim_{C^1}(X, p)$.

The integer $\max_{p\in X} \dim_{C^1}(X, p)$ is called *differential dimension* (or *C^1-dimension*) of X and it is denoted by $\dim_{C^1}(X)$.

Elementary properties of $\dim_{C^1}(X, p)$:

- $\dim_{C^1}(X, p) = \dim_{C^1}(\bar{X}, p)$.
- The map $f(p) = \dim_{C^1}(X, p) : \bar{X} \to \mathbf{N}$ is upper semicontinuous (therefore, since it takes its values in $\mathbf{N}$, every point is a point of local maximum).
- If $f : X \to Y$ is a diffeomorphism then $\dim_{C^1}(X, p) = \dim_{C^1}(Y, f(p))$ for every $p \in X$.

Problem If $\dim_{C^1}(X) = n$, is it possible to find an n-dimensional C^1-submanifold of $\mathbb{R}^m$ containing X?

The answer seems to be obviously positive but we couldn't find the way to construct the required manifold.

We now use the differential dimension to prove that a set $X \subset \mathbb{R}^m$ is a C^1-submanifold of dimension n if and only if it is locally compact and $C_p(X) = S_p(X) \simeq \mathbb{R}^n$ for every $p \in X$. This result was previously proved in Tierno [9] using weaker tools than the one we use here (in particular in [9] there's no use of the paratingent space), however we give this new proof since it is an easy consequence of our main result.

We need the following characterization of open sets in terms of contingent cones.

Proposition 3.6. *1) A set $A \subset \mathbb{R}^n$ is open if and only if it is locally compact and $C_x(A) = \mathbb{R}^n$ for every $x \in A$.*

2) More generally, let M be a C^1-submanifold of $\mathbb{R}^m$. A subset A of M is open in M if and only if it is locally compact and $C_x(A) = C_x(M)$ for every $x \in A$.

3) Let $A, B \subset \mathbb{R}^n$ and let $f : A \to B$ be a diffeomorphism. If A is open then B is open.

Proof. 1) The proof can be found in Tierno [9, Prop. 7].

2) This is an easy generalization of part 1).

3) Since A is open we have that it is locally compact and $C_p(A) = \mathbb{R}^n$ for every $p \in A$. Therefore B is locally compact and $C_y(B) = \mathbb{R}^n$ for every $y \in B$. This, by part 1), implies that B is open. □

Remark Proposition 3.6-3 can also be proved using the Inverse Function Theorem but it is important for us not to use it because we want to obtain it as a consequence of our results (Section 3.3).

We have the following characterization of the C^1-submanifolds of $\mathbb{R}^m$.

Theorem 3.7. *A set $X \subset \mathbb{R}^m$ is a C^1-submanifold of $\mathbb{R}^m$ of dimension n if and only if it is locally compact and $C_p(X) = S_p(X) \simeq \mathbb{R}^n$ for every $p \in X$.*

Proof. We only have to prove the "if-part" (the contrary is easy and follows from the property of stability of cones shown in Proposition 4.5).

Since X is locally compact and $S_p(X) \simeq \mathbb{R}^n$ for every $p \in X$ we have that $\Delta_p(X) = S_p(X)$ for every $p \in X$. Now fix $p \in X$. Since $\dim(\Delta_p(X)) = n$, there is, by Corollary 3.5, a C^1-submanifold M of $\mathbb{R}^m$ of dimension n that contains an open neighborhood U of p in X. Since U is locally compact and $C_x(U) \simeq \mathbb{R}^n$ for every $x \in U$, we have, by Proposition 3.6, that U is open in M and therefore it is a C^1-submanifold of $\mathbb{R}^m$ of dimension n. It follows that each point of X has a neighborhood diffeomorphic to $\mathbb{R}^n$, that is, X is an n-dimensional C^1-submanifold of $\mathbb{R}^m$. □

As we pointed out in [9] what is interesting in Theorem 3.7 is that it gives a geometrical characterization of the C^1-submanifolds of $\mathbb{R}^m$ that does not rely upon the concept of diffeomorphism.

We also want to observe that the hypotheses of Theorem 3.7 are not redundant. In a sense $C_p(X) \simeq \mathbb{R}^n$ avoids the presence of "boundary lines" (where $C_p(X)$ is not isomorphic to $\mathbb{R}^n$): think, for instance, of the set $\{x \in \mathbb{R}^2 : \|x\| \leq 1\}$. The condition $S_p(X) \simeq \mathbb{R}^n$ avoids the presence of "bifurcation points" (where

$C_p(X) \simeq \mathbb{R}^n$ but $\dim(T_p(X)) > n$): think, for example, of the set $\{(x, y) \in \mathbb{R}^2 : y = \pm x^2\}$. The local compactness of X avoids the presence of infinitely many "holes" accumulating at a point of X (a "hole" is a point of $\mathrm{bd}X \cap \mathcal{C}X$), think, for instance, of the set $\{x \in \mathbb{R} : -1 < x < 1,\ x \neq 1/n,\ n \in \mathbf{N}\}$ or the set $\mathbf{Q} \subset \mathbb{R}$.

We now give an application of our characterization of manifolds.

Theorem 3.8. *Let $\delta > 0$, $A \subset \mathbb{R}^n$ and set $X = bdB(A, \delta) = \{x \in \mathbb{R}^n : d(x, A) = \delta\}$. Assume that for every $x \in X$ there exists a unique $y \in \bar{A}$ (denoted $\pi_A(x)$) such that $d(x, y) = \delta$. Then X is a C^1-submanifold of $\mathbb{R}^n$ of dimension $n - 1$.*

Proof. Since X is closed we only need to show that $C_x(X) = S_x(X) \simeq \mathbb{R}^{n-1}$ for every $x \in X$. Let (x_k) and (y_k) be two sequences of points of X converging to $x \in X$, such that $x_k \neq y_k$ and $(x_k - y_k)/\|x_k - y_k\| \to v \in S_x(X)$. Set $p_k = \pi_A(x_k)$, $q_k = \pi_A(y_k)$ and $p = \pi_A(x)$. From the uniqueness of the "projection" it easily follows that $p_k, q_k \to p$ (that is, π_A is continuous).

We have

$$p_k \in \mathrm{bd}B(x_k, \delta) \cap \mathcal{C}B(y_k, \delta), \qquad q_k \in \mathrm{bd}B(y_k, \delta) \cap \mathcal{C}B(x_k, \delta),$$

therefore

$$\langle p_k - \frac{x_k + y_k}{2}, \frac{x_k - y_k}{\|x_k - y_k\|} \rangle \geq 0, \qquad \langle q_k - \frac{x_k + y_k}{2}, \frac{x_k - y_k}{\|x_k - y_k\|} \rangle \leq 0,$$

and taking the limit (for $k \to \infty$)

$$\langle p - x, v \rangle \geq 0, \qquad \langle p - x, v \rangle \leq 0,$$

hence $\langle p - x, v \rangle = 0$ and therefore $S_x(X) \subset \langle p - x \rangle^{\perp}$.

We now prove that $C_x(X) \supset \langle p - x \rangle^{\perp}$. Let $v \in \langle p - x \rangle^{\perp}$ and consider points of the form $x_{t\lambda} = x + tv + \lambda(x - p)$ for $t > 0$ and $\lambda \in \mathbb{R}$. We have that for any sufficiently small t there exists λ so that $d(x_{t\lambda}, A) > \delta$ and there exists λ so that $d(x_{t\lambda}, A) < \delta$, hence there exists also λ_t so that $d(x_{t\lambda_t}, A) = \delta$, that is, $x_{t\lambda_t} \in X$. Let $t_n \to 0+$ and set $\lambda_n = \lambda_{t_n}$ and $x_n = x_{t_n \lambda_n}$. Now observe that λ_n / t_n must converge to 0 because otherwise, by extracting a subsequence, we could assume $t_n / \lambda_n \to a \in [0, +\infty)$ so that $\frac{1}{\lambda_n}(x_n - x) = \frac{t_n}{\lambda_n} v + (x - p) \to av + (x - p) \in C_x(X)$ and this would contradict what we showed above. Hence $\frac{1}{t_n}(x_n - x) = v + \frac{\lambda_n}{t_n}(x - p) \to v \in C_x(X)$.

This concludes the proof since $\dim(\langle p - x \rangle^{\perp}) = n - 1$. □

Remark Every convex set satisfies the hypotheses of Theorem 3.8 for any positive δ.

A similar and related result has been previously proved by Federer [4, Theorem 9.5]: *Let d denote the distance function from the set A, let $U(A)$ denote the set $\{x \in \mathbb{R}^n : x$ has a unique closest point in $\bar{A}\}$ and let $\pi : U(A) \to A$ denote the "projection" on A; then d is C^1 on the interior part of $U(A) \setminus A$.*

This result is related to ours since it implies that the set $d^{-1}(\delta) = \{x \in \mathbb{R}^n : d(x, A) = \delta\}$ is a C^1-submanifold of $\mathbb{R}^m$ whenever $\delta > 0$ is such that $d^{-1}(\delta)$ is contained in the interior part of $U(A) \setminus A$ (the result derives from the Implicit Function Theorem observing that δ must be a regular value of d).

We can observe that our hypotheses are somewhat weaker since we don't require $d^{-1}(\delta)$ to be contained in the set where we know that d is C^1 but only on the set where we have unique projection.

3.3. **Application 3: The differential for C^1-maps defined on arbitrary sets and the Inverse Function Theorem.** In this Section we use the properties of the paratingent space to define the differential for C^1-maps defined on arbitrary sets and to show that the definition we give is the best possible. We then use this generalized definition of differential to prove a generalized version of the Inverse Function Theorem. Moreover, we don't use the Inverse Function Theorem to reach this generalization so that we are able to obtain it as a true corollary.

Definition Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ and let $f : X \to Y$ be a C^1_X-map. We define the differential, $df_p^{X,Y} : \Delta_p(X) \to \Delta_{f(p)}(Y)$, of f at $p \in X$, as the restriction to $\Delta_p(A)$ of the differential of a C^1-map defined on an open neighborhood U of p and coinciding with f on $U \cap X$.

Remark 1) By the property shown in Proposition 4.5 the above definition does not depend on the choice of the map extending f. Moreover, again by Proposition 4.5, the image of $df_p^{X,Y}$ is actually contained in $\Delta_{f(p)}(Y)$ so that the definition makes sense.

2) Observe also that this definition of differential extends the usual one.

3) We shall see that this differential admits also a constructive definition (property 3 below).

4) We shall show in a while that the differential of a C^1_X-map $f : X \to Y$ can not be defined on a bigger space than $\Delta_p(X)$.

Some properties of the generalized differential:

1) The usual (functorial) properties of the differential can easily be proved:

- If I is the identity map of $X \subset \mathbb{R}^n$ then $dI_x^{X,X}$ is the identity map of $\Delta_x(X)$,
- $d(g \circ f)_p^{X,Z} = dg_{f(p)}^{Y,Z} \circ df_p^{X,Y}$ (where f is a C^1_X-map, g is a C^1_Y-map and $f(X) \subset Y$).

2) Using the properties above and Proposition 4.5 we obtain that if $f : X \to Y$ is a diffeomorphism, then the differential $df_x^{X,Y} : \Delta_x(X) \to \Delta_{f(x)}(Y)$ is an isomorphism such that $df_x^{X,Y}(C_x(X)) = C_{f(x)}(Y)$ and $df_x^{X,Y}(S_x(X)) = S_{f(x)}(Y)$.

If, in particular, M is an n-dimensional C^1-submanifold of $\mathbb{R}^m$ we have

$$\Delta_x(M) = S_x(M) = C_x(M) \simeq \mathbb{R}^n$$

for every $x \in M$.

3) Consider a C^1_X-map $f : X \to Y$. In this case $x \mapsto (x, f(x))$ is a diffeomorphism between X and G_f and therefore $\Delta_{(p,f(p))}(G_f)$ is the graph of $df_p^{X,Y}$, $S_{(p,f(p))}(G_f)$ is the graph of $df_p^{X,Y}$ restricted to $S_p(X)$ and $C_{(p,f(p))}(G_f)$ is the graph of $df_p^{X,Y}$ restricted to $C_p(X)$. Since the paratingent space can be constructed by topological-geometrical means (Section 4.1) so does $\Delta_{(p,f(p))}(G_f)$ and therefore this shows that our generalized differential admits a constructive definition.

We now give a new characterization of the paratingent space. This characterization will also show that we cannot define the differential of a given map on a bigger space than the paratingent space. To this end we need the following definition.

Definition Let $A \subset \mathbb{R}^m$ and $p \in A$. Set

$$Z(A,p) = \Big\{ f : U \to \mathbb{R} : \text{ U is a nbd of } p \text{ in } \mathbb{R}^m,\ f \text{ is } C^1,\ f(A \cap U) = \{0\} \Big\}.$$

The set

$$T^1_p(A) = \bigcap_{f \in Z(A,p)} \operatorname{Ker}(df_p)$$

is called the C^1-*tangent space* of A at p.

Remark $T^1_p(A)$ is a subspace of $\mathbb{R}^m$.

Remark If $f \in Z(A,p)$, then the set-valued map $H(x) = \operatorname{Ker}(df_x)$ has a closed graph, is subspace-valued and $H(x) \supset S_x(A)$ for every x on a neighborhood of p in $\bar{A}$, therefore, by definition of $\Delta_p(A)$, we have that $\Delta_p(A) \subset T^1_p(A)$.

(This fact also follows from Proposition 4.5. Indeed if $f(A \cap U) = \{0\}$, then $\Delta_p(A) \subset \operatorname{Ker}(df_p)$.)

Proposition 3.9. *Let* $f : U \to \mathbb{R}^m$ *be a* C^1*-map defined on an open set* U *of* $\mathbb{R}^n$*. If* $X \subset U$ *and* $p \in X$ *then*

$$df_p(T^1_p(X)) \subset T^1_{f(p)}(f(X)).$$

Proof. Let $v \in T^1_p(X)$. We have to prove that if $g : V \to \mathbb{R}$ is a C^1-map defined on a neighborhood V of $f(p)$ in $\mathbb{R}^m$ such that $g(V \cap f(X)) = \{0\}$ then $dg_{f(p)}(df_p(v)) = 0$. But $dg_{f(p)}(df_p(v)) = d(g \circ f)_p(v) = 0$, since $g \circ f$ is C^1 on a neighborhood of p and zero on X. □

Remark If M is an n-dimensional C^1-submanifold of $\mathbb{R}^m$, then $T^1_p(M) \simeq \mathbb{R}^n$ for every $p \in M$.

We are now able, using the differential dimension, to obtain the announced characterization of the paratingent space. We prove that $T^1_p(A)$ and $\Delta_p(A)$ always coincide.

Theorem 3.10. *Let* $A \subset \mathbb{R}^m$ *and* $p \in \bar{A}$*, then*

$$T^1_p(A) = \Delta_p(A).$$

Proof. Set $n = \dim(\Delta_p(A))$ and let M be an n-dimensional C^1-submanifold of $\mathbb{R}^m$ containing a neighborhood of p in A (see Corollary 3.5). We have

$$\Delta_p(A) \ = \ \Delta_p(M) \ = \ T^1_p(M) \ \supset \ T^1_p(A).$$

The reversed inclusion is the content of a prevoius Remark. □

Theorem 3.10 tells us that a C^1-map that is zero on A is not obliged, by this fact, to have zero differential on a bigger space than $\Delta_p(A)$.

The discussion above shows also that it is not possible to define the differential of a C^1_X-map $f : X \to Y$ on a bigger space than $\Delta_p(X)$. Indeed $\Delta_p(X) = T^1_p(X)$ and $T^1_p(X)$ is just the space where all the differentials of the C^1-extensions of f coincide.

The definition of differential we have given, in connection with our main result, leads to a generalized Inverse Function Theorem. The next result generalizes the Inverse Function Theorem to C^1-maps defined on arbitrary subsets of $\mathbb{R}^n$.

Theorem 3.11 (Generalized Inverse Function Theorem)**.** *Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$, $p \in X$ and let $f : X \to Y$ be a C^1_X-map. If $df^{X,Y}_p : \Delta_p(X) \to \Delta_{f(p)}(Y)$ is injective, there exists a neighborhood U of p in X such that $f_{|U} : U \to f(U)$ is a diffeomorphism.*

Proof. Set $A = \mathbb{R}^n \times 0$ and $B = 0 \times \mathbb{R}^m$. Since $df^{X,Y}_p$ is injective we have that $G_{df^{X,Y}_p} \cap A = \{0\}$ and since $G_{df^{X,Y}_p} = \Delta_{(p,f(p))}(G_f)$, we have $\Delta_{(p,f(p))}(G_f) \cap A = \{0\}$. Therefore, by Theorem 3.4, there is a neighborhood of $(p, f(p))$ in G_f that may be regarded as the graph of a map $g : V \to A$ defined on a subset V of B and of class C^1_V. The map g is a C^1_V-inverse of f and $U = f^{-1}(V)$ is a neighborhood of p in A. □

Theorem 3.11 is a veritable generalization of the Inverse Function Theorem in the sense that the Inverse Function Theorem can be deduced from it (note that we never used the Inverse Function Theorem in our path and that we will not use it in the proof of Theorem 3.1).

Corollary 3.12 (Inverse Function Theorem)**.** *Let $f : A \to \mathbb{R}^n$ be a C^1-map defined on an open set $A \subset \mathbb{R}^n$ and let $p \in A$. If $df_p : \mathbb{R}^n \to \mathbb{R}^n$ is injective, then there exists an open neighborhood U of p such that $f(U)$ is open and $f_{|U} : U \to f(U)$ is a diffeomorphism.*

Proof. According to Theorem 3.11 there exists an open neighborhood U of p (in A) such that $f_{|U} : U \to f(U)$ is a diffeomorphism and by Proposition 3.6-3 $f(U)$ is open. □

It is interesting to observe that Theorem 3.11 and Proposition 3.6-3 split the Inverse Function Theorem in two different statements. The first gives conditions for a C^1-map to be a local diffeomorphism. The second specifies conditions ensuring that this map is locally open.

Observe also that our proof of the Inverse Function Theorem is not based, as usual, on any fixed point theorem but on the use of tangent cones and, implicitly, on Whitney's Extension Theorem.

In [9] we gave another proof of the Inverse Function Theorem based on the use of tangent cones. That proof had the advantage of being very simple (we only used there the contingent and paratingent cones) but it was not extendible to the case of arbitrary subset of $\mathbb{R}^n$.

It is not difficult to see that the only way to extend the Inverse Function Theorem to C^1-maps defined on arbitrary sets is to use the generalized definition of differential we gave (i.e. if we define the differential on a smaller space than $\Delta_p(X)$, a result as Theorem 3.11 does not hold).

4. Properties of the contingent cone, the paratingent cone and the paratingent space

In this Section we prove many basic properties of tangent cones. Some of them have already been used and this Section should stay among the first ones but since it is quite long and somewhat technical we have postponed it in order to reach the main results sooner.

4.1. **Construction of the Paratingent Space.**
In this section we give a geometrical characterization of $\Delta_p(A)$.

Let $H : X \rightrightarrows Y$ be a set-valued map of a set X into a set Y, the graph of H (as a set-valued map) is the set $\Gamma_H = \{(x, y) \in X \times Y : \ y \in H(x)\}$.

Let now X be a topological space, Y a topological vector space and set $\mathcal{H} = \{H : X \rightrightarrows Y\}$. We define the following maps:

- $\Lambda : \mathcal{H} \to \mathcal{H}, \ \Lambda(H)(x) = \langle H(x)\rangle$ ($\langle H(x)\rangle$ is the subspace spanned by $H(x)$ in Y).
- $\Omega : \mathcal{H} \to \mathcal{H}, \ \Omega(H)(x) = \{y \in Y : \ (x, y) \in \bar{\Gamma}_H\}$ ($\bar{\Gamma}_H$ is the closure of Γ_H in $X \times Y$; $\Omega(H)$ is “the closed-graph regularization” of H).
- $\Phi : \mathcal{H} \to \mathcal{H}, \ \Phi(H) = \Lambda(\Omega(H)), \ \Phi^0(H) = H, \ \Phi^{j+1}(H) = \Phi(\Phi^j(H))$.

We obviously have that $\Lambda(H)$ is subspace-valued, $\Omega(H)$ has a closed graph, $\Omega(H) \supset H$ and $\Lambda(H) \supset H$.

The next proposition gives a somewhat constructive definition of $\Delta_p(A)$. It shows in fact that $\Delta_p(A)$ can be obtained by repeatedly “closing and spanning” the set-valued map $p \mapsto S_p(A)$ a finite number of times, where “closing and spanning” means “applying Ω and applying Λ”(compare Glaeser [6, Chapter 2, Prop. 7]).

Theorem 4.1. *Let $A \subset \mathbb{R}^m$ and set $H(p) = S_p(A)$ for $p \in \bar{A}$. Set also $R_k = \{x \in \bar{A} : \ \Phi^k(H)(x) \neq \Phi^{k-1}(H)(x)\}$ for $k \geq 2$. Then*

1. *$R_k \subset \bar{R}_j$ for $k \geq j$.*
2. *If $x \in R_{2k}$ then $\dim(\Phi^{2k}(H)(x)) \geq k + 1$.*
3. *$\Delta_x(A) = \Phi^{2m-1}(H)(x)$ for $x \in \bar{A}$.*

Proof. 1) If $x \in \bar{A} \setminus \bar{R}_j$ then there is an open neighborhood U of x in $\bar{A}$ such that $\Phi^j(H)(y) = \Phi^{j-1}(H)(y)$ for $y \in U$ and therefore $\Phi^{j-1}(H)$ has a closed graph and is subspace-valued in U. It follows that $\Phi^k(H)(y) = \Phi^{j-1}(H)(y)$ for $k \geq j - 1$ and $y \in U$ and therefore $x \notin R_k$ for $k \geq j$. Thus $R_k \subset \bar{R}_j$ for $k \geq j$.

2) We prove the result by induction. If $x \in R_2$ then x is not an isolated point of $\bar{A}$ (since in that case $S_x(A) = \{0\} = \Phi^j(H)(x)$ for every $j \in \mathbf{N}$), therefore $\dim(\Phi^2(H)(x)) > \dim(\Phi^1(H)(x)) \geq 1$, that is, $\dim(\Phi^2(H)(x)) \geq 2$. If $x \in R_{2k}$ then, according to 1) $x \in \bar{R}_{2k-2}$. From the induction hypothesis we have $\dim(\Phi^{2k-2}(H)(y)) \geq k$ for $y \in R_{2k-2}$ and since $\Omega(\Phi^{2k-2}(H))$ has closed graph we obtain $\dim(\Phi^{2k-1}(H)(y)) \geq k$ for $y \in \bar{R}_{2k-2}$. Therefore

$$\dim(\Phi^{2k}(H)(x)) > \dim(\Phi^{2k-1}(H)(x)) \geq k,$$

that is, $\dim(\Phi^{2k}(H)(x)) \geq k + 1$.

3) Since $\dim(\Phi^{2k}(H)(x)) \geq k+1$ for $x \in R_{2k}$ we have $R_{2m} = \emptyset$. Therefore $\Phi^{2m}(H)(x) = \Phi^{2m-1}(H)(x)$ for $x \in \bar{A}$, so that $\Phi^{2m-1}(H)(x)$ is with closed graph and subspace-valued. This implies that $\Phi^{2m-1}(H)(x) \supset \Delta_x(A)$.

The reversed inclusion is obtained considering that from $\Delta_x(A) \supset S_x(A)$ it easily follows that $\Delta_x(A) \supset \Phi^j(H)(x)$ for every $j \in \mathbf{N}$. □

4.2. **Tangent cones to graphs.** We prove now some results relating the paratingent cone with the properties of maps.

Recall that a map $f : X \to \mathbb{R}^m$ defined on $X \subset \mathbb{R}^n$ is said to be locally Lipschitz (respectively, locally radially Lipschitz) at $x \in X$ if there exist $L, \varrho > 0$ such that $\|f(y) - f(z)\| \leq L\|y - z\|$ for every $y, z \in B(x, \varrho) \cap X$ (respectively, $\|f(y) - f(x)\| \leq L\|y - x\|$ for every $y \in B(x, \varrho) \cap X$).

Proposition 4.2. *Let $X \subset \mathbb{R}^n$ and assume that $f : X \to \mathbb{R}^m$ is continuous at $p \in X$. If $S_{(p,f(p))}(G_f)$ (respectively, $C_{(p,f(p))}(G_f)$) is a graph, then f is locally Lipschitz (respectively, locally radially Lipschitz) at p.*

Proof. Let us assume that f is not locally Lipschitz at p. There exist then two sequences of elements $x_k, y_k \in X$ converging to p, such that $\|f(x_k) - f(y_k)\| > k\|x_k - y_k\|$ for every $k \in \mathbf{N}$. By extracting a subsequence we may assume that

$$\frac{1}{\|f(x_k) - f(y_k)\|}(x_k - y_k\,,\ f(x_k) - f(y_k)) \to (0, w) \in S_{(p,f(p))}(G_f),$$

with $\|w\| = 1$, so that $S_{(p,f(p))}(G_f)$ is not a graph. This contradicts our hypotheses.

In the same way one proves the result concerning $C_{(p,f(p))}(G_f)$. □

The following proposition gives an interesting condition for a C^1-map to have a local continuous inverse.

Proposition 4.3. *Let $f : U \to \mathbb{R}^m$ be a C^1-map defined on an open set $U \subset \mathbb{R}^n$. Let $X \subset U$ and $p \in X$. If df_p is injective on $S_p(X)$ then there exists a neighborhood V of p in X such that $f_{|V} : V \to f(V)$ is a homeomorphism.*

Proof. First we want to show that there exists a neighborhood W of p in $\bar{X}$ on which f is injective. Assume it is false and let (p_k) and (q_k) be two sequences in $\bar{X}$ converging to p such that $p_k \neq q_k$ and $f(p_k) = f(q_k)$ for every $k \in \mathbf{N}$. We may also suppose that $(p_k - q_k)/\|p_k - q_k\| \to v \in S_p(\bar{X}) = S_p(X)$. Since f is C^1 we have

$$0 = \frac{f(p_k) - f(q_k)}{\|p_k - q_k\|} \to df_p(v)$$

and this contradicts the fact that df_p is injective on $S_p(X)$.

Let now W be a compact neighborhood of p in $\bar{X}$ on which f is injective. The map $f_{|W} : W \to f(W)$ is continuous and invertible on a compact set, hence it is a homeomorphism. We complete the proof by setting $V = W \cap X$. □

From Proposition 4.3 we derive the following condition for a map to be continuous on a neighborhood of a given point. We have seen, or at least stated, that if a map $f : X \to \mathbb{R}^m$ is continuous at $p \in X$ and $\Delta_{(p,f(p))}(G_f)$ is a graph then that map is C^1 on a neighborhood of p in X. Analogously, Proposition 4.4 below asserts that if the map f is continuous at p and $S_{(p,f(p))}(G_f)$ is a graph, then f is actually continuous on a neighborhood of p. This constitutes a first step in proving Theorem 3.1. In both cases it is interesting that we can deduce information about the local behavior of a map from the fact that a tangent cone nicely behaves at a single point.

In the next proposition we also show that when $S_{(p,f(p))}(G_f)$ is the graph of a map, that map may be considered a sort of differential of f at p.

Proposition 4.4. *Let* $f : X \to \mathbb{R}^m$ *be a map defined on* $X \subset \mathbb{R}^n$ *and let* $p \in X$. *If* f *is continuous at* p *and* $S_{(p,f(p))}(G_f)$ *is the graph of a map* L*, then*

(1) *there exist a neighborhood* K *of* p *in* $\bar{X}$ *and a continuous map* $\bar{f} : K \to \mathbb{R}^m$ *such that* $\bar{f} = f$ *on* $K \cap X$. *(Consequently, because of Tiezte's Theorem (see Section 5), there exists a continuous map* $g : \mathbb{R}^n \to \mathbb{R}^m$ *coinciding with* f *on a neighborhood of* p.*)*

(2) *if*

$$X \ni x_k, y_k \to p \quad \textit{and} \quad \frac{x_k - y_k}{\|x_k - y_k\|} \to v,$$

we have that

$$\frac{f(x_k) - f(y_k)}{\|x_k - y_k\|} \to L(v).$$

Proof. 1) Let $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $\mu : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be the projections onto the first and the second factor, respectively. Since π is injective on $S_{(p,f(p))}(G_f)$, there exists, by Proposition 4.3, a compact neighborhood C of $(p, f(p))$ in $\bar{G}_f$ such that $g = \pi_{|C} : C \to \pi(C)$ is a homeomorphism. Thus, if we set $K = \pi(C)$ and $\bar{f} = \mu \circ g^{-1} : K \to \mathbb{R}^m$ we have that $\bar{f}$ is continuous and that C is the graph of $\bar{f}$. Considering that f is bounded on a neighborhood of p, it is not difficult to see that K is a neighborhood of p in $\bar{X}$.

2) Set

$$B_k = \frac{f(x_k) - f(y_k)}{\|x_k - y_k\|}.$$

By Proposition 4.2, the sequence (B_k) is bounded. To prove that $B_k \to L(v)$, we prove that every convergent subsequence of (B_k) converges to $L(v)$. In fact, if $B_{k_r} \to z \in \mathbb{R}^m$ then

$$\frac{1}{\|x_{k_r} - y_{k_r}\|} \Big(x_{k_r} - y_{k_r} \, , \; f(x_{k_r}) - f(y_{k_r}) \Big) \to (v, z) \in S_{(p,f(p))}(G_f)$$

and therefore $L(v) = z = \lim_{r \to \infty} B_{k_r}$. □

It would be interesting to study the relations between some results of this work (for instance, Prop. 4.4 or Theorem 3.1) and results of Bessis [2] (for instance, Prop. 1.2.8).

4.3. **Tangent cones to direct images.** We now consider a C^1-map f and we study the relationships between $\Delta_p(A)$ and $\Delta_{f(p)}(f(A))$. That is, roughly speaking, we deform A and we ask what happens to $\Delta_p(A)$.

Proposition 4.5. *Let* $f : U \to \mathbb{R}^m$ *be a* C^1*-map defined on an open set* $U \subset \mathbb{R}^n$ *and let* $X \subset \mathbb{R}^n$*. If* $\bar{X} \subset U$ *and* $p \in \bar{X}$ *then* $df_p(C_p(X)) \subset C_{f(p)}(f(X))$, $df_p(S_p(X)) \subset S_{f(p)}(f(X))$ *and* $df_p(\Delta_p(X)) \subset \Delta_{f(p)}(f(X))$.

Consequently, if $g : U \to \mathbb{R}^m$ *is* C^1 *and coincides with* f *on* X*, then* $df_x(v) = dg_x(v)$ *for every* $x \in X$ *and* $v \in \Delta_x(X)$.

Remark Since f is continuous we have $f(\bar{X}) \subset \overline{f(X)}$. In particular $f(p) \in \overline{f(X)}$ so that we can talk of $\Delta_{f(p)}(f(X))$.

Proof. We begin by treating the case of the paratingent cone. Let $v \in S_p(X)$. We have to prove that $df_p(v) \in S_{f(p)}(f(X))$, so we may assume that $\|v\| = 1$. There exist then two sequences of elements $x_k, y_k \in X$ converging to p, such that $(x_k - y_k)/\|x_k - y_k\| \to v$. Since f is C^1 we have

$$\frac{f(x_k) - f(y_k)}{\|x_k - y_k\|} \to df_p(v),$$

and therefore $df_p(v) \in S_{f(p)}(f(X))$.

In the same way one proves $df_p(C_p(X)) \subset C_{f(p)}(f(X))$.

Let's now turn our attention to the paratingent space. We set $H(x) = S_x(X)$ and $N(y) = S_y(f(X))$ and we use the notations of Section 4.1. The result is easily obtained considering that $\Delta_p(X) = \Phi^{2n-1}(H)(p)$, $\Delta_{f(p)}(f(X)) = \Phi^{2m-1}(N)(p)$ and

$$df_p(\Phi^j(H)(p)) \subset \Phi^j\big(df_p(H(p))\big) \subset \Phi^j(N)(f(p)),$$

for every $j \in \mathbf{N}$. The first inclusion can be proved by induction considering that $df_p(\Omega(H)(p)) \subset \Omega(df_p(H(p)))$; the second derives from Proposition 4.5.

This result can also be proved using the very definition of $\Delta_p(X)$ (the proof below has also the advantage that it works in arbitrary normed spaces, provided we extend the definition of $\Delta_p(X)$).

Set $Y = f(X)$ and

$$\begin{aligned}\Psi(X) &= \Big\{H : \bar{X} \rightrightarrows \mathbb{R}^n : H \text{ has closed graph, is subspace-valued} \\ &\qquad \text{and } H(x) \supset S_x(X) \text{ for every } x \in \bar{X}\Big\}, \\ \Psi(Y) &= \Big\{G : \bar{Y} \rightrightarrows \mathbb{R}^m : G \text{ has closed graph, is subspace-valued} \\ &\qquad \text{and } G(y) \supset S_y(Y) \text{ for every } y \in \bar{Y}\Big\}.\end{aligned}$$

Let $G \in \Psi(Y)$ and consider the set-valued map $H_G(x) = df_x^{-1}(G(f(x)))$. It is not difficult to see that $H_G \in \Psi(X)$. Thus

$$\Delta_p(X) = \bigcap_{H \in \Psi(X)} H(p) \subset \bigcap_{G \in \Psi(Y)} H_G(p) = \bigcap_{G \in \Psi(Y)} df_p^{-1}\Big(G\big(f(p)\big)\Big)$$

and hence

$$df_p(\Delta_p(X)) \subset df_p\Bigg\{\bigcap_{G \in \Psi(Y)} H_G(p)\Bigg\} \subset \bigcap_{G \in \Psi(Y)} df_p\Big\{df_p^{-1}\Big(G\big(f(p)\big)\Big)\Big\}$$

$$\subset \bigcap_{G\in\Psi(Y)} G\big(f(p)\big) \;=\; \Delta_{f(p)}(Y). \qquad \square$$

We now identify conditions ensuring that $df_p(\Delta_p(X)) = \Delta_{f(p)}(f(X))$. Let us begin with $C_p(X)$ and $S_p(X)$.

Let X and Y be metric spaces. A set valued map $H : X \rightrightarrows Y$ is said to be *pseudo-continuous* at $(x, y) \in \Gamma_H$ if for any sequence of elements $x_k \in X$ converging to x there exists a sequence of elements $y_k \in H(x_k)$ such that $y_k \to y$. (The map H is lower-semicontinuous at $x \in X$ if it is pseudo-continuous at (x, y) for every $y \in H(x)$.)

A selection of a set-valued map $H : X \rightrightarrows Y$ is a map $h : X \to Y$ such that $h(x) \in H(x)$ for every $x \in X$.

Proposition 4.6. *Let X, Y be metric spaces. Let $H : X \rightrightarrows Y$ be a set-valued map and let $(p, y) \in \Gamma_H$. The following properties are equivalent*

(1) *H is pseudo-continuous at (p, y),*
(2) $\lim\limits_{z\to p} d(H(z), y) = 0$,
(3) *H has a selection h continuous at p and with $h(p) = y$,*
(4) *For every neighborhood W of y in Y, $H^{-1}(W) = \{x \in X : \; H(x) \cap W \neq \emptyset\}$ is a neighborhood of p in X.*

The proof is easy and left to the reader.

Proposition 4.7. *Let $f : U \to \mathbb{R}^m$ be a C^1-map defined on an open set U of $\mathbb{R}^n$. Let $X \subset U$ and $p \in X$. Assume that the set-valued map $(f_{|X})^{-1}$ (where $(f_{|X})^{-1}(y) = \{f^{-1}(y)\cap X\}$) is pseudo-continuous at $(f(p), p)$ and that df_p is injective on $C_p(X)$ (respectively, on $S_p(X)$). Then $df_p(C_p(X)) = C_{f(p)}(f(X))$ (respectively $df_p(S_p(X)) = S_{f(p)}(f(X))$).*

Proof. Let $g : f(X) \to X$ denote a selection of $(f_{|X})^{-1}$ continuous at $f(p)$ and with $g(f(p)) = p$ (see Proposition 4.6).

Let $w \in S_{f(p)}(f(X))$ with $\|w\| = 1$ and let $a_k, b_k \in f(X)$ be two sequences converging to $f(p)$ such that $(a_k - b_k)/\|a_k - b_k\| \to w$.

Set $x_k = g(a_k)$ and $y_k = g(b_k)$. We have that $x_k, y_k \to p$ and therefore we may assume that $(x_k - y_k)/\|x_k - y_k\|$ converges to a vector $v \in S_p(X)$. Since f is C^1 we have

$$\frac{f(x_k) - f(y_k)}{\|x_k - y_k\|} = \frac{a_k - b_k}{\|x_k - y_k\|} \to df_p(v).$$

Considering that df_p is injective on $S_p(X)$ and that $\|v\| = 1$ we deduce that $df_p(v) \neq 0$ and therefore

$$w = \frac{df_p(v)}{\|df_p(v)\|} = df_p\Big(\frac{v}{\|df_p(v)\|}\Big) \in df_p\big(S_p(X)\big).$$

The result concerning $C_p(X)$ can be proved in a similar way. $\square$

To prove the analogous result for $\Delta_p(X)$ we need some preliminaries.

Lemma 4.8. *Let* $f : U \to \mathbb{R}^m$ *be a* C^1*-map defined on an open set* $U \subset \mathbb{R}^n$*. Let* $X \subset U$ *be locally compact. If for every* $x \in X$*,* $(f_{|X})^{-1}$ *is pseudo-continuous at* $(f(x), x)$ *and* df_x *is injective on* $\Delta_x(X)$*, then* $df_x(\Delta_x(X)) = \Delta_{f(x)}(f(X))$*.*

Proof. (Notations as in Section 4.1) We prove by induction that

$$df_x(\Phi^j S_x(X)) = \Phi^j S_{f(x)}(f(X)),$$

for $x \in X$ and $j \in \mathbf{N}$; the result will then follow from Proposition 4.1. The base of the induction derives from Proposition 4.7.

Set $H(x) = \Phi^j S_x(X)$ and $N(y) = \Phi^j S_y(f(X))$, then fix $x \in X$ and let $g : f(X) \to X$ be a selection of $(f_{|X})^{-1}$ continuous at $f(x)$ and with $g(f(x)) = x$. We first want to show that $df_x(\Omega H(x)) = \Omega N(f(x))$. We know that $df_x(\Omega H(x)) \subset \Omega N(f(x))$; let then $v \in \Omega N(f(x))$ with $v \neq 0$.

Observe that N is defined on $\overline{f(X)}$ but since $f(X)$ is locally compact (use Proposition 4.6(4)) there exist $y_k \in f(X)$ and $v_k \in N(y_k)$ such that $(y_k, v_k) \to (f(x), v)$. Set $x_k = g(y_k)$ and observe that $x_k \to x$. From the induction hypothesis we have that $df_{x_k}(H(x_k)) = N(f(x_k))$, so that v_k may be written as $v_k = df_{x_k}(w_k)$ with $w_k \in H(x_k)$. By extracting a subsequence we may assume that

- $\|w_k\| \to a \in (0, +\infty) \cup \{+\infty\}$ ($a = 0$ is excluded since $v \neq 0$),
- $\dfrac{w_k}{\|w_k\|} \to w \in \Omega H(x)$.

Therefore

$$df_{x_k}\Big(\frac{w_k}{\|w_k\|}\Big) = \frac{df_{x_k}(w_k)}{\|w_k\|} = \frac{v_k}{\|w_k\|} \to df_x(w) = \begin{cases} 0 & \text{if } a = +\infty, \\ \frac{v}{a} & \text{if } a \neq +\infty. \end{cases}$$

Since $\|w\| = 1$ and df_x is injective on $\Delta_x(X)$ we have $df_x(w) \neq 0$ and therefore $v = df_x(aw) \in df_x(\Omega H(x))$.

To conclude the proof observe that

$$\begin{aligned} df_x(\Phi^{j+1} S_x(X)) &= df_x(\Lambda\Omega H(x)) = \Lambda df_x(\Omega H(x)) = \\ &= \Lambda\Omega N(f(x)) = \Phi^{j+1} S_{f(x)}(f(X)). \end{aligned}$$

□

As for Proposition 4.5, Lemma 4.8 may also be proved using the definition of $\Delta_p(X)$ and not relying upon Proposition 4.1. We don't give such a proof (which unfortunately, and contrarily to the second proof of Proposition 4.5, does not hold in arbitrary normed spaces).

We also need the following simple lemmas.

Lemma 4.9. *Let* $U \subset \mathbb{R}^n$ *be open and let* $f : U \to \mathbb{R}^m$ *be a* C^1*-map. Let* X *be a subset of* $\mathbb{R}^m$ *such that* $\bar{X} \subset U$ *and let* $p \in X$*. Assume that* $(f_{|X})^{-1}$ *is pseudo-continuous at* $(f(p), p)$ *and that* df_p *is injective on* $S_p(X)$*, then there exists a neighborhood* V *of* p *in* $\bar{X}$ *such that* $(f_{|\bar{X}})^{-1}$ *is pseudo-continuous at* $(f(x), x)$ *for every* $x \in V$*.*

Proof. By Proposition 4.3, there exists a neighborhood V of p in $\bar{X}$ such that $f_{|V} : V \to f(V)$ is a homeomorphism. By Proposition 4.6, $f(V)$ is a neighborhood of $f(p)$ in $f(\bar{X})$, we may therefore assume that it is open in $f(\bar{X})$. Let $h : f(\bar{X}) \to$

$\bar{X}$ be a selection of $(f_{|\bar{X}})^{-1}$ and define $g : f(\bar{X}) \to \bar{X}$ with $g(y) = (f_{|V})^{-1}(y)$ if $y \in f(V)$ and $g(y) = h(y)$ otherwise. We have that g is a selection of $(f_{|\bar{X}})^{-1}$, continuous on $f(V)$ and with $g(f(x)) = x$ for every $x \in V$. By Proposition 4.6, this concludes the proof. □

Lemma 4.10. *Let $f : X \to \mathbb{R}^m$ be defined on $X \subset \mathbb{R}^n$ and let $p \in X$. Assume that f^{-1} is pseudo-continuous at $(f(p), p)$. If V is a neighborhood of p in X then the map $(f_{|V})^{-1}$ is pseudo-continuous at $(f(p), p)$.*

Proof. Let $g : f(X) \to X$ be a selection of f^{-1}, continuous at $f(p)$ and with $g(f(p)) = p$ and let $h : f(V) \to V$ be a selection of $(f_{|V})^{-1}$. Define $r : f(V) \to V$ with $r(y) = g(y)$ if $g(y) \in V$ and $r(y) = h(y)$ otherwise. The map r is a selection of $(f_{|V})^{-1}$, moreover it is continuous at $f(p)$, in fact $g^{-1}(V)$ is a neighborhood of $f(p)$ in $f(X)$ and therefore $r(y) = g(y)$ on a neighborhood of $f(p)$. □

And now the announced result for $\Delta_p(X)$.

Theorem 4.11. *Let $f : U \to \mathbb{R}^m$ be a C^1-map defined on an open set $U \subset \mathbb{R}^n$, let $X \subset U$ and $p \in X$ and assume that $(f_{|X})^{-1}$ is pseudo-continuous at $(f(p), p)$. If df_p is injective on $\Delta_p(X)$ then $df_p(\Delta_p(X)) = \Delta_{f(p)}(f(X))$.*

Proof. Let V be an open neighborhood of p in $\bar{X}$ such that

- $(f_{|\bar{X}})^{-1}$ is pseudo-continuous at $(f(x), x)$ for every $x \in V$ (Lemma 4.9),
- df_x is injective on $\Delta_x(X)$ for every $x \in V$ (observe that $\mathrm{Ker}(df_x)$ is a subspace-valued map with a closed graph and the existence of such a neighborhood easily follows).

By Lemma 4.10, $(f_{|V})^{-1}$ is pseudo-continuous at $(f(x), x)$ for every $x \in V$ and by Proposition 4.6, $f(V)$ is open in $f(\bar{X})$.

Since V is locally compact we have, by Lemma 4.8, that

$$df_p(\Delta_p(X)) = df_p(\Delta_p(V)) = \Delta_{f(p)}(f(V)) = \Delta_{f(p)}(f(X)). \qquad \square$$

The next proposition points out a particular case of Proposition 4.7 and Theorem 4.11.

Proposition 4.12. *Let $f : X \to \mathbb{R}^m$ be a map defined on $X \subset \mathbb{R}^n$ and let $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ denote the projection onto the first factor. If f is continuous at $p \in X$ and $C_{(p,f(p))}(G_f)$ (respectively, $S_{(p,f(p))}(G_f)$ or $\Delta_{(p,f(p))}(G_f)$) is a graph then*

$$\pi(C_{(p,f(p))}(G_f)) = C_p(X)$$

(respectively, $\pi(S_{(p,f(p))}(G_f)) = S_p(X)$ or $\pi(\Delta_{(p,f(p))}(G_f)) = \Delta_p(X)$).

Proof. Just observe that π is C^1 and that $(\pi_{|G_f})^{-1}(x) = (x, f(x))$ is (pseudo-) continuous at p. Then apply Proposition 4.7 and Theorem 4.11. □

5. Proof of the main result and Whitney's Extension Theorem

Using the properties of the paratingent space obtained in the previous Section and Whitney's Extension Theorem we are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1

We begin by recalling Tiezte's Theorem: *Let X be a metric space (or, more generally, a normal topological space) and let $f : A \to [0, 1]$ be a continuous map defined on a closed subset A of X. Then f can be continuously extended to X.*

From Tiezte's Theorem one easily derives the following result: *Let X be a metric space and let $f : A \to M$ be a continuous map defined on a closed subset A of X and with values on a topological manifold M. Let $p \in A$. Then there exist a neighborhood U of p in X and a continuous map $g : U \to M$ such that $g = f$ on $U \cap A$.*

We also need the following proposition due to Glaeser [6, Chapter 2, Prop. 3]. The proof of Glaeser uses the Inverse Function Theorem. We modified it a little to avoid the use of the Inverse Function Theorem that we want to obtain as a corollary (Section 3.3).

Denote by $G(m, s)$ the topological manifold (called *Grassmann Manifold*) of the s-dimensional subspaces of $\mathbb{R}^m$. We consider it also as the quotient space of the space of s-tuples of indipendent vectors of $\mathbb{R}^m$ under the equivalence relation that identifies two such s-tuples if they span the same subspace of $\mathbb{R}^m$(for more details on Grassmann Manifolds see, for instance, Wells [10]).

Lemma 5.1. *If $G : X \rightrightarrows \mathbb{R}^m$ is a subspace-valued map with a closed graph whose values are subspaces of $\mathbb{R}^m$ of a constant dimension s, then G is continuous (as a set-valued map).*

Moreover if we define $H : X \rightrightarrows G(m, s)$ by setting $H(x) = G(x) \in G(m, s)$ we have that H is continuous too.

Proof. Fix $x \in X$ and observe that the orthogonal projection of $\mathbb{R}^m$ into $G(x)$ (denoted $\pi_{G(x)}$) is injective on $G(y)$ for any y in a neighborhood of x: if not, there would exist a sequence $x_n \in X$ converging to x and a sequence $v_n \in G(x_n)$ such that $\|v_n\| = 1$ and $\pi_{G(x)}(v_n) = 0$ for every n; we may also assume that v_n converges to a vector v which, since G has a closed graph, belongs to $G(x)$ but then $\pi_{G(x)}(v) = v = 0$ (by the continuity of $\pi_{G(x)}$) and this contradicts $\|v\| = 1$.

Since the values of G have all the same dimension we have that the orthogonal projection of $G(y)$ into $G(x)$ is an isomorphism for every y in a neighborhood of x.

We have to prove that G is lower semicontinuous at x. Let $v \in G(x)$ $(v \neq 0)$ and let $x_n \in X$, $x_n \to x$. For every n let $v_n \in G(x_n)$ be such that $\pi_{G(x)}(v_n) = v$ (this is possible for any large enogh n by what we showed above). We want to prove that $v_n \to v$. First observe that $\|v_n\|$ is bounded: otherwise we could consider a subsequence v_{k_n} such that $\|v_{k_n}\| \to \infty$ and $(v_{k_n}/\,\|v_{k_n}\|) \to w \in G(x)$ but then we would have $w = \pi_{G(x)}(w) = \pi_{G(x)}(\lim \frac{v_{k_n}}{\|v_{k_n}\|}) = \lim \frac{1}{\|v_{k_n}\|}\pi_{G(x)}(v_{k_n}) = \lim \frac{v}{v_{k_n}} = 0$ which contradicts $\|w\| = 1$. We now prove that every convergent subsequence of v_n converges to v and this proves the claim. Let then v_{k_n} be a convergent subsequence of v_n. We have $v_{k_n} \in G(x_{k_n})$, $v_{k_n} \to w$, $w \in G(x)$, $\pi_{G(x)}(v_{k_n}) = v$ and taking the limit $w = \pi_{G(x)}(w) = v$.

Let us now prove that H is continuous. Let $v_1^x, \cdots, v_s^x \in G(x)$ be a basis of $G(x)$ and set $v_i^y = \pi_{G(y)}(v_i^x)$. For any y in a neighborhood of x, $v_1^y, \cdots, v_s^y$ is a basis of $G(y)$. Reasoning as before we have that v_i^y is a continuous function of y and this is enogh to prove our claim since $H(y) = [(v_1^y, \cdots, v_s^y)]$ (where the brackets denote the class of equivalence with respect to the relation that gives rise to $G(m, s)$). □

Proposition 5.2. *Let $C \subset \mathbb{R}^m$ be closed and let $H : C \rightrightarrows \mathbb{R}^m$ be a subspace-valued map with a closed graph. Let $p \in C$ and assume $\dim(H(p)) = s$. Then there exist a neighborhood U of p in C and a subspace-valued map $\bar{H} : U \rightrightarrows \mathbb{R}^m$ with a closed graph such that $\dim(\bar{H}(x)) = s$ and $\bar{H}(x) \supset H(x)$ for every $x \in U$.*

Proof. 1) If $\dim(H(x)) = s$ on a neighborhood of p in C then there is nothing to be proved.

2) Suppose $\dim(H(x))$ takes on only two values on a neighborhood V of p in C and let them be s and s_1 (since H has a closed graph we obviously have $s_1 < s$). We may suppose that V is closed. Set

$$C_d = \{ x \in V : \dim(H(x)) = d \}$$

and observe that C_s is closed. The set valued map $H_{|C_s}$ may be regarded as a map from C_s into $G(m, s)$, we hence define $G = H_{|C_s} : C_s \to G(m, s)$. Since H has a closed graph and $\dim(H(x))$ is constant on C_s the map G is continuous in virtue of the Lemma above. There exist then (use the above generalization of Tiezte's Theorem), a closed neighborhood V' of p in C and a continuous map $L : V' \to G(m, s)$ that extends G.

It is not difficult (reasoning as in Lemma 5.1) to see that $L^\perp$ has a closed graph and since H has a closed graph too, we have that $H(x) \cap L(x)^\perp = \{0\}$ on a neighborhood $U \subset V'$ of p in C.

Let $\pi_{L(x)}$ denote the orthogonal projection onto $L(x)$. It can be easily shown that $\pi_{L(x)}(H(x))^\perp \cap L(x) = H(x)^\perp \cap L(x)$. We hence define

$$\begin{cases} \bar{H} : U \rightrightarrows \mathbb{R}^m \\ \bar{H}(x) = \langle H(x) \, , \, \pi_{L(x)}(H(x))^\perp \cap L(x) \rangle = \langle H(x) \, , \, H(x)^\perp \cap L(x) \rangle \end{cases}$$

Observe that $\bar{H}(x) = H(x)$ for $x \in U \cap C_s$.

Let $x \in U$. Since the projection of $H(x)$ into $L(x)$ is injective, we have that $\pi_{L(x)}(H(x))^\perp \cap L(x)$ has dimension $s - \dim(H(x))$ and since $H(x)$ and $H(x)^\perp \cap L(x)$ are orthogonal spaces, we deduce that $\bar{H}(x) = \langle H(x) \, , \, H(x)^\perp \cap L(x) \rangle$ has dimension s.

We have to prove that $\bar{H}$ has a closed graph. Assume that $p_k \in U$, $p_k \to p \in U$, $v_k \in \bar{H}(p_k)$ and $v_k \to v$. We have to show that $v \in \bar{H}(p)$. If $p \in C_{s_1}$ then on a neighborhood of p we have $\dim(H(x)) = s_1$ and therefore H is continuous at p and this easily implies that $\langle H(x) \, , \, H(x)^\perp \cap L(x) \rangle$ is continuous at p. If $p \in C_s$ we write $v_k = v_k' + v_k''$ with $v_k' \in H(p_k)$ and $v_k'' \in H(p_k)^\perp \cap L(p_k)$. Since $H(p_k)$ and $H(p_k)^\perp \cap L(p_k)$ are orthogonal spaces we have that $\|v_k\| = \|v_k'\| + \|v_k''\|$ so that v_k' and v_k'' are bounded. We may then assume, by extracting a subsequence, that

$v'_k \to v'$ and $v''_k \to v''$. Since H and L have a closed graph, we derive that $v' \in H(p)$ and $v'' \in L(p) = H(p)$ and therefore $v = v' + v'' \in H(p)$.

3) Suppose that $\dim(H(x))$ takes on the values $s > s_1 > \ldots > s_N$ on a closed neighborhood V of p in C. Set $C_1 = \{x \in V : \dim(H(x)) \geq s_1\}$ and observe that it is closed. We now apply part 2) to the map $H_{|C_1}$ to construct a map $H' : V' \to G(m, s)$, defined on a neighborhood V' of p in C_1, such that $H'(x) \supset H(x)$ and $\dim(H'(x)) = s$ for every $x \in V'$. We then set $V_1 = V' \cup (V \setminus C_1)$ and define $H_1 : V_1 \rightrightarrows \mathbb{R}^m$ by

$$H_1(x) = \begin{cases} H' & on \;\; V', \\ H & on \;\; V \setminus C_1. \end{cases}$$

The set V_1 is a neighborhood of p in C and the map H_1 has a closed graph on V_1 and takes on only the values $s > s_2 > \ldots > s_N$.

Proceeding this way we construct the required map $\bar{H}$. □

We shall need Whitney's Extension Theorem which gives conditions for a map defined on a closed subset of $\mathbb{R}^n$ to have a C^1-extension defined on $\mathbb{R}^n$. We state it here in the form we shall use it (for a proof see, for instance, Evans-Gariepy [3]).

Whitney's Extension Theorem *Let $K \subset \mathbb{R}^n$ be compact, $f : K \to \mathbb{R}^m$ a continuous map and $d : K \times \mathbb{R}^n \to \mathbb{R}^m$ a continuous map that is linear with respect to the second variable. Set*

$$\varrho(\delta) = \sup \left\{ \frac{\|f(y) - f(x) - d(x, y - x)\|}{\|y - x\|} \;:\; x, y \in K, \; 0 < \|x - y\| \leq \delta \right\}.$$

If $\varrho(\delta) \to 0$ as $\delta \to 0$, then there exists a map $\bar{f} : \mathbb{R}^n \to \mathbb{R}^m$ such that

- *$\bar{f}$ is C^1,*
- *$\bar{f} = f$ on K,*
- *$d\bar{f}_x(v) = d(x, v)$ for every $(x, v) \in K \times \mathbb{R}^n$.*

We are now in a position to complete the proof of Theorem 3.1. The idea is as follows: Proposition 4.4-2 says, in a sense, that the map L_x whose graph is $S_{(x,f(x))}(G_f)$ is the differential of f at x restricted to $S_x(X)$. We then use Proposition 5.2 to extend L to a map d defined on $X \times \mathbb{R}^n$ and we apply Whitney's Extension Theorem to f and d to get the required C^1-extension of f.

Assume now that we are under the hypotheses of Theorem 3.1 and let $g : K \to \mathbb{R}^m$ be a continuous map defined on a compact neighborhood K of p in $\bar{X}$ such that $g = f$ on $K \cap X$ and $\Delta_{(x,g(x))}(G_g)$ is the graph of a map $L_x : \Delta_x(K) \to \mathbb{R}^m$ for every $x \in K$ (see Propositions 4.4, 4.12). Consider now the set-valued map $L : K \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$ defined by $L(x) = \Delta_{(x,g(x))}(G_g)$.

If $\dim(L(p)) < n$ we find an n-dimensional subspace S of $\mathbb{R}^n \times \mathbb{R}^m$ that contains $\Delta_{(x,g(x))}(G_g)$ and that injectively projects onto the first factor and we set, with little abuse of notation, $L(p) = S$.

The map L thus defined satisfies the hypotheses of Proposition 5.2, there exist therefore a compact neighborhood C of p in K and a subspace-valued map with closed graph $\bar{L} : C \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$ such that $\dim(\bar{L}(x)) = n$ and $\bar{L}(x) \supset L(x)$ for every $x \in C$. Since $\bar{L}$ has a closed graph and $\bar{L}(p)$ is a graph, we may also suppose that for every $x \in C$, $\bar{L}(x)$ is the graph of a map $\bar{L}_x : \mathbb{R}^n \to \mathbb{R}^m$. Consider now the map $d : K \times \mathbb{R}^n \to \mathbb{R}^m$ defined by $d(x, v) = \bar{L}_x(v)$ and observe that d is continuous ($\bar{L}$ has a closed graph and values of constant dimension, hence it is continuous) and linear with respect to the second variable.

We want to show that g and d satisfy the hypotheses of Whitney's Extension Theorem. Set

$$\varrho(\delta) = \sup\left\{ \frac{\|g(y) - g(x) - d(x, y - x)\|}{\|y - x\|} \; : \; x, y \in C, \; 0 < \|x - y\| \leq \delta \right\}.$$

Suppose that $\varrho(\delta)$ does not converge to zero as $\delta \to 0$. There exist then $\varepsilon > 0$ and two sequences of elements $x_k, y_k \in C$ such that $\|x_k - y_k\| < 1/k$ and

$$\frac{\|g(y_k) - g(x_k) - d(x_k, y_k - x_k)\|}{\|y_k - x_k\|} > \varepsilon,$$

for every $k \in \mathbf{N}$. We may also suppose that $x_k, y_k \to x \in C$ and that

$$\frac{y_k - x_k}{\|y_k - x_k\|} \to v.$$

By Proposition 4.4 we have

$$\left\| \frac{g(y_k) - g(x_k) - d(x_k, y_k - x_k)}{\|y_k - x_k\|} \right\| =$$

$$= \left\| \frac{g(y_k) - g(x_k)}{\|y_k - x_k\|} - d\left(x_k, \frac{y_k - x_k}{\|y_k - x_k\|} \right) \right\| \to \|L(x, v) - d(x, v)\| = 0,$$

which contradicts our assumption.

We conclude the proof invoking Whitney's Extension Theorem. □

As we have already observed, this theorem allows us to decide if a map is C^1 by checking only the properties of its graph (i.e. by checking the properties of its values) and not relying upon the existence of other functions as Whitney's Extension Theorem does. In this sense Theorem 3.1 may be considered an improvement of Whitney's Extension Theorem itself. The next result shows a deeper and surprising connection between these two theorems.

Theorem 5.3. *The following assertions are equivalent*

(1) *The statement of Whitney's Extension Theorem holds true.*
(2) *Let $A \subset \mathbb{R}^m$ and $p \in A$. Let F be a subspace of $\mathbb{R}^m$ such that $\Delta_p(A) \cap F = \{0\}$. Then in a neighborhood of p the set A may be regarded as the graph of a C^1-map defined on a subset of $F^\perp$ (Theorem 3.4).*
(3) *$T^1_p(A) = \Delta_p(A)$ for every $A \subset \mathbb{R}^m$ and $p \in \bar{A}$.*

Proof. We have seen that 1) ⇒ 2) ⇒ 3).

3) ⇒ 2). Set $k = \dim(F)$ and let $F = \langle v_1, \cdots, v_k \rangle$. For $i = 1, \cdots, k$ let $f_i : U_i \to \mathbb{R}$ be a C^1-map defined on an open neighborhood U_i of p and null on $A \cap U_i$, such that $v_i \notin \mathrm{Ker}(df_{i_p})$ (this can be done since $v_i \notin \Delta_p(A) = T^1_p(A)$). Set $f = (f_1, \cdots, f_k)$. We have that $v_1, \cdots, v_k \notin \mathrm{Ker}(df_p) = \cap_i \mathrm{Ker}(df_{i_p})$. Therefore, by the Implicit-Function Theorem, $f^{-1}(0)$ is, around p, the graph of a C^1-map defined on a subset of $F^\perp$. This concludes the proof since $A \subset f^{-1}(0)$.

2) ⇒ 1). (Sketch) We leave to the reader to check that the hypotheses of Whitney's Extension Theorem imply that $\Delta_{(p,f(p))}(G_f)$ is contained in the graph of $d(p,\cdot)$ (the map in the statement of Whitney's Extension Theorem). Therefore f has a C^1 extension around p. The proof can be concluded by means of the partitions of unity. □

In view of the previous theorem, proving that $T^1_p(A) = \Delta_p(A)$ without using Whitney's Extension Theorem would lead to a new proof of Whitney's Theorem. Proving that $T^1_p(A) = \Delta_p(A)$ consists in proving that if $v \notin \Delta_p(A)$ then there exists a C^1-map f defined on an open neighborhood U of p such that $f(A \cap U) = \{0\}$ and $df_p(v) \neq 0$.

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