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CORE EQUIVALENCE IN TOPOLOGICAL RIESZ SPACES WITHOUT CONVEXITY

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Abstract. The paper studies exchange economies with infinite-dimensional commodity spaces in the setting of Riesz spaces. A core equivalence theorem is presented under conditions without convexity of preferences of consumers.

1. Introduction

Debreu and Scarf[3] proved a limit theorem on the core of an economy rigorously for an arbitrary finite number of consumers with a convexity condition for preferences of consumers. The limit theorem asserts the following: Consider exchange economies consisting of r consumers of each type for $r = 1, 2, 3 \ldots$; then an allocation which assigns the same commodity bundles to all consumers of the same type and which is in the core for all r must be a Walrasian equilibrium.

McKenzie[5] pointed out that the convexity assumption supposed in [3] is not necessarily essential to obtain the limit theorem on the core and proved it under some types of conditions without convexity, see also [6] and [7].

It is our purpose to remove convexity assumption for the limit theorem on the core in infinite-dimensional commodity spaces in the setting of Riesz spaces.

2. Definitions

Let E be a locally solid-convex topological Riesz space with the lattice partial order $\geq$, and let m be a positive integer. For each $i = 1, \ldots, m$, let $\omega_i > 0$ and $\succeq_i$ a symmetric, complete and transitive binary relation in E. We call the pair $(E, \{\omega_i, \succeq_i\colon i = 1, \ldots, m\})$ of E and $\{\omega_i, \succeq_i\colon i = 1, \ldots, m\}$ an *exchange economy* and denote it by $\mathcal{E}$. We call $i = 1, \ldots, m$ a consumer of the economy $\mathcal{E}$, the binary relation $\succeq_i$ the *preference* of the consumer i, and ω_i the initial endowment of the consumer i. The sum $\omega = \sum_{i=1}^{m} \omega_i$ is called the *total endowment* of the exchange economy $\mathcal{E}$. Define a new binary relation $\succ_i$ in E by $x \succ_i y \Leftrightarrow x \succeq_i y$ and $y \not\succeq_i x$.

We list some properties of preferences $\succeq_i$:

(1) Monotonicity
$x \geq y$ implies $x \succeq_i y$ for each $x, y \in E$.
(2) Strict monotonicity
$x > y$ implies $x \succ_i y$ for each $x, y \in E$.
(3) Algebraic continuity
the sets $\{y \in E : y \succ_i x\}$ is algebraicly open in E for all $x \in E_+$.

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A preference $\succeq_i$ on E is said to be *uniformly proper* whenever there exists some $v_i > 0$ and some neighborhood V_i of 0 such that for any arbitrary $x \in E$ satisfying $x - \alpha v_i + z \succeq x$ in E with $\alpha > 0$ we have $z \notin \alpha V_i$.

For a positive integer r, the *r-fold replica economy* $\mathcal{E}_r$ of an exchange economy $\mathcal{E}$ is a new exchange economy having the following characteristics. The economy $\mathcal{E}_r$ has rm consumers indexed by (i,j), $(i = 1,\dots,m; j = 1,\dots,r)$ such that each consumer (i,j) has

(1) a preference $\succeq_{ij}$ equal to $\succeq_i$; and
(2) an initial endowment ω_{ij} equal to ω_i, and so the total endowment of the r-fold replica economy $\mathcal{E}_r$ is $\sum_{j=1}^{r}\sum_{i=1}^{m}\omega_{ij} = r\omega$.

A tuple $(x_{ij} : i = 1,\dots,m; j = 1,\dots,r)$ of elements of E is said to be an *allocation* of r-fold replica economy $\mathcal{E}_r$ whenever

$$\mathrm{co}\{x_{ij} : j = 1,\dots,r\} \cap E_+ \neq \emptyset \text{ and } \sum_{i=1}^{m}\sum_{j=1}^{r} x_{ij} = r\sum_{i=1}^{m}\omega_i = r\omega.$$

Note that an allocation is not restricted to be non-negative elements in our definition and only required that, for each i, some convex combination of $x'_{ij}s$ is non-negative.

An allocation $(x_{ij} : i = 1,\dots,m; j = 1,\dots,r)$ of $\mathcal{E}_r$ is said to be a *core allocation* if there are no subsets $S_1,\dots,S_m$ of $\{1,\dots,r\}$ and no allocation $(y_{ij} : i = 1,\dots,m; j = 1,\dots,r)$ of $\mathcal{E}_r$ such that $\cup_{i=1}^{m} S_i \neq \emptyset$, and $y_{ij} \succ_i x_{ij}$ for $i = 1,\dots,m$ and $j \in S_i$.

¿From any allocation $(x_1,\dots,x_m)$ of an exchange economy $\mathcal{E}$ we can construct an allocation of any r-fold replica economy $\mathcal{E}_r$ of $\mathcal{E}$ by letting $x_{ij} = x_i$ for $i = 1,\dots,m$ and $j = 1,\dots,r$. This type of allocations are referred to as *r-equal treatment allocations* and denoted by $et_r(x_1,\dots,x_m)$, that is,

$$et_r(x_1,\dots,x_m) = (\overbrace{\underbrace{x_1,\dots,x_1}_{r \text{ terms}},\dots,\underbrace{x_m,\dots,x_m}_{r \text{ terms}}}^{rm \text{ terms}}).$$

An allocation in an exchange economy is said to be a *strong Edgeworth equilibrium* whenever its r-equal treatment allocation is a core allocation of the r-fold replica economy $\mathcal{E}_r$ for every $r = 1,2,3,\dots$.

An allocation $(x_1,\dots,x_m)$ of an exchange economy $\mathcal{E}$ is said to be a *Walrasian (or competitive) equilibrium* of $\mathcal{E}$ if there is a non-zero continuous linear functional p on E such that

$$p(x) \leq p(\omega_i) \text{ implies } x_i \succeq_i x,$$

and a *quasiequilibrium* of $\mathcal{E}$ if there is a non-zero continuous linear functional p on E such that

$$x \succeq_i x_i \text{ implies } p(x) \geq p(\omega_i).$$

3. Results

The following theorem is our main result. It is easily seen that if an allocation is a Walrasian equilibrium of an exchange economy then it is a strong Edgeworth equilibrium. The theorem with its corollary asserts the inverse of this result and this type of theorems are sometimes called core equivalence theorems as well as limit theorems of cores.

Theorem 1. *Suppose that in an exchange economy $\mathcal{E} = (E, \{\omega_i, \succeq_i \ i = 1, \ldots, m\})$ all preferences $\succeq_i$ are uniformly proper and monotone. Then every strong Edgeworth equilibrium is a quasiequilibrium.*

Proof. Suppose that $(x_1, \ldots, x_m)$ is a strong Edgeworth equilibrium of the exchange economy $\mathcal{E}$. Define

$$P_i = \{x \in E_+ : x \succeq_i x_i\} \text{ and } G_i = P_i - \omega_i = \{x \in E : x + \omega_i \succeq_i x_i\},$$

and let $G = \operatorname{co} \bigcup_{i=1}^m G_i$.

By the uniform properness of the preferences, for each i there is a convex, solid, open neighborhood V_i of 0 and some $v_i > 0$ such that $x - \alpha v_i + z \succeq_i x$ in E with $\alpha > 0$ imply $z \notin \alpha V_i$. Put $V = \cap_{i=1}^m V_i$ and $v = v_1 + \cdots + v_m$, and consider a non-empty, convex, open cone C

$$C = \bigcup_{\alpha > 0} \alpha(\frac{1}{2}V - v)$$

We claim that $C \cap G = \emptyset$. Assume by way of contradiction that $C \cap G \neq \emptyset$ and let $a \in C \cap G$. Since $a \in C$, there is $\alpha > 0$ such that $a + \alpha v \in (\alpha/2)V$. On the other hand, since $a \in G$ There are a positive integer l, $z_{ij} \succeq_i x_i (i = 1, \ldots, m; j = 1, \ldots, l)$, $\lambda_j \geq 0 (j = 1, \ldots, l)$, $\mu_{ij} \geq 0 (i = 1, \ldots, m; j = 1, \ldots, l)$, such that $a = \sum_{i=1}^m \sum_{j=1}^l \lambda_i \mu_{ij}(z_{ij} - \omega_i)$, $\sum_{i=1}^q \lambda_i = 1$, $\sum_{j=1}^l \mu_{ij} = 1 (i = 1, \ldots, m)$. By approximating λ_i and μ_{ij} by rational numbers, we can find positive rational numbers $b_{ij} (i = 1, \ldots, m; j = 1, \ldots, l)$, positive integers n and $n_i (i = 1, \ldots, m)$ such that

$$\sum_{i=1}^m \sum_{j=1}^l \frac{n_i}{n} b_{ij}(z_{ij} - \omega_i) - a \in \frac{\alpha}{2}V,$$

$\sum_{i=1}^m n_i = n$, $\sum_{j=1}^l b_{ij} = 1 (i = 1, \ldots, m)$, and each $n_i b_{ij}$ is a positive integers.

Consequently, it follows that

$$\sum_{i=1}^m \sum_{j=1}^l n_i b_{ij}(z_{ij} - \omega_i) + \alpha n v \in \alpha n V,$$

and put

$$y = \sum_{i=1}^m \sum_{j=1}^l n_i b_{ij} \omega_i - \alpha n v \text{ and } z = \sum_{i=1}^m \sum_{j=1}^l n_i b_{ij} z_{ij}.$$

Since $z - y = z + \alpha n v - \sum_{i=1}^m \sum_{j=1}^l n_i b_{ij} \omega_i \leq z + \alpha n v$, it follows that

$$(y - z)^- = (z - y)^+ \leq z + \alpha n v = \sum_{i=1}^m \big(\sum_{j=1}^l n_i b_{ij} z_{ij} + \alpha n v_i \big).$$

Therefore, by the Riesz decomposition property, there are $u_i \in E_+ (i = 1, \ldots, m)$ with $0 \leq u_i \leq \sum_{j=1}^l n_i b_{ij} z_{ij} + \alpha n v_i$ and $\sum_{i=1}^m u_i = (y - z)^-$. Now let

$$y_{ij} = z_{ij} + \frac{\alpha n}{n_i} v_i - \frac{1}{n_i} u_i \text{ for } i = 1, \ldots, m \text{ and } j = 1, \ldots, l.$$

It is easily seen that $\sum_{j=1}^{l} b_{ij} y_{ij} \geq 0$ for all i, that is, $\mathrm{co}\{y_{ij} : j = 1, \ldots, l\} \cap E_+ \neq \emptyset$. It follows that $y_{ij} \succ_i z_{ij}$ for all i and j. Indeed, if this is not true, then we must have $y_{ij} \preceq_i z_{ij} = y_{ij} - (\alpha n/n_i)v_i + (1/n_i)u_i$ for some i and j, which implies $\frac{1}{n_i}u_i \notin \frac{\alpha n}{n_i}V$, or $u_i \notin \alpha nV$. On the other hand, since

$$0 \leq u_i \leq (y-z)^- \leq |y-z| = \left| \sum_{i=1}^{m}\sum_{j=1}^{l} n_i b_{ij}\omega_i - \alpha n v - \sum_{i=1}^{m}\sum_{j=1}^{l} n_i b_{ij} z_{ij} \right| \in \alpha nV,$$

we see that $u_i \in \alpha nV$, which contradicts $u_i \notin \alpha nV$. Thus, $y_{ij} \succ_i z_{ij}$ holds for all i and j.

Then we have $y_{ij} \succ_i z_{ij} \succeq_i x_i$ and

$$\begin{aligned}
\sum_{i=1}^{m}\sum_{j=1}^{l} n_i b_{ij} y_{ij} &= \sum_{i=1}^{m}\sum_{j=1}^{l} n_i b_{ij} z_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{l} \alpha n b_{ij} v_i - \sum_{i=1}^{m}\sum_{j=1}^{l} b_{ij} u_i \\
&= z + \sum_{i=1}^{m} \alpha n v_i - \sum_{i=1}^{m} u_i \\
&= z + \sum_{i=1}^{m} \alpha n v_i - (y-z)^- \\
&\leq z + \alpha n v - (z-y) \\
&= y + \alpha n v \\
&= \sum_{i=1}^{m}\sum_{j=1}^{l} n_i b_{ij} \omega_i.
\end{aligned}$$

Therefore, setting

$$\begin{aligned}
e &= \left(\sum_{i=1}^{m}\sum_{j=1}^{l} n_i b_{ij}\omega_i - \sum_{i=1}^{m}\sum_{j=1}^{l} n_i b_{ij} y_{ij}\right) / \sum_{i=1}^{m}\sum_{j=1}^{l} n_i b_{ij} \\
&= \left(\sum_{i=1}^{m} n_i \omega_i - \sum_{i=1}^{m}\sum_{j=1}^{l} n_i b_{ij} y_{ij}\right)/n \geq 0,
\end{aligned}$$

we have $y_{ij} + e \succ_i x_i$ for $i = 1, \ldots, m$ and $j = 1, \ldots, l$ and

$$\sum_{i=1}^{m}\sum_{j=1}^{l} n_i b_{ij}(y_{ij} + e) = \sum_{i=1}^{m}\sum_{j=1}^{l} n_i b_{ij}\omega_i.$$

Moreover, it follows that

$$\frac{1}{n_i}\sum_{j=1}^{l} n_i b_{ij}(y_{ij} + e) = \sum_{j=1}^{l} b_{ij} y_{ij} + e \geq 0,$$

for $i = 1, \ldots, l$. Therefore, putting $r = \max_{1\leq i\leq m} \sum_{j=1}^{l} n_i b_{ij} = \max_{1\leq i\leq m} n_i$, we have constructed an allocation (y_{ij}) of $\mathcal{E}_r$ which prevent $et_r(x_1, \ldots, x_m)$ from being a core allocation of $\mathcal{E}_r$, and this contradicts the hypothesis that $(x_1, \ldots, x_m)$ is a strong Edgeworth equilibrium and hence $C \cap G = \emptyset$.

Since $C \cap G = \emptyset$ and C is open, it follows from the separation theorem that there is some non-zero continuous linear functional p satisfying $p \cdot g \geq p \cdot w$ for all $g \in G$ and $w \in C$. Since $w \in C$ implies $\alpha w \in C$ for all $\alpha > 0$, we see that $p \cdot g \geq 0$ holds for each $g \in G$. Thus, if $x \succeq_i x_i$, then $x - \omega_i \in G$, and so $p \cdot (x - \omega_i) = p \cdot x - p \cdot \omega_i \geq 0$ implies $p \cdot x \geq p \cdot \omega_i$. This means that $(x_1, \ldots, x_m)$ is a quasiequilibrium of the exchange economy $\mathcal{E}$. □

Corollary 1. *Suppose that in an exchange economy* $\mathcal{E} = (E, \{\omega_i, \succeq_i \ i = 1, \ldots, m\})$ *all preferences* $\succeq_i$ *are uniformly proper and monotone. If each preferences* $\succeq_i$ *is algebraicly open and one of the following assumptions is satisfied,*

(1) *each initial endowment* ω_i *is strictly positive;*
(2) *each preference* $\succeq_i$ *is strictly monotone and the total endowment* $\omega = \sum_{i=1}^m \omega_i$ *is strictly positive;*

then every strong Edgeworth equilibrium is a Walrasian equilibrium.

Proof. Let $(x_1, \ldots, x_m)$ be a strong Edgeworth equilibrium. By Theorem 1, $(x_1, \ldots, x_m)$ is a quasiequilibrium and there is a non-zero continuous linear functional p on E such that $x \succeq_i x_i$ implies $p(x) \geq p(\omega_i)$ for $i = 1, \ldots, m$. Then, since $p(x_i) \geq p(\omega_i)$ and $\sum_{i=1}^m x_i = \sum_{i=1}^m \omega_i$, it follows that $p(x_i) = p(\omega_i)$ for $i = 1, \ldots, m$. Moreover, we have $p \geq 0$ by the monotonicity of $\succeq_i$.

Note that, if $p(\omega_i) > 0$ for some i, then it follows that $p(x) \leq p(\omega_i)$ implies $x \preceq_i x_i$. Indeed, if there were $x \in E$ such that $p(x) \leq p(\omega_i)$, but $x \succ_i x_i$. Since $\{y \in E : y \succ_i x_i\}$ is algebraicly open, there is a number α such that $0 < \alpha < 1$ and $\alpha x \succ_i x_i$. Therefore, $p(\alpha x) \geq p(\omega_i) \geq p(x)$ and hence $p(x) \leq 0$ and $p(\omega_i) \leq 0$, which is a contradiction.

In case of (1), it is clear that $p(\omega_i) > 0$ for all $i = 1, \ldots, m$ and $(x_1, \ldots, x_m)$ is a Walrasian equilibrium.

In case of (2), it is clear that $p(\omega_i) > 0$ for some i. For any i' with $i' \neq i$, we have $x_i + \omega_{i'} \succ_i x_i$ and hence $p(x_i + \omega_{i'}) > p(x_i)$. Therefore, we have $p(\omega_{i'}) > 0$ and $(x_1, \ldots, x_m)$ is a Walrasian equilibrium. □

A preference $\succeq_i$ on E is said to be *irreducible* whenever, for any allocation $(x_1, \ldots, x_m)$ of $\mathcal{E}$ and for each pair of nonempty subsets I_1 and I_2 of $\{1, \ldots, m\}$ with $I_1 \cup I_2 = \{1, \ldots, m\}$ and $I_1 \cap I_2 = \emptyset$, there exists a subset $\{y_j : j \in I_2\}$ of E_+ such that $\sum_{j \in I_2} y_j \leq \sum_{i \in I_1} \omega_i$ and $x_j + y_j \succ_j x_j$ for some $j \in I_2$.

Corollary 2. *Suppose that in an exchange economy* $\mathcal{E} = (E, \{\omega_i, \succeq_i \ i = 1, \ldots, m\})$ *all preferences* $\succeq_i$ *are uniformly proper, monotone and irreducible. If each preferences* $\succeq_i$ *is algebraicly open and the total endowment* $\omega = \sum_{i=1}^m \omega_i$ *is strictly positive, then every strong Edgeworth equilibrium is a Walrasian equilibrium.*

Proof. Let $(x_1, \ldots, x_m)$ be a strong Edgeworth equilibrium. By Theorem 1, $(x_1, \ldots, x_m)$ is a quasiequilibrium and there is a non-zero continuous linear functional p on E such that $x \succeq_i x_i$ implies $p(x) \geq p(\omega_i)$ for $i = 1, \ldots, m$. it follows that $p(x_i) = p(\omega_i)$ for $i = 1, \ldots, m$. Moreover, we have $p \geq 0$ by the monotonicity of $\succeq_i$. To prove this corollary, by the proof of Corollary 1, it is sufficient that $p(\omega_i) > 0$ for each $i \in \{1, \ldots, m\}$. Since $p(\omega) > 0$, then there exists $i_0 \in \{1, \ldots, m\}$ such that $p(\omega_{i_0}) > 0$. Let $I_1 = \{i \in \{1, \ldots, m\} : p(\omega_i) = 0\}$ and $I_2 = \{j \in \{1, \ldots, m\} : p(\omega_j) > 0\}$. We assume that I_1 is nonempty. Since I_1 and I_2 are nonempty with $I_1 \cup I_2 = \{1, \ldots, m\}$ and $I_1 \cap I_2 = \emptyset$, by the irreducibility assumption, there exists a subset $\{y_j : j \in I_2\}$ of E_+ such that $\sum_{j \in I_2} y_j \leq \sum_{i \in I_1} \omega_i$ and $x_{j_0} + y_{j_0} \succ_{j_0} x_{j_0}$ for some $j_0 \in I_2$. Since, for each $j \in I_2$, $p(y_j) \geq 0$ and

$$0 \leq \sum_{j \in I_2} p(y_j) \leq \sum_{j \in I_1} p(\omega_i) = 0,$$

then we have $p(y_j) = 0$ for all $j \in I_2$. Since $p(\omega_{j_0}) > 0$ and $x_{j_0} + y_{j_0} \succ_{j_0} x_{j_0}$, by the same method with the proof of Corollary 1, we have

$$p(x_{j_0}) < p(x_{j_0} + y_{j_0}) = p(x_{j_0}) + p(y_{j_0}) = p(x_{j_0}).$$

This is contradiction. Then we have I_1 is empty. Therefore we obtain that $p(\omega_j) > 0$ for each $j \in \{1, \ldots, m\}$. □

Remark 1. *Note that if $\succeq_i$ is strictly monotone for each $i \in \{1, \ldots, m\}$, then $\succeq_i$ is irreducible. Indeed, let $(x_1, \ldots, x_m)$ be any allocation of $\mathcal{E}$, and I_1 and I_2 be nonempty subsets of $\{1, \ldots, m\}$ such that $I_1 \cup I_2 = \{1, \ldots, m\}$ and $I_1 \cap I_2 = \emptyset$. Put $y_j = \frac{1}{\# I_2} \sum_{i \in I_1} \omega_i > 0$ for all $j \in I_2$. The set $\{y_j : j \in I_2\}$ satisfies that $\sum_{j \in I_2} y_j \leq \sum_{i \in I_1} \omega_i$ and $x_j + y_j \succ_j x_j$ for each $j \in I_2$. Thus the second half of Corollary 1 is a corollary of Corollary 2, but we have listed the two kinds of assumptions of Corollary 1 in order to make a comparison between the two assumptions.*

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