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EXISTENCE OF PERIODIC SOLUTIONS UNDER SADDLE POINT TYPE CONDITIONS

NORIMICHI HIRANO AND NAOKI SHIOJI

Abstract. We study the existence of periodic solutions for nonlinear evolution equations. We do not assume either coercive conditions or variational structures. Under saddle point type conditions, we show the existence of periodic solutions by degree theory.

1. Introduction

Let H be a Hilbert space, let A be a maximal monotone subset of $H \times H$, let $f : [0,T] \times V \to H$ be a function, where V is a suitable subset of H, and let $h \in L^1(0,T;H)$. We study the existence of T-periodic solution for the equation

$$u'(t) + Au(t) \ni f(t,u(t)) + h(t) \quad \text{for } 0 \le t \le T. \tag{1.1}$$

Problems of this kind have been studied by many authors; see [1, 3, 8, 10, 12, 13, 14, 17]. To get the existence of periodic solutions, it is usually assumed that $A-f$ is coercive. In the case that $A-f$ is not coercive, the first author and Mizoguchi [10] obtained an existence result under the assumption that $A-f$ is the derivative of a functional on H and $A-f$ satisfies a saddle point type condition. That is the conditions for A and f were very restricted.

In this paper, we study the case that $A-f$ is not coercive and $A-f$ is not the derivative of a functional but $A-f$ satisfies some kind of saddle point type condition. In the elliptic case, the first author obtained an existence result under these conditions in [9]. Our typical result is the following which is a direct consequence of Propositions 1, 2 and Remark in Section 3:

Theorem 1. *Let $(V, \|\cdot\|)$ be a Hilbert space which is densely and compactly imbedded into a Hilbert space $(H, |\cdot|)$ with an inner product $\langle \cdot,\cdot \rangle$ and let L be the canonical isomorphism from V onto its topological dual $(V^*, \|\cdot\|_*)$. Let $A : D(A) \to H$ be a single-valued, maximal monotone operator such that $0 \in D(A)$, $D(A)$ is a dense subset of V with respect to the topology of V and*

$$\langle Ax - Ay, x-y \rangle \ge \omega \|x-y\|^2 \quad \text{and} \quad \|Ax\|_* \le c(\|x\|+1)$$

for every $x, y \in D(A)$, where ω and c are some positive constants. Let $T > 0$ and let f be a mapping from $[0,T] \times V$ into H such that $f(t,\cdot)$ is continuous for almost every $t \in (0,T)$, $f(\cdot,x)$ is strongly measurable for every $x \in V$ and

$$|f(t,x)| \le a_1 \|x\|^{2-\rho} + a_2(t) \quad \text{for almost every } t \in (0,T) \text{ and for every } x \in V,$$

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where ρ is some constant with $0 < \rho < 2$, a_1 is some positive constant and a_2 is some function in $L^1(0,T;\mathbb{R}_+)$. Assume that there exist a finite dimensional subspace H_1 of H and $b \in L^1(0,T;\mathbb{R}_+)$ such that

$$H_1 \subset D(L \cap (H \times H)), \qquad LH_1 \subset H_1$$

and for every $x \in D(A)$,

$$\langle Ax - f(t,x), x - 2Px\rangle \geq \omega\|x\|^2 - b(t) \quad \text{for almost every } t \in (0,T),$$

where P is the orthogonal projection from H onto H_1. Then for every $g \in L^2(0,T;H)$, there exists $\delta > 0$ such that for every $h \in L^1(0,T;H)$ with $\int_0^T |h(t) - g(t)|\,dt \leq \delta$, there exists at least one T-periodic integral solution of (1.1)*. Further, if $A0 = 0$, $f(\cdot,0) \equiv 0$ and there exist a finite dimensional subspace H_2 of H and $\varepsilon > 0$ such that*

$$H_2 \subset D(L \cap (H \times H)), \qquad LH_2 \subset H_2,$$

for every $x \in D(A)$ with $|x| \leq \varepsilon$,

$$\langle Ax - f(t,x), x - 2Qx\rangle \geq \omega\|x\|^2 \quad \text{for almost every } t \in (0,T),$$

where Q is the orthogonal projection from H onto H_2, and $\dim H_2 - \dim H_1$ is odd, then there exists $\delta > 0$ such that for every $h \in L^1(0,T;H)$ with $\int_0^T |h(t)|\,dt \leq \delta$, there exist at least two T-periodic integral solutions of (1.1)*.*

This paper is organized as follows: Section 2 is devoted to some preliminaries and notations. We state our main results in Section 3 and we prove them in Section 4. Finally, we study an example to which our results are applicable.

2. Preliminaries

Throughout this paper, all vector spaces are real. Let X be a Banach space, let Ω be an open, bounded subset of X, let Γ be a compact mapping from Ω into X and let $y \in X$ with $y \notin (I - \Gamma)(\partial\Omega)$. We denote by $\deg(I - \Gamma, \Omega, y)$ the Leray-Schauder degree. For the Leray-Schauder degree, see [6].

Let $(H, |\cdot|)$ be a Hilbert space with an inner product $\langle\cdot,\cdot\rangle$ and let A be a maximal monotone subset of $H \times H$. We know from [2, 5, 11] that the negative of A generates a semigroup $\{S(t) : t \geq 0\}$. We say the semigroup $\{S(t)\}$ is compact if for every $t > 0$, $S(t) : \overline{D(A)} \to \overline{D(A)}$ is compact. Let $f \in L^1(a,b;H)$ and let $x \in \overline{D(A)}$. We say a function $u : [a,b] \to H$ is an integral solution of the initial value problem

$$u(a) = x, \quad u'(t) + Au(t) \ni f(t) \quad \text{for } a \leq t \leq b, \tag{2.1}$$

if u is continuous on $[a,b]$, $u(a) = x$, $u(t) \in \overline{D(A)}$ for every $a \leq t \leq b$ and

$$|u(t) - y|^2 \leq |u(s) - y|^2 + 2\int_s^t \langle f(\tau) - z, u(\tau) - y\rangle\,d\tau$$

for every $(y,z) \in A$ and s,t with $a \leq s \leq t \leq b$. We know from [2, 4] that the initial value problem (2.1) has a unique integral solution.

Let $(V, \|\cdot\|)$ be a reflexive Banach space which is continuously imbedded into H. We identify V with a subspace of H. Let $\omega > 0$ and let A be a maximal monotone subset of $H \times H$ such that $D(A) \subset V$ and $\langle y - q, x - p\rangle \geq \omega\|x - p\|^2$ for every $(x,y),(p,q) \in A$. In this case, if u and v are the integral solutions of (2.1) corresponding to $(x,f),(y,g) \in \overline{D(A)} \times L^1(a,b;H)$ respectively, then

$$|u(t)-v(t)| \le e^{-\omega\eta(t-s)}|u(s)-v(s)| + \int_s^t e^{-\omega\eta(t-\tau)}|f(\tau)-g(\tau)|\,d\tau \tag{2.2}$$

for $a \le s \le t \le b$, where η is a positive constant satisfying $|\cdot\| \le \eta\|\cdot\|$, and

$$\begin{aligned} &|u(t)-v(t)|^2 - |u(s)-v(s)|^2 + 2\omega\int_s^t \|u(\tau)-v(\tau)\|^2\,d\tau \\ &\qquad \le 2\int_s^t \langle u(\tau)-v(\tau), f(\tau)-g(\tau)\rangle\,d\tau \end{aligned} \tag{2.3}$$

for $a \le s \le t \le b$.

To prove our theorems, we use the following, which are special cases of [6, Theorem 8.10] and [17, Theorem 2], respectively.

Theorem A (Leray and Schauder)**.** *Let X be a Banach space, let Γ be a compact linear operator on X such that one is not an eigenvalue of Γ and let Ω be a bounded, open subset of X with $0 \in \Omega$. Then* $\deg(I-\Gamma,\Omega,0) = (-1)^n$*, where n is the sum of the algebraic multiplicities of the eigenvalues μ satisfying $\mu > 1$ if Γ has eigenvalues μ of this kind and $n = 0$ if Γ does not have those.*

Theorem B (Vrabie)**.** *Let H be a Hilbert space and let A be a maximal monotone subset of $H \times H$ whose negative generates a compact semigroup. Let B be a bounded subset of $\overline{D(A)}$, let $T > 0$ and let G be a uniformly integrable subset of $L^1(0,T;H)$. Let $\mathcal{S}$ be the set of all integral solutions u of*

$$u(0) = x, \qquad u'(t) + Au(t) \ni f(t), \quad 0 \le t \le T$$

for $x \in B$ and $f \in G$. Then $\{u(T) : u \in \mathcal{S}\}$ is relatively compact in H. Further, if B is relatively compact in H, then $\mathcal{S}$ is relatively compact in $C(0,T;H)$.

3. Main results

We begin this section with hypotheses and notations which we will use in the sequel:

- (H1) $(V,\|\cdot\|)$ is a reflexive Banach space which is densely and compactly imbedded into a Hilbert space $(H,|\cdot|)$ with an inner product $\langle\cdot,\cdot\rangle$;
- (H2) $\omega > 0$, $c > 0$ and $L \subset H \times H$ is a symmetric linear operator such that $D(L)$ is a dense subset of V with respect to the topology of V and

$$\langle Lx,x\rangle \ge \omega\|x\|^2 \quad \text{and} \quad \|Lx\|_* \le c\|x\| \qquad \text{for every } x \in D(L), \tag{3.1}$$

  where $\|\cdot\|_*$ is the norm of the topological dual V^* of V;
- (H3) A is a maximal monotone subset of $H \times H$ such that $D(A)$ is a dense subset of V with respect to the topology of V, $D(A) \cap D(L) \ne \emptyset$,

$$\langle y-q, x-p\rangle \ge \omega\|x-p\|^2 \quad \text{for every } (x,y),(p,q) \in A \tag{3.2}$$

  and

$$\|y\|_* \le c(\|x\|+1) \quad \text{for every } (x,y) \in A; \tag{3.3}$$

(H4) $T > 0$ and f is a mapping from $[0,T] \times V$ into H such that the mapping $u(\cdot) \mapsto f(\cdot, u(\cdot)) : L^2(0,T;V) \to L^1(0,T;H)$ is continuous and it maps a bounded subset of $L^2(0,T;V)$ to a uniformly integrable subset of $L^1(0,T;H)$.

Now we show our main results.

Theorem 2. *Assume* (H1)–(H4). *Assume also that there exist a finite dimensional subspace H_1 of H and $b \in L^1(0,T;\mathbb{R}_+)$ which satisfy*

$$H_1 \subset D(L), \qquad LH_1 \subset H_1 \tag{3.4}$$

and for every $(x,y) \in A$,

$$\langle y - f(t,x), x - 2Px \rangle \geq \omega \|x\|^2 - b(t) \quad \text{for almost every } t \in (0,T), \tag{3.5}$$

where P is the orthogonal projection from H onto H_1. Then for every $g \in L^2(0,T;H)$, there exists $\delta > 0$ such that for every $h \in L^1(0,T;H)$ with $\int_0^T |h(t) - g(t)|\,dt \leq \delta$, there exists at least one T-periodic integral solution of (1.1).

Theorem 3. *Assume* (H1)–(H4). *Assume also that $(0,0) \in A$, $f(\cdot, 0) \equiv 0$ and there exist a finite dimensional subspace H_2 of H and $\varepsilon > 0$ such that*

$$H_2 \subset D(L), \qquad LH_2 \subset H_2$$

and for every $(x,y) \in A$ with $|x| \leq \varepsilon$,

$$\langle y - f(t,x), x - 2Qx \rangle \geq \omega \|x\|^2 \quad \text{for almost every } t \in (0,T),$$

where Q is the orthogonal projection from H onto H_2. Then there exists $\delta > 0$ such that for every $h \in L^1(0,T;H)$ with $\int_0^T |h(t)|\,dt \leq \delta$, there exists at least one T-periodic integral solution of (1.1).

Remark. In the theorems above, the condition that $D(A)$ and $D(L)$ are dense in V can be replaced by the condition that $\alpha A + (1-\alpha)L$ is maximal monotone in $H \times H$ for every $\alpha \in [0,1]$, since the proofs in the next section work with $\alpha A + (1-\alpha)L$ instead of $A(\alpha)$ in (4.2). The condition (H2) or (H4) can be also replaced by the following condition (H2′) or (H4′), respectively:

(H2′) $\omega > 0$, $c > 0$ and $L \subset H \times H$ has an extension $L_V \subset V \times V^*$ such that $D(L_V) = V$, L_V is symmetric linear and

$$\langle L_V x, x \rangle \geq \omega \|x\|^2 \quad \text{and} \quad \|L_V x\|_* \leq c\|x\| \qquad \text{for every } x \in D(L_V);$$

(H4′) $T > 0$ and f is a mapping from $[0,T] \times V$ into H such that $f(t,\cdot)$ is continuous for almost every $t \in (0,T)$, $f(\cdot,x)$ is strongly measurable for every $x \in V$ and $|f(t,x)| \leq a_1\|x\|^{2-\rho} + a_2(t)$ for almost every $t \in (0,T)$ and for every $x \in V$, where ρ is some constant with $0 < \rho < 2$, a_1 is some positive constant and a_2 is some function in $L^1(0,T;\mathbb{R}_+)$.

As a consequence of Theorem 2, we can get a solution of an elliptic problem as follows:

Corollary. *Assume* (H1)–(H3). *Let f be a continuous mapping from V into H such that $|f(x)| \leq a(\|x\|^{2-\rho} + 1)$ for every $x \in V$, where ρ and a are some positive* *constants with* $0 < \rho < 2$. *Assume also that there exist a finite dimensional subspace* H_1 *of* H *and* $b > 0$ *which satisfy* (3.4) *and*

$$\langle y - f(x), x - 2Px\rangle \geq \omega\|x\|^2 - b \quad \text{for every } (x, y) \in A,$$

where P *is the orthogonal projection from* H *onto* H_1. *Then for every* $y \in H$, *there exists* $x \in D(A)$ *which satisfies* $Ax \ni f(x) + y$.

4. Proofs of Theorems

In this section, we give the proofs for our results.

Lemma 1. *Let* $T > 0$, *let* n *be a natural number and let* M *be a positive, symmetric* $n \times n$*-matrix. Let* Γ *be a function from* $L^2(0, T; \mathbb{R}^n)$ *into itself defined by* $\Gamma u = v$ *if* v *is the unique* T*-periodic solution of* $v'(t) + Mv(t) = 2Mu(t)$ *for* $0 \leq t \leq T$. *Let* Ω *be an open, bounded subset in* $L^2(0, T; \mathbb{R}^n)$ *containing* 0. *Then* $\deg(I - \Gamma, \Omega, 0) = (-1)^n$.

Proof. Since M is positive and symmetric, we may assume

$$M = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \quad \text{with } \lambda_1, \dots, \lambda_n > 0.$$

It is easy to see that Γ is a compact operator. We will show one is not an eigenvalue of Γ. Assume that $u \in L^2(0, T; \mathbb{R}^n)$ satisfies $\Gamma u = u$. Then for every i with $1 \leq i \leq n$, the i-th coordinate u_i of u is T-periodic and it satisfies $u_i'(t) - \lambda_i u_i(t) = 0$ for $0 \leq t \leq T$. So we have $u_i \equiv 0$ for every i, i.e., $u \equiv 0$. Hence one is not an eigenvalue of Γ. Similarly, we can show that two is the only eigenvalue of Γ greater than one and its corresponding eigenspace consists of all constant functions on $[0, T]$. So the dimension of the eigenspace is just n. Next we will show

$$\{u \in L^2(0, T; \mathbb{R}^n) : (2I - \Gamma)^2 u = 0\} = \{u \in L^2(0, T; \mathbb{R}^n) : (2I - \Gamma)u = 0\}. \tag{4.1}$$

Let u be a function such that $(2I - \Gamma)^2 u = 0$. Put $v = (2I - \Gamma)u$. Then v is a constant function and $2u' - Mv = 0$ for $0 \leq t \leq T$. Since v is constant and u is T-periodic, we have $v \equiv 0$, i.e., $(2I - \Gamma)u = 0$. So we have shown (4.1) and the algebraic multiplicity of the eigenvalue two is n. Hence by Theorem A, we obtain the desired result. □

We show a condition that the negative of a maximal monotone operator generates a compact semigroup; see also [15, Theorem 6.3].

Lemma 2. *Let* $(V, \|\cdot\|)$ *be a reflexive Banach space which is compactly imbedded into a Hilbert space* $(H, |\cdot|)$ *with an inner product* $\langle\cdot,\cdot\rangle$ *and let* A *be a maximal monotone subset of* $H \times H$ *which satisfies* $D(A) \subset V$ *and*

$$\langle y - q, x - p\rangle \geq \omega\|x - p\|^2 \quad \text{for every } (x, y), (p, q) \in A.$$

Then the negative of A *generates a compact semigroup.*

Proof. Let $\{S(t) : t \geq 0\}$ be the semigroup generated by $-A$ and let F be the closure of $D(A)$ with respect to the topology of H. Fix $t > 0$ and $y \in D(A)$. Let $\{x_n\}$ be a sequence in F which is bounded in H. By (2.3), there exists $M > 0$ such that $\int_0^t \|S(\tau)x_n - S(\tau)y\|^2 \, d\tau \leq M$ for every n. Set $E_n = \{\tau \in [0, T] : \|S(\tau)x_n - S(\tau)y\|^2 \leq 2M/t\}$. Since each Lebesgue measure of E_n is greater than or equal to $t/2$, $\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m$ has a positive Lebesgue measure. Let $\tau \in \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m$. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\|S(\tau)x_{n_i} - S(\tau)y\|^2 \leq 2M/t$ for every i. Since V is compactly imbedded into H, there exists $z \in H$ which is a cluster point of $\{S(\tau)x_{n_i} - S(\tau)y\}$ with respect to the topology of H. So we may assume $|S(\tau)x_{n_i} - S(\tau)y - z| \to 0$ as $i \to \infty$. Since $S(\tau)y - z \in F$, we know $|S(t)x_{n_i} - S(t-\tau)(S(\tau)y - z)| \to 0$ as $i \to \infty$. Hence $S(t)$ is compact with respect to the topology of H. □

Till the end of this section, we assume the conditions (H1)–(H4) and we also assume $|\cdot| \leq \|\cdot\|$ for the sake of simplicity. We identify H with its topological dual and we denote by $\langle \cdot, \cdot \rangle$ not only the inner product on H but also the dual pair of $V^* \times V$. We know that the natural mapping from H into V^* is one to one and continuous. So we identify H with a subset of V^*.

We define monotone subsets L_V and A_V of $V \times V^*$ as follows: L_V is the closure of L in $V \times V^*$ with respect to the strong topology of $V \times V^*$, $D(A_V) = V$ and for each $x \in D(A_V)$, $A_V x$ is the closure of the convex hull of the set which consists of all limit points of $\{y_n\}$ with respect to the weak topology of V^* such that $(x_n, y_n) \in A$ for every n and $\{x_n\}$ converges strongly to x in V.

Lemma 3. *L_V is single-valued, linear, maximal monotone in $V \times V^*$, A_V is maximal monotone in $V \times V^*$, and* (3.1), (3.2) *and* (3.3) *are satisfied with the replacement of L and A by L_V and A_V, respectively.*

Proof. First, we remark that $\|y\|_* \leq c(\|x\| + 1)$ for every $(x, y) \in A_V$. Assume that there exists $(x, y) \notin A_V$ such that $\langle q - y, p - x \rangle \geq 0$ for every $(p, q) \in A_V$. Then there exist $z \in V$ and $C > 0$ such that $\langle q - y, z \rangle \leq -C$ for every $q \in A_V x$. Let $u = z + x$. For every $t \in (0, 1)$, set $p_t = tx + (1-t)u$ and choose $q_t \in A_V p_t$. We may assume $\{q_t\}$ converges weakly to $q \in A_V x$ as $t \uparrow 1$. Since $0 \leq \langle q_t - y, p_t - x \rangle = (1-t)\langle q_t - y, u - x \rangle$, we have

$$0 \leq \langle q - y, u - x \rangle = \langle q - y, z \rangle \leq -C,$$

which is a contradiction. Hence A_V is maximal monotone in $V \times V^*$. It is easy to see that other assertions hold. □

We set a subset $A(\alpha)$ of $H \times H$ by

$$A(\alpha) = (\alpha A_V + (1 - \alpha) L_V) \cap (H \times H) \qquad \text{for } \alpha \in [0, 1]. \tag{4.2}$$

Lemma 4. *For every $\alpha \in [0, 1]$, $A(\alpha)$ is maximal monotone in $H \times H$ and its negative generates a compact semigroup.*

Proof. Let $\alpha \in [0, 1]$ and let $y \in H$. Since $\alpha A_V + (1 - \alpha) L_V$ is maximal monotone in $V \times V^*$, there exists $x \in D(\alpha A_V + (1 - \alpha) L_V)$ with $y \in x + \alpha A_V x + (1 - \alpha) L_V x$. Then we have $(x, y - x) \in A(\alpha)$. So we get $y \in R(I + A(\alpha))$. Since $y \in H$ is arbitrary, $A(\alpha)$ is maximal monotone in $H \times H$. By Lemma 2, we know that $-A(\alpha)$ generates a compact semigroup. □

Lemma 5. *For every $\alpha \in [0,1]$ and $g \in L^1(0,T;H)$, there exists a unique T-periodic, integral solution u of*

$$u'(t) + A(\alpha)u(t) \ni g(t) \qquad \text{for } 0 \le t \le T. \tag{4.3}$$

Proof. Let $\alpha \in [0,1]$ and let $g \in L^1(0,T;H)$. Let F be the closure of $D(A(\alpha))$ with respect to the topology of H. We define a mapping $U : F \to F$ by $Ux = u(T)$ for $x \in F$, where u is the unique integral solution of the initial value problem (4.3) with $u(0) = x$. From (2.2), we have $|Ux - Uy| \le e^{-\omega T}|x - y|$ for every $x, y \in F$. By the Banach contraction principle, U has the unique fixed point x. Then the integral solution u of (4.3) with $u(0) = x$ satisfies $u(0) = u(T)$. □

We define a mapping $G : L^1(0,T;H) \times [0,1] \to C(0,T;H) \cap L^2(0,T;V)$ by

$$G(g,\alpha) = u \qquad \text{for } (g,\alpha) \in L^1(0,T;H) \times [0,1],$$

where u is the unique T-periodic, integral solution of (4.3).

Let $C(0,T;H) \cap L^2(0,T;V)$ be endowed with the norm $\|\cdot\| = \|\cdot\|_{C(0,T;H)} + \|\cdot\|_{L^2(0,T;V)}$.

Lemma 6. *For every bounded subset B of $L^1(0,T;H)$, $G(B \times [0,1])$ is bounded in $C(0,T;H) \cap L^2(0,T;V)$.*

Proof. Let B be a bounded subset of $L^1(0,T;H)$. Fix $x \in D(A) \cap D(L)$ and $y \in Ax$. Let $(g,\alpha) \in B \times [0,1]$ and let $u = G(g,\alpha)$. Set $m = \min\{|u(t) - x| : 0 \le t \le T\}$ and $M = \max\{|u(t) - x| : 0 \le t \le T\}$. Then we have $M \le m + T\max\{|y|, |Lx|\} + \int_0^T |g(t)|\,dt$ and

$$\begin{aligned}
\omega T m^2 &\le \omega \int_0^T |u(t) - x|^2\,dt \le \omega \int_0^T \|u(t) - x\|^2\,dt \\
&\le \int_0^T \langle g(t) - (\alpha y + (1-\alpha)Lx), u(t) - x\rangle\,dt \\
&\le M\Big(T\max\{|y|, |Lx|\} + \int_0^T |g(t)|\,dt\Big).
\end{aligned}$$

So $C \equiv \sup\{|G(g,\alpha)(t)| : g \in B, \alpha \in [0,1], t \in [0,T]\} < \infty$. Since we have

$$\begin{aligned}
\omega \int_0^T \|u(t) - x\|^2\,dt &\le \int_0^T \langle g(t) - (\alpha y + (1-\alpha)Lx), u(t) - x\rangle\,dt \\
&\le (C + |x|)\Big(T\max\{|y|, |Lx|\} + \int_0^T |g(t)|\,dt\Big),
\end{aligned}$$

$G(B \times [0,1])$ is bounded in $C(0,T;H) \cap L^2(0,T;V)$. □

Lemma 7. *For every bounded subset B of $L^1(0,T;H)$, G is a uniformly continuous mapping from $B \times [0,1]$ into $C(0,T;H) \cap L^2(0,T;V)$.*

Proof. Let $g, h \in L^1(0,T;H)$ and let $\alpha, \beta \in [0,1]$. Set $u = G(g,\alpha)$ and $v = G(h,\beta)$. Let $\eta > 0$. Choose $x_\eta \in D(A(\alpha))$, $y_\eta \in D(A(\beta))$ and $g_\eta, h_\eta \in W^{1,1}(0,T;H)$ such that

$$|u(0) - x_\eta| < \eta, |v(0) - y_\eta| < \eta, \int_0^T |g(t) - g_\eta(t)|\,dt < \eta \text{ and } \int_0^T |h(t) - h_\eta(t)|\,dt < \eta.$$

Let u_η and v_η be the strong solutions of the initial value problems

$$\begin{cases} u_\eta(0) = x_\eta, & u'_\eta(t) + A(\alpha)u_\eta(t) \ni g_\eta(t) \quad \text{for } 0 \le t \le T; \\ v_\eta(0) = y_\eta, & v'_\eta(t) + A(\beta)v_\eta(t) \ni h_\eta(t) \quad \text{for } 0 \le t \le T, \end{cases}$$

respectively. We choose functions w_η and z_η which satisfy

$$\begin{cases} w_\eta(t) \in A_V u_\eta(t), & u'_\eta(t) + \alpha w_\eta(t) + (1-\alpha)L_V u_\eta(t) = g_\eta(t); \\ z_\eta(t) \in A_V v_\eta(t), & v'_\eta(t) + \beta z_\eta(t) + (1-\beta)L_V v_\eta(t) = h_\eta(t) \end{cases}$$

for almost every $t \in [0,T]$, respectively. Since

$$\begin{aligned} u'_\eta(t) - v'_\eta(t) + \alpha(w_\eta(t) - z_\eta(t)) + (1-\alpha)(L_V u_\eta(t) - L_V v_\eta(t)) \\ = g_\eta(t) - h_\eta(t) + (\beta - \alpha)(z_\eta(t) - L_V v_\eta(t)) \end{aligned}$$

for almost every $t \in [0,T]$, we have

$$\begin{aligned} &\frac{1}{2}|u_\eta(T) - v_\eta(T)|^2 - \frac{1}{2}|x_\eta - y_\eta|^2 + \omega \int_0^T \|u_\eta(t) - v_\eta(t)\|^2\,dt \\ &\quad \le \sup_{0\le\tau\le T} |u_\eta(\tau) - v_\eta(\tau)| \int_0^T |g_\eta(t) - h_\eta(t)|\,dt \\ &\qquad + 2|\alpha-\beta| \left(\int_0^T (c\|v_\eta(t)\| + 1)^2\,dt \right)^{\frac{1}{2}} \left(\int_0^T \|u_\eta(t) - v_\eta(t)\|^2\,dt \right)^{\frac{1}{2}}. \end{aligned}$$

Since u is T-periodic and $\eta > 0$ is arbitrary, we obtain

$$\begin{aligned} &\omega \int_0^T \|u(t) - v(t)\|^2\,dt \\ &\qquad \le \sup_{0\le\tau\le T} |u(\tau) - v(\tau)| \int_0^T |g(t) - h(t)|\,dt \\ &\qquad\quad + 2|\alpha-\beta| \left(\int_0^T (c\|v(t)\| + 1)^2\,dt \right)^{\frac{1}{2}} \left(\int_0^T \|u(t) - v(t)\|^2\,dt \right)^{\frac{1}{2}} \\ &\qquad \le C \int_0^T |g(t) - h(t)|\,dt + C|\alpha - \beta|, \end{aligned}$$

where C is some constant which can be chosen by Lemma 6. So $G : B \times [0,1] \to L^2(0,T;V)$ is uniformly continuous.

By the similar way, we have

$$\begin{aligned} |u(0) - v(0)|^2 &= |u(T) - v(T)|^2 \\ &\le e^{-2\omega T}|u(0) - v(0)|^2 + C \int_0^T |g(t) - h(t)|\,dt + C|\alpha - \beta|, \end{aligned}$$

and hence we obtain

$$|u(t) - v(t)|^2 \le C \left(\frac{e^{-2\omega t}}{1 - e^{-2\omega T}} + 1 \right) \left(\int_0^T |g(t) - h(t)|\,dt + |\alpha - \beta| \right)$$

for every $t \in [0,T]$. Hence $G : B \times [0,1] \to C(0,T;H)$ is also uniformly continuous. $\square$

Lemma 8. *For every uniformly integrable subset B of $L^1(0,T;H)$, $G(B \times [0,1])$ is relatively compact in $C(0,T;H) \cap L^2(0,T;V)$.*

Proof. Let B be a uniformly integrable subset of $L^1(0,T;H)$. Fix $\alpha \in [0,1]$. We know that $\{G(g,\alpha)(0) : g \in B\}$ is bounded in H by Lemma 6. By Theorem B, $\{G(g,\alpha)(0) : g \in B\} = \{G(g,\alpha)(T) : g \in B\}$ is relatively compact in H. Using Theorem B again, we know $G(B,\alpha)$ is relatively compact in $C(0,T;H)$. Next, we will show that $G(B,\alpha)$ is relatively compact in $L^2(0,T;V)$ by the method employed in the proof of [16, Theorem 3.1]. Fix $\eta > 0$. Then there exists $\{f_1, \ldots, f_n\} \subset B$ such that for every $g \in B$, there exists i such that $|G(g,\alpha)(t) - G(f_i,\alpha)(t)| \leq \eta$ for every $t \in [0,T]$. Since $\omega \int_0^T \|G(g,\alpha)(t) - G(f_i,\alpha)(t)\|^2 \, dt \leq \eta \int_0^T |g(t) - f_i(t)| \, dt$ and B is bounded in $L^1(0,T;H)$, $G(B,\alpha)$ is totally bounded in $L^2(0,T;V)$. So $G(B,\alpha)$ is relatively compact in $C(0,T;H) \cap L^2(0,T;V)$. From the previous lemma, $G(B \times [0,1])$ is relatively compact in $C(0,T;H) \cap L^2(0,T;V)$. $\square$

We denote by B_r the open ball in $L^2(0,T;V)$ with center 0 and radius $r > 0$. Theorem 2 is a direct consequence of the following proposition.

Proposition 1. *Assume the assumptions in Theorem 2. Then for every $g \in L^2(0,T;H)$, there exists $R_0 > 0$ such that for every $R \geq R_0$, there exists $\delta > 0$ such that*

$$\deg(I - \mathcal{H}_h(\cdot,1), B_R, 0) = (-1)^{\dim H_1}$$

for every $h \in L^1(0,T;H)$ with $\int_0^T |h(t) - g(t)| \, dt \leq \delta$, where $\mathcal{H}_h : L^2(0,T;V) \times [0,1] \to L^2(0,T;V)$ is defined by

$$\mathcal{H}_h(u,\alpha)(t) = G(\alpha f(\cdot,u) + (1-\alpha) 2LPu + \alpha h, \alpha)(t)$$

for $(u,\alpha) \in L^2(0,T;V) \times [0,1]$.

Proof. First, we remark that $\mathcal{H}_h : L^2(0,T;V) \times [0,1] \to L^2(0,T;V)$ is compact from the assumption (H4) and Lemma 8 and that (3.5) is satisfied by A_V with the replacement of A. We also remark that $\langle L_V x - 2LPx, x - 2Px \rangle \geq \omega \|x\|^2/2$ for every $x \in V$. Indeed, we have

$$\langle Lx - 2LPx, x - 2Px \rangle = \langle L(x - Px), x - Px \rangle + \langle LPx, Px \rangle \geq \frac{\omega}{2} \|x\|^2$$

for every $x \in D(L)$ and P is continuous with respect to the strong topology of V by the finite dimensionality of H_1. Let $g \in L^2(0,T;H)$. Choose $R_0 > 0$ such that $\omega R_0^2/4 - \int_0^T |b(t)| \, dt - \int_0^T |g(t)|^2 \, dt/\omega > 0$. Fix $R \geq R_0$. By Lemma 6, there exists $M > 0$ such that $\sup_t |G(\alpha f(\cdot,u) + (1-\alpha)2LPu + \alpha g, \alpha)(t)| \leq M/2$ for every $(u,\alpha) \in B_R \times [0,1]$. Fix $\delta > 0$ satisfying

$$\frac{\omega}{4} R^2 - \int_0^T |b(t)| \, dt - \frac{1}{\omega} \int_0^T |g(t)|^2 \, dt - M\delta > 0$$

and

$$\sup_{0 \leq t \leq T} \left| G(\alpha f(\cdot,u) + (1-\alpha)2LPu + \alpha h, \alpha)(t) \right| \leq M$$

for every $(u,\alpha,h) \in B_R \times [0,1] \times L^1(0,T;H)$ with $\int_0^T |h(t) - g(t)| \, dt \leq \delta$. Fix $h \in L^1(0,T;H)$ with $\int_0^T |h(t) - g(t)| \, dt \leq \delta$. We will show that there is no $(u,\alpha) \in$

$\partial B_R \times [0,1]$ with $\mathcal{H}_h(u,\alpha) = u$. Suppose not. Then there exists $(u,\alpha) \in \partial B_R \times [0,1]$ such that u is a T-periodic, integral solution of $u'(t) + A(\alpha)u(t) \ni \alpha f(t,u(t)) + (1-\alpha)2LPu(t) + \alpha h(t)$. We remark $\sup_t |u(t)| \le M$. Let $\eta > 0$. We can choose $x_\eta \in D(A(\alpha))$ and $k_\eta \in W^{1,1}(0,T;H)$ such that $|x_\eta - u(0)| < \eta$ and $\int_0^T |\alpha f(t,u(t)) + (1-\alpha)2LPu(t) + \alpha h(t) - k_\eta(t)|\,dt < \eta$. For this pair (x_η, k_η), there exists the strong solution u_η of the initial value problem

$$u_\eta(0) = x_\eta, \quad u'_\eta(t) + A(\alpha)u_\eta(t) \ni k_\eta(t) \quad \text{for } 0 \le t \le T. \tag{4.4}$$

By (2.2), we have $|u_\eta(t)| \le |u(t)| + |u(t) - u_\eta(t)| \le M + 2\eta$ for every $t \in [0,T]$. We set $u_\eta^2 = (I-P)u_\eta$ and $u_\eta^1 = Pu_\eta$. Since u_η is the strong solution of (4.4), there exists a function w_η such that $w_\eta(t) \in A_V u_\eta(t)$ and $u'_\eta(t) + \alpha w_\eta(t) + (1-\alpha)L_V u_\eta(t) - k_\eta(t) = 0$ almost everywhere on $[0,T]$. From

$$\begin{aligned}
&\big\langle u'_\eta(t) + \alpha w_\eta(t) + (1-\alpha)L_V u_\eta(t) - k_\eta(t), u_\eta^2(t) - u_\eta^1(t)\big\rangle \\
&= \frac{1}{2}\big(|u_\eta^2(t)|^2 - |u_\eta^1(t)|^2\big)' + \alpha\langle w_\eta(t) - f(t,u_\eta(t)), u_\eta^2(t) - u_\eta^1(t)\rangle \\
&\quad + (1-\alpha)\langle L_V u_\eta(t) - 2LPu_\eta(t), u_\eta^2(t) - u_\eta^1(t)\rangle \\
&\quad + \big\langle \alpha f(t,u_\eta(t)) + (1-\alpha)2LPu_\eta(t) - \alpha f(t,u(t)) - (1-\alpha)2LPu(t), u_\eta^2(t) - u_\eta^1(t)\big\rangle \\
&\quad + \big\langle \alpha f(t,u(t)) + (1-\alpha)2LPu(t) + \alpha h(t) - k_\eta(t), u_\eta^2(t) - u_\eta^1(t)\big\rangle \\
&\quad - \alpha\langle h(t) - g(t), u_\eta^2(t) - u_\eta^1(t)\rangle - \alpha\langle g(t), u_\eta^2(t) - u_\eta^1(t)\rangle,
\end{aligned}$$

we have

$$\begin{aligned}
0 \ge{}& \frac{1}{2}|u_\eta^2(T)|^2 - \frac{1}{2}|u_\eta^2(0)|^2 + \frac{1}{2}|u_\eta^1(0)|^2 - \frac{1}{2}|u_\eta^1(T)|^2 \\
&+ \frac{\omega}{2}\int_0^T \|u_\eta(t)\|^2\,dt - \int_0^T |b(t)|\,dt \\
&- (M+2\eta)\bigg(\int_0^T |f(t,u_\eta(t)) - f(t,u(t))|\,dt + 2\int_0^T |LPu_\eta(t) - LPu(t)|\,dt + \eta\bigg) \\
&- (M+2\eta)\delta - \frac{\omega}{4}\int_0^T \|u_\eta(t)\|^2\,dt - \frac{1}{\omega}\int_0^T |g(t)|^2\,dt.
\end{aligned}$$

Since u is T-periodic and $\eta > 0$ is arbitrary, we have

$$0 \ge \frac{\omega}{4}R^2 - \int_0^T |b(t)|\,dt - \frac{1}{\omega}\int_0^T |g(t)|^2\,dt - M\delta > 0,$$

which is a contradiction. So we have shown that for every $(u,\alpha) \in \partial B_R \times [0,1]$, $\mathcal{H}_h(u,\alpha) \ne u$. By the properties of the Leray-Schauder degree and Lemma 1, we obtain

$$\deg(I - \mathcal{H}_h(\cdot,1), B_R, 0) = \deg(I - \mathcal{H}_h(\cdot,0), B_R, 0) = (-1)^{\dim H_1},$$

which is the desired result. □

Theorem 3 is a direct consequence of the following proposition.

Proposition 2. *Assume the assumptions in Theorem 3. Then there exists* $r_0 > 0$ *such that for every* $r \in (0, r_0]$, *there exists* $\rho > 0$ *such that*

$$\deg(I - \mathcal{K}_h(\cdot, 1), B_r, 0) = (-1)^{\dim H_2}$$

for every $h \in L^1(0, T; H)$ *with* $\int_0^T |h(t)|\, dt \leq \rho$, *where* $\mathcal{K}_h : L^2(0, T; V) \times [0, 1] \to L^2(0, T; V)$ *is defined by*

$$\mathcal{K}_h(u, \alpha)(t) = G(\alpha f(\cdot, u) + (1 - \alpha)2LQu + \alpha h, \alpha)(t)$$

for $(u, \alpha) \in L^2(0, T; V) \times [0, 1]$.

Proof. There exists $r_0 > 0$ such that $\sup_t |G(\alpha f(\cdot, u) + (1 - \alpha)2LQu, \alpha)(t)| \leq \varepsilon/2$ for every $(u, \alpha) \in B_{r_0} \times [0, 1]$ by Lemma 7. Fix $r \in (0, r_0]$. Choose $\rho > 0$ such that $\omega r^2/2 - \varepsilon\rho > 0$ and $\sup_t |G(\alpha f(\cdot, u) + (1 - \alpha)2LQu + \alpha h, \alpha)(t)| \leq \varepsilon$ for every $(u, \alpha, h) \in B_{r_0} \times [0, 1] \times L^1(0, T; H)$ with $\int_0^T |h(t)|\, dt \leq \rho$. Fix $h \in L^1(0, T; H)$ with $\int_0^T |h(t)|\, dt \leq \rho$. We will show that there is no $(u, \alpha) \in \partial B_r \times [0, 1]$ with $\mathcal{K}_h(u, \alpha) = u$. Suppose not. Then there exists $(u, \alpha) \in \partial B_r \times [0, 1]$ such that u is a T-periodic, integral solution of $u'(t) + A(\alpha)u(t) \ni \alpha f(t, u(t)) + (1 - \alpha)2LQu(t) + \alpha h(t)$. We remark $\sup_t |u(t)| \leq \varepsilon$. By the same lines as those in the proof of Proposition 1, we obtain

$$0 \geq \frac{\omega}{2} \int_0^T \|u(t)\|^2\, dt - \varepsilon \int_0^T |h(t)|\, dt \geq \frac{\omega}{2} r^2 - \varepsilon\rho > 0,$$

which is a contradiction. Hence there is no $(u, \alpha) \in \partial B_r \times [0, 1]$ with $\mathcal{K}_h(u, \alpha) = u$. Therefore we have

$$\deg(I - \mathcal{K}_h(\cdot, 1), B_r, 0) = \deg(I - \mathcal{K}_h(\cdot, 0), B_r, 0) = (-1)^{\dim H_2}$$

by the properties of the Leray-Schauder degree and Lemma 1. □

We give the proof of Corollary.

Proof of Corollary. Let $y \in H$. From Theorem 2, for every n, there exists a $1/2^n$-periodic integral solution u_n of

$$u_n'(t) + Au_n(t) \ni f(u_n(t)) + y \quad \text{for } 0 \leq t \leq 1.$$

By the same lines as those in the proof of Proposition 1, we have $\int_0^1 \|u_n(t)\|^2\, dt \leq |y|/\omega^2 + 2b/\omega$ for every n, and hence $\{f(u_n(\cdot)) + y\}$ is uniformly integrable in $L^1(0, 1; H)$. So $\{u_n\}$ is relatively compact in $C(0, 1; H) \cap L^2(0, 1; V)$ by Lemma 8. Hence there exists a constant function $u \in C(0, 1; H) \cap L^2(0, 1; V)$ which is a cluster point of $\{u_n\}$. Set $x = u(0)$. Since

$$\langle f(x) + y - q, x - p\rangle = \int_0^1 \langle f(u(t)) + y - q, u(t) - p\rangle\, dt \geq |u(1) - p| - |u(0) - p| = 0$$

for every $(p, q) \in A$, we have $(x, f(x) + y) \in A$. □

5. An example

Let $\Omega(\subset \mathbb{R}^N)$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the nonlinear differential equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(\frac{\partial u}{\partial x_i}\right) + g(t,x,u,\nabla u) + h(t,x) & \text{in } [0,T]\times\Omega, \\ u(t,x) = 0 & \text{on } [0,T]\times\partial\Omega, \end{cases}$$

where $a_i \in C^1(\mathbb{R})$ such that $a_i(0) = 0$ and $0 < \inf_s a_i'(s) \le \sup_s a_i'(s) < \infty$, $h : [0,T]\times\Omega \to \mathbb{R}$ is measurable with $\int_0^T (\int_\Omega |h(t,x)|^2\,dx)^{1/2}\,dt < \infty$, and $g : [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ such that for almost every $(t,x) \in [0,T]\times\Omega$, $g(t,x,\cdot,\cdot)$ is continuous and for every $(u,v) \in \mathbb{R}\times\mathbb{R}^N$, $g(\cdot,\cdot,u,v)$ is measurable.

Let $\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$ be the eigenvalues of the operator $-\Delta$ with homogeneous Dirichlet boundary condition.

Theorem 4. *Let $\varepsilon > 0$. Assume that*

$$\begin{aligned} &1-\varepsilon < a_i'(s) < 1+\varepsilon \quad \textit{for every } i = 1,\dots,N \textit{ and } s \in \mathbb{R}, \\ &g(\cdot,\cdot,0,\cdot) \equiv 0, \textit{ and} \\ &\alpha \le \frac{g(t,x,u,v)}{u} \le \beta \quad \textit{for every } (t,x,u,v) \in [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^N \textit{ with } u \ne 0, \end{aligned}$$

where α and β are some constants. Assume also that there exist natural numbers k and l such that $k-l$ is odd,

$$\begin{aligned} &(1+2\varepsilon)\lambda_k < \lim_{|u|\to\infty} \frac{g(t,x,u,v)}{u} < (1-2\varepsilon)\lambda_{k+1} \textit{ uniformly in } (t,x,v), \textit{ and} \\ &(1+2\varepsilon)\lambda_l < \lim_{|u|\to 0} \frac{g(t,x,u,v)}{u} < (1-2\varepsilon)\lambda_{l+1} \textit{ uniformly in } (t,x,v). \end{aligned}$$

If $\int_0^T (\int_\Omega |h(x,t)|^2\,dx)^{1/2}\,dt$ is sufficiently small, then the problem has at least two T-periodic solutions.

Proof. We define operators A and L on $(L^2(\Omega), |\cdot|)$ and $f : [0,T]\times H_0^1(\Omega) \to L^2(\Omega)$ as follows:

$$\begin{cases} Au = -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(\frac{\partial u}{\partial x_i}\right), & u \in D(A) = H^2(\Omega)\cap H_0^1(\Omega); \\ Lu = -\Delta u, & u \in D(L) = H^2(\Omega)\cap H_0^1(\Omega); \\ f(t,u)(x) = g(t,x,u(x),\nabla u(x)), & (t,u) \in [0,T]\times H_0^1(\Omega). \end{cases}$$

Let $\varphi_1, \varphi_2, \varphi_3, \cdots$ be eigenfunctions for L corresponding to $\lambda_1, \lambda_2, \lambda_3, \cdots$, respectively. Let $H_1 = \operatorname{span}\{\varphi_1, \dots, \varphi_k\}$ and let P be the orthogonal projection from $L^2(\Omega)$ onto H_1. Let $u \in H^2(\Omega)\cap H_0^1(\Omega)$. Then, putting $u_2 = u - Pu$ and $u_1 = Pu$, we have

$$\begin{aligned}\langle Au, u_2 - u_1\rangle &\geq (1-\varepsilon)|\nabla u_2|^2 - (1+\varepsilon)|\nabla u_1|^2 \\ &= \varepsilon|\nabla u|^2 + (1-2\varepsilon)|\nabla u_2|^2 - (1+2\varepsilon)|\nabla u_1|^2 \\ &\geq \varepsilon|\nabla u|^2 + (1-2\varepsilon)\lambda_{k+1}|u_2|^2 - (1+2\varepsilon)\lambda_k|u_1|^2\end{aligned}$$

and

$$-\int_\Omega g(t,x,u,\nabla u)(u_2-u_1)\,dx \geq (1+2\varepsilon)\lambda_k|u_1|^2 - (1-2\varepsilon)\lambda_{k+1}|u_2|^2 - C,$$

where C is some constant which is independent of u. So we have

$$\langle Au - f(t,u), u_2 - u_1\rangle \geq \varepsilon|\nabla u|^2 - C.$$

Let $H_2 = \operatorname{span}\{\varphi_1, \dots, \varphi_l\}$ and let Q be the orthogonal projection from $L^2(\Omega)$ onto H_2. Fix $m \geq l$ with $(1-2\varepsilon)\lambda_{m+1} > \beta$ and let R be the orthogonal projection from $L^2(\Omega)$ onto $\operatorname{span}\{\varphi_{l+1}, \dots, \varphi_m\}$. For each $v \in H^2(\Omega) \cap H_0^1(\Omega)$, we set $v_1 = Qv$, $v_2 = Rv$ and $v_3 = v - Qv - Rv$. Then we can show

$$\begin{aligned}\langle Av, v_3 + v_2 - v_1\rangle \geq \varepsilon|\nabla v|^2 &+ (1-2\varepsilon)\lambda_{m+1}|v_3|^2 \\ &+ (1-2\varepsilon)\lambda_{l+1}|v_2|^2 - (1+2\varepsilon)\lambda_l|v_1|^2\end{aligned}$$

and

$$\begin{aligned}\varliminf_{|v|\to 0} \frac{1}{|v|^2}\Big(&-\int_\Omega g(t,x,v,\nabla v)(v_3+v_2-v_1)\,dx \\ &- (1+2\varepsilon)\lambda_l|v_1|^2 + (1-2\varepsilon)\lambda_{l+1}|v_2|^2 + \beta|v_3|^2\Big) \geq 0\end{aligned}$$

by the same lines as those in the proof of [7, Lemma 4]. So we have

$$\langle Av - f(t,v), v_3 + v_2 - v_1\rangle \geq \frac{\varepsilon}{2}|\nabla v|^2$$

if $|v|$ is sufficiently small. Hence from Theorem 1, we obtain the desired result. □

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Norimichi Hirano
Department of Mathematics, Faculty of Engineering, Yokohama National University, Tokiwadai, Hodogaya-ku, Yokohama 240-8501, Japan
E-mail address: `hirano@math.sci.ynu.ac.jp`

Naoki Shioji
Department of Mathematics, Faculty of Engineering, Yokohama National University, Tokiwadai, Hodogaya-ku, Yokohama 240-8501, Japan
E-mail address: `shioji@math.sci.ynu.ac.jp`