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STRONG CONVERGENCE THEOREMS FOR ASYMPTOTICALLY NONEXPANSIVE SEMIGROUPS IN BANACH SPACES

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Abstract. Let S be a semigroup and let C be a closed, convex subset of a Banach space E. Let $\{a_n\}$ and $\{b_n\}$ be real sequences such that $0 < a_n \leq 1$, $a_n \to 0$, $0 \leq b_n \leq 1$ and $b_n \to 0$, let $\{\mu_n\}$ be a sequence of means on a subspace X of the Banach space of all bounded real-valued functions on S, and let $\mathcal{S} = \{T_t : t \in S\}$ be an asymptotically nonexpansive semigroup on C such that the set of common fixed points of $\mathcal{S}$ is nonempty. Let x and y_0 be elements of C. In this paper, we study the strong convergence of the sequences $\{x_n\}$ and $\{y_n\}$ respectively defined by

$$x_n = a_n x + (1 - a_n) T_{\mu_n} x_n \qquad \text{for all sufficiently large } n,$$

and

$$y_{n+1} = b_n x + (1 - b_n) T_{\mu_n} y_n \qquad \text{for } n = 0, 1, 2, \ldots,$$

where for $u \in C$ and a mean μ on X, $T_\mu u$ is a unique element of C satisfying $\langle T_\mu u, u^* \rangle = \mu_t \langle T_t u, u^* \rangle$ for all $u^* \in E^*$.

1. Introduction

Let C be a closed, convex subset of a Hilbert space and let T be a nonexpansive mapping from C into itself such that the set $F(T)$ of fixed points of T is nonempty. Let x be an element of C and for each t with $0 < t < 1$, let x_t be a unique point of C satisfying

$$x_t = tx + (1 - t) T x_t.$$

Browder [3] showed that $\{x_t\}$ converges strongly to the element of $F(T)$ which is nearest to x in $F(T)$ as $t \downarrow 0$. This result was extended to those of a Banach space by Reich [11] and Takahashi and Ueda [25]. Since $\{x_t\}$ converges strongly, Halpern [7] and Reich [12] considered the following iteration process:

$$y_{n+1} = b_n x + (1 - b_n) T y_n \qquad \text{for } n = 0, 1, 2, \ldots,$$

where y_0 is an element of C and $\{b_n\}$ is a real sequence satisfying $0 \leq b_n \leq 1$ and $b_n \to 0$. Recently, Wittmann [26] showed that $\{y_n\}$ converges strongly to the element of $F(T)$ which is nearest to x if $\sum_{n=0}^{\infty} b_n = \infty$ and $\sum_{n=0}^{\infty} |b_{n+1} - b_n| < \infty$. The authors [16] extended his result to that of a Banach space, which gives an answer to Reich's problem [12]. On the other hand, using ideas of Browder [3] and Wittmann [26], Shimizu and Takahashi [14, 15] studied the convergence of the

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following sequences $\{x_n\}$ and $\{y_n\}$:

$$x_n = a_n x + (1 - a_n)\frac{1}{n+1}\sum_{i=0}^{n} T^i x_n \qquad \text{for } n = 0, 1, 2, \ldots; \tag{1.1}$$

$$y_{n+1} = b_n x + (1 - b_n)\frac{1}{n+1}\sum_{i=0}^{n} T^i y_n \qquad \text{for } n = 0, 1, 2, \ldots, \tag{1.2}$$

where $\{a_n\}$ is a real sequence satisfying $0 < a_n \leq 1$ and $a_n \to 0$. The authors [19, 20] also extended these results to those of a Banach space.

In this paper, we study strong convergence theorems for an asymptotically nonexpansive semigroup on a Banach space by the use of a sequence of means, which has been developed in the study of nonlinear ergodic theorems (cf. [1, 5, 6, 8, 13, 22, 23]). In the framework of a Hilbert space, we introduced two iteration processes which generalize (1.1) and (1.2) and we discussed strong convergence of the iterative processes in [18]. Though the proofs in this paper and those in [18] are considerably different, Theorems 2 and 3 below are straight generalizations of those in [18]. However, to prove Theorems 4 and 5 below, we need the concept of monotone convergence for means. The reason is that the duality mapping is not weakly continuous on a Banach space.

This paper is organized as follows: Section 2 is devoted to some preliminaries and notations. In Section 3, we state our main results and in Section 4, we prove them. Finally, we investigate some theorems which can be deduced from our main results.

2. Preliminaries and notations

Throughout this paper, all vector spaces are real and we denote by $\mathbb{N}$ the set of all nonnegative integers. For a real number a, we also denote $\max\{a, 0\}$ by $(a)_+$.

Let E be a Banach space and let $r > 0$. We denote by B_r the closed ball in E with center 0 and radius r. E is said to be uniformly convex if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|(x + y)/2\| \leq 1 - \delta$ for each $x, y \in B_1$ with $\|x - y\| \geq \varepsilon$. Let C be a subset of E, let T be a mapping from C into E and let $\varepsilon > 0$. We denote by $\overline{\text{co}}C$ the closed convex hull of C and we denote by $F(T)$ and $F_\varepsilon(T)$ the sets $\{x \in C : x = Tx\}$ and $\{x \in C : \|x - Tx\| \leq \varepsilon\}$, respectively.

Let E^* be the topological dual of E. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^* \rangle$. We also denote by J the duality mapping from E into 2^{E^*}, i.e.,

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} \qquad \text{for each } x \in E.$$

Let $U = \{x \in E : \|x\| = 1\}$. E is said to be smooth if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit (2.1) exists uniformly for $x \in U$. We know that if E is smooth then the duality mapping is single-valued and norm to weak star continuous and that if the norm of E is uniformly Gâteaux differentiable then the duality mapping is norm to weak star uniformly continuous on each bounded subset of E.

Let C be a convex subset of E, let K be a nonempty subset of C and let P be a retraction from C onto K, i.e., $Px = x$ for each $x \in K$. P is said to be sunny if $P(Px + t(x - Px)) = Px$ for each $x \in C$ and $t \geq 0$ with $Px + t(x - Px) \in C$. We know from [4, Theorem 3] or [10, Lemma 2.7] that if E is smooth, then a retraction P from C onto K is sunny and nonexpansive if and only if

$$\langle x - Px, J(y - Px)\rangle \leq 0 \qquad \text{for all } x \in C \text{ and } y \in K, \tag{2.2}$$

and hence there is at most one sunny, nonexpansive retraction from C onto K. If there is a sunny, nonexpansive retraction from C onto K, K is said to be a sunny, nonexpansive retract of C.

Let S be a semigroup. Let $B(S)$ be the space of all bounded real-valued functions defined on S with supremum norm. For $s \in S$ and $f \in B(S)$, we define an element $l_s f$ in $B(S)$ by

$$(l_s f)(t) = f(st) \qquad \text{for each } t \in S.$$

Let X be a subspace of $B(S)$ containing 1 and let X^* be its topological dual. An element μ of X^* is said to be a mean on X if $\|\mu\| = \mu(1) = 1$. We know that μ is a mean on X if and only if $\inf f(S) \leq \mu(f) \leq \sup f(S)$ for all $f \in X$. We often write $\mu_t(f(t))$ instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$. Let X be l_s-invariant, i.e., $l_s(X) \subset X$ for each $s \in S$. A mean μ on X is said to be left invariant if $\mu(l_s f) = \mu(f)$ for each $s \in S$ and $f \in X$. A sequence of means $\{\mu_n\}$ on X is said to be strongly left regular if $\|\mu_n - l_s^*\mu_n\| \to 0$ for each $s \in S$, where l_s^* is the adjoint operator of l_s. In the case when S is commutative, a left invariant mean is said to be an invariant mean and a strongly left regular sequence is said to be a strongly regular sequence [8, 9]. We remark that an invariant mean on $B(\mathbb{N})$ is said to be a Banach limit [2]. Further, let X be satisfied that for each bounded subset $\{f_n : n \in \mathbb{N}\}$ of X, the mappings $t \mapsto \inf_n f_n(t)$ and $t \mapsto \sup_n f_n(t)$ are elements of X. A mean μ on X is said to be monotone convergent if $\mu_t(\lim_n f_n(t)) = \lim_n \mu_t(f_n(t))$ for each bounded sequence $\{f_n : n \in \mathbb{N}\}$ of X such that $0 \leq f_1 \leq f_2 \leq \cdots$.

Let E be a reflexive Banach space, let X be a subspace of $B(S)$ containing 1 and let μ be a mean on X. Let f be a bounded function from S into E such that the mapping $t \mapsto \langle f(t), x^*\rangle$ is an element of X for each $x^* \in E^*$. We know from [8] that there exists a unique element $x_0 \in E$ such that $\langle x_0, x^*\rangle = \mu_t\langle f(t), x^*\rangle$ for all $x^* \in E^*$. Following [8], we denote such x_0 by $\int f(t)\, d\mu(t)$.

Let C be a closed, convex subset of a reflexive Banach space E. A family $\mathcal{S} = \{T_t : t \in S\}$ is said to be a uniformly Lipschitzian semigroup on C with Lipschitz constants $\{k_t : t \in S\}$ if

(i) k_t is a nonnegative real number for each $t \in S$ and $\sup_{t\in S} k_t < \infty$;
(ii) for each $t \in S$, T_t is a mapping from C into itself and $\|T_t x - T_t y\| \leq k_t\|x - y\|$ for each $x, y \in C$;
(iii) $T_{ts} = T_t T_s$ for each $t, s \in S$.

A uniformly Lipschitzian semigroup $\mathcal{S} = \{T_t : t \in S\}$ on C with Lipschitz constants $\{k_t : t \in S\}$ is said to be asymptotically nonexpansive if $\inf_{s\in S}\sup_{t\in S} k_{st} \leq 1$, and it is said to be nonexpansive if $k_t = 1$ for all $t \in S$. We denote by $F(\mathcal{S})$ the set of common fixed points of $\mathcal{S}$, i.e., $\bigcap_{t\in S}\{x \in C : T_t x = x\}$. Let $\mathcal{S} = \{T_t : t \in S\}$ be a uniformly Lipschitzian semigroup on C with Lipschitz constants $\{k_t : t \in S\}$ such that $\{T_t u : t \in S\}$ is bounded for some $u \in C$ and let X be a subspace of $B(S)$ such that $1 \in X$ and the mappings $t \mapsto k_t$ and $t \mapsto \langle T_t x, x^* \rangle$ are elements of X for each $x \in C$ and $x^* \in E^*$. Following [13], we also write $T_\mu x$ instead of $\int T_t x \, d\mu(t)$ for a mean μ on X and $x \in C$.

3. Main results

Now we state our main results.

Theorem 1. *Let C be a closed, convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable. Let S be a semigroup and let $\mathcal{S} = \{T_t : t \in S\}$ be an asymptotically nonexpansive semigroup on C with Lipschitz constants $\{k_t : t \in S\}$ such that $F(\mathcal{S})$ is nonempty. Let X be a subspace of $B(S)$ such that $1 \in X$, X is l_s-invariant for each $s \in S$ and the mappings $t \mapsto k_t$ and $t \mapsto \langle T_t x, x^* \rangle$ are elements of X for each $x \in C$ and $x^* \in E^*$. If there is a left invariant mean on X, then $F(\mathcal{S})$ is a sunny, nonexpansive retract of C.*

We show a strong convergence theorem which generalizes the results in [14, 19, 18]:

Theorem 2. *Let C, E, S, $\mathcal{S}$ and X be as in Theorem 1. Assume that there is a left invariant mean on X. Let P be the sunny, nonexpansive retraction from C onto $F(\mathcal{S})$ and let $\{\mu_n\}$ be a strongly left regular sequence of means on X. Let $\{a_n\}$ be a real sequence satisfying $0 < a_n \leq 1$, $a_n \to 0$ and*

$$\varlimsup_{n \to \infty} \frac{(\mu_n)_t(k_t) - 1}{a_n} < 1. \tag{3.1}$$

Let x be an element of C and let $\{x_n\}$ be the sequence defined by

$$x_n = a_n x + (1 - a_n) T_{\mu_n} x_n \tag{3.2}$$

for $n \geq n_0$, where n_0 is a sufficiently large natural number. Then $\{x_n\}$ converges strongly to Px.

Remark 1. The inequality (3.1) implies that there exists $n_0 \in \mathbb{N}$ such that $(1 - a_n)\,(\mu_n)_t(k_t) < 1$ for $n \geq n_0$. So for $n \geq n_0$, there exists a unique point $x_n \in C$ satisfying $x_n = a_n x + (1 - a_n) T_{\mu_n} x_n$, since the mapping T_n from C into itself defined by $T_n u = a_n x + (1 - a_n) T_{\mu_n} u$ is a contraction, i.e., $\|T_n u - T_n v\| \leq (1 - a_n)\,(\mu_n)_t(k_t)\,\|u - v\|$ for each $u, v \in C$.

Remark 2. By [24], we know that the condition $F(\mathcal{S}) \neq \emptyset$ can be replaced by the condition that there exists $u \in C$ such that $\{T_t u : t \in S\}$ is bounded.

In the case when $\mathcal{S}$ is nonexpansive, we have the following:

Theorem 3. *Let C, E, S, $\mathcal{S}$, X, P and $\{\mu_n\}$ be as in Theorem 2. Assume that $\mathcal{S}$ is nonexpansive. Let $\{a_n\}$ be a real sequence satisfying $0 < a_n \leq 1$ and $a_n \to 0$. Let x be an element of C and let $\{x_n\}$ be the sequence defined by (3.2) for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to Px.*

Next, we show another strong convergence theorem which generalizes the results in [15, 20, 18]:

Theorem 4. *Let C, E, S, $\mathcal{S}$, X and P be as in Theorem 2. Assume that for each bounded subset $\{f_n : n \in \mathbb{N}\}$ of X, the mappings $t \mapsto \sup_n f_n(t)$ and $t \mapsto \inf_n f_n(t)$ are elements of X. Let $\{\mu_n\}$ be a strongly left regular sequence of monotone convergent means on X. Let $\{b_n\}$ be a real sequence satisfying $0 \le b_n \le 1$, $b_n \to 0$, $\sum_{n=0}^{\infty} b_n = \infty$ and*

$$\sum_{n=0}^{\infty} \big((1-b_n)((\mu_n)_t(k_t))^2 - 1\big)_+ < \infty. \tag{3.3}$$

Let x and y_0 be elements of C and let $\{y_n\}$ be the sequence defined by

$$y_{n+1} = b_n x + (1-b_n) T_{\mu_n} y_n \qquad \text{for } n \in \mathbb{N}. \tag{3.4}$$

Then $\{y_n\}$ converges strongly to Px.

In the case when $\mathcal{S}$ is nonexpansive, we also have the following:

Theorem 5. *Let C, E, S, $\mathcal{S}$, X, P and $\{\mu_n\}$ be as in Theorem 4. Assume that $\mathcal{S}$ is nonexpansive. Let $\{b_n\}$ be a real sequence satisfying $0 \le b_n \le 1$, $b_n \to 0$ and $\sum_{n=0}^{\infty} b_n = \infty$. Let x and y_0 be elements of C and let $\{y_n\}$ be the sequence defined by* (3.4). *Then $\{y_n\}$ converges strongly to Px.*

4. Proofs of Theorems

For the sake of completeness, we give the proof of the following lemma.

Lemma 1. *Let C be a closed, convex subset of a reflexive Banach space E. Let S be a semigroup and let $\mathcal{S} = \{T_t : t \in S\}$ be a uniformly Lipschitzian semigroup on C with Lipschitz constants $\{k_t : t \in S\}$ such that $F(\mathcal{S})$ is nonempty. Let X be a subspace of $B(S)$ such that $1 \in X$, X is l_s-invariant for each $s \in S$ and the mappings $t \mapsto k_t$ and $t \mapsto \langle T_t x, x^* \rangle$ are elements of X for each $x \in C$ and $x^* \in E^*$. Let μ be a mean on X. Then*

(i) *if $x \in F(\mathcal{S})$, then $T_\mu x = x$;*
(ii) *$\|T_\mu y - T_\mu z\| \le \mu_t(k_t)\|y - z\|$ for each $y, z \in C$.*

Proof. Let $x \in F(\mathcal{S})$. Then we have $\langle T_\mu x, x^* \rangle = \mu_t \langle T_t x, x^* \rangle = \langle x, x^* \rangle$ for all $x^* \in E^*$, and hence we get (i). Let $y, z \in C$ and let $x^* \in J(T_\mu y - T_\mu z)$. Then we have

$$\|T_\mu y - T_\mu z\|^2 = \mu_t \langle T_t y - T_t z, x^* \rangle \le \mu_t(k_t)\|y - z\|\|x^*\|,$$

and hence we get (ii). □

The following is crucial to prove our theorems.

Lemma 2. *Let C be a closed, convex subset of a uniformly convex Banach space E. Let S be a semigroup and let $\mathcal{S} = \{T_t : t \in S\}$ be an asymptotically nonexpansive semigroup on C with Lipschitz constants $\{k_t : t \in S\}$ such that $F(\mathcal{S})$ is nonempty. Let X be a subspace of $B(S)$ such that $1 \in X$, X is l_s-invariant for each $s \in S$ and the mappings $t \mapsto k_t$ and $t \mapsto \langle T_t x, x^* \rangle$ are elements of X for each $x \in C$ and $x^* \in E^*$. Let $\{\mu_n\}$ be a strongly left regular sequence of means on X. Then for each $r > 0$,*

$$\inf_{s \in S} \max\Big\{\sup_{t \in S}(k_{st} - 1)_+,\ \sup_{t \in S} \overline{\lim_{n\to\infty}} \sup_{u \in C \cap B_r} \|T_{\mu_n} u - T_{st}(T_{\mu_n} u)\|\Big\} = 0.$$

Proof. Let $r > 0$. Set $d = \sup\{\|T_t u\| : u \in C \cap B_r, t \in S\}$ and set $R = \max\{d, \sup\{\|T_t u\| : u \in C \cap B_d, t \in S\}\}$. We may assume $d > 0$. Fix $\varepsilon > 0$. From [19, Lemma 1], there exists $\delta > 0$ satisfying

$$\big(\overline{\mathrm{co}}(F_\delta(U) \cap B_R) + B_\delta\big) \cap C \subset F_\varepsilon(U)$$

for all mappings U from C into E such that $\|Ux - Uy\| \le (1+\delta)\|x - y\|$ for each $x, y \in C$; see also [6, Theorem 1.2]. From [19, Lemma 3], there also exist $\eta > 0$ and a positive natural number N such that for each mapping U from C into itself satisfying $\sup\{\|U^n x\| : n \in \mathbb{N}, x \in C \cap B_d\} \le R$ and $\|Ux - Uy\| \le (1+\eta)\|x - y\|$ for each $x, y \in C$, there holds

$$\left\| \frac{1}{m+1} \sum_{i=0}^{m} U^i x - U\left(\frac{1}{m+1} \sum_{i=0}^{m} U^i x \right) \right\| \le \delta$$

for all $m \ge N$ and $x \in C \cap B_d$. We may assume $\max\{\delta, \eta\} \le \varepsilon$. From $\inf_s \sup_t k_{st} \le 1$, there exists $s_0 \in S$ such that $k_{s_0 t} \le 1 + \min\{\delta, \eta\}$ for all $t \in S$. Fix $t \in S$. Then we have

$$\left\| \frac{1}{N+1} \sum_{i=0}^{N} (T_{s_0 t})^i (T_s u) - T_{s_0 t}\left(\frac{1}{N+1} \sum_{i=0}^{N} (T_{s_0 t})^i (T_s u) \right) \right\| \le \delta$$

for all $s \in S$ and $u \in C \cap B_r$. Hence for each mean μ on X, we have

$$\int \frac{1}{N+1} \sum_{i=0}^{N} T_{(s_0 t)^i s} u \, d\mu(s) \in \overline{\mathrm{co}} \left\{ \frac{1}{N+1} \sum_{i=0}^{N} T_{(s_0 t)^i s} u : s \in S \right\} \subset \overline{\mathrm{co}} F_\delta(T_{s_0 t}) \cap B_R$$

for all $u \in C \cap B_r$, where $(s_0 t)^0 s$ represents s. From the strong left regularity of $\{\mu_n\}$, there exists $n_{s_0 t}^N \ge n_0$ such that $\|\mu_n - l^*_{(s_0 t)^i} \mu_n\| < \delta/d$ for all $n \ge n_{s_0 t}^N$ and $i = 1, \dots, N$. Since

$$\begin{aligned}
&\left\| T_{\mu_n} u - \int \frac{1}{N+1} \sum_{i=0}^{N} T_{(s_0 t)^i s} u \, d\mu_n(s) \right\| \\
&\qquad = \sup_{\|u^*\|=1} \left| (\mu_n)_s \langle T_s u, u^* \rangle - \frac{1}{N+1} \sum_{i=0}^{N} (\mu_n)_s \langle T_{(s_0 t)^i s} u, u^* \rangle \right| \\
&\qquad \le \frac{1}{N+1} \sum_{i=1}^{N} \sup_{\|u^*\|=1} \left| (\mu_n)_s \langle T_s u, u^* \rangle - (l^*_{(s_0 t)^i} \mu_n)_s \langle T_s u, u^* \rangle \right| \\
&\qquad \le \frac{1}{N+1} \sum_{i=1}^{N} \|\mu_n - l^*_{(s_0 t)^i} \mu_n\| \cdot d \le \delta,
\end{aligned}$$

we get $T_{\mu_n} u \in (\overline{\mathrm{co}}(F_\delta(T_{s_0 t}) \cap B_R) + B_\delta) \cap C$ for all $u \in C \cap B_r$ and $n \ge n_{s_0 t}^N$. So we have $T_{\mu_n} u \in F_\varepsilon(T_{s_0 t})$ for all $u \in C \cap B_r$ and $n \ge n_{s_0 t}^N$, and hence we get

$$\begin{aligned}
&\inf_{s \in S} \max\Big\{ \sup_{t \in S} (k_{st} - 1)_+, \ \sup_{t \in S} \varlimsup_{n \to \infty} \sup_{u \in C \cap B_r} \|T_{\mu_n} u - T_{st}(T_{\mu_n} u)\| \Big\} \\
&\qquad \le \max\Big\{ \sup_{t \in S} (k_{s_0 t} - 1)_+, \ \sup_{t \in S} \varlimsup_{n \to \infty} \sup_{u \in C \cap B_r} \|T_{\mu_n} u - T_{s_0 t}(T_{\mu_n} u)\| \Big\} \le \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain the conclusion. $\square$

Till the end of Lemma 5, we assume that C, E, S, $\mathcal{S}$, X, P, $\{\mu_n\}$, $\{a_n\}$, n_0, x and $\{x_n\}$ are as in Theorem 2 and we set $a = \overline{\lim}_n \big((\mu_n)_t(k_t) - 1\big)/a_n$. For $n \in \mathbb{N}$ with $1 \le n \le n_0 - 1$, we set $x_n = x$.

Lemma 3. *Let ν be a Banach limit and let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$. Then there exists a unique point z of C satisfying*

$$\nu_i \|x_{n_i} - z\|^2 = \min_{y \in C} \nu_i \|x_{n_i} - y\|^2 \tag{4.1}$$

and the point z is an element of $F(\mathcal{S})$.

Proof. From Lemma in [24], there exists a unique point z of C satisfying (4.1). We shall show $\inf_{s\in S} \sup_{t\in S} \|T_{st}z - z\| = 0$. Suppose not. Then there exists $\varepsilon > 0$ such that for each $s \in S$, there exists $t \in S$ satisfying $\|T_{st}z - z\| \ge \varepsilon$. Set $R = \max\{\|z\|, \sup\{\|T_t z\| : t \in S\}\}$. From Lemma in [24] and its proof, we can choose $\delta > 0$ such that

$$\nu_i \left\| x_{n_i} - \frac{u+v}{2} \right\|^2 \le \frac{1}{2}\big(\nu_i \|x_{n_i} - u\|^2 + \nu_i \|x_{n_i} - v\|^2\big) - \delta$$

for all $u, v \in C \cap B_R$ with $\|u - v\| \ge \varepsilon$. By the property of ε, $\inf_s \sup_t k_{st} \le 1$ and Lemma 2, there also exists $s \in S$ such that $\|T_s z - z\| \ge \varepsilon$, $(k_s^2 - 1)\nu_i \|x_{n_i} - z\|^2 < \delta$ and $\nu_i \|x_{n_i} - T_s z\|^2 \le \nu_i \|T_s x_{n_i} - T_s z\|^2 + \delta$. Then we have

$$\begin{aligned}
\nu_i \left\| x_{n_i} - \frac{T_s z + z}{2} \right\|^2 &\le \frac{1}{2}\big(\nu_i \|x_{n_i} - T_s z\|^2 + \nu_i \|x_{n_i} - z\|^2\big) - \delta \\
&\le \nu_i \|x_{n_i} - z\|^2 + \frac{1}{2}\big((k_s^2 - 1)\nu_i \|x_{n_i} - z\|^2 - \delta\big) \\
&< \nu_i \|x_{n_i} - z\|^2.
\end{aligned}$$

So we get a contradiction. Hence we have $\inf_s \sup_t \|T_{st}z - z\| = 0$. From the strong left regularity of $\{\mu_n\}$, there is a left invariant mean μ on X. Fix $w \in S$. For each $s \in S$, we have

$$\begin{aligned}
\|T_w z - z\|^2 &= \langle T_w z - z, J(T_w z - z)\rangle \\
&= \mu_t \langle T_w z - T_t z, J(T_w z - z)\rangle + \mu_t \langle T_t z - z, J(T_w z - z)\rangle \\
&= \mu_t \langle T_w z - T_{wst} z, J(T_w z - z)\rangle + \mu_t \langle T_{st} z - z, J(T_w z - z)\rangle \\
&\le \sup_{t\in S} \|T_w z - T_{wst} z\| \|J(T_w z - z)\| + \sup_{t \in S} \|T_{st} z - z\| \|J(T_w z - z)\| \\
&\le (k_w + 1) \sup_{t\in S} \|T_{st} z - z\| \|J(T_w z - z)\|.
\end{aligned}$$

Since $\inf_s \sup_t \|T_{st}z - z\| = 0$, we get $\|T_w z - z\|^2 = 0$. Therefore we obtain $z \in F(\mathcal{S})$. □

Lemma 4.

$$\langle x_n - x, J(x_n - z)\rangle \le \frac{((\mu_n)_t(k_t) - 1)_+}{a_n} \|x_n - z\|^2 \qquad \textit{for all } n \ge n_0 \textit{ and } z \in F(\mathcal{S}).$$

Proof. Let $n \geq n_0$ and let $z \in F(\mathcal{S})$. Since $a_n(x_n - x) = (1 - a_n)(T_{\mu_n} x_n - x_n)$ and $T_{\mu_n} z = z$, we get

$$\begin{aligned}
\langle x_n - x, J(x_n - z)\rangle &= \frac{1 - a_n}{a_n}\langle T_{\mu_n} x_n - x_n, J(x_n - z)\rangle \\
&= \frac{1 - a_n}{a_n}\big(\langle T_{\mu_n} x_n - T_{\mu_n} z, J(x_n - z)\rangle + \langle z - x_n, J(x_n - z)\rangle\big) \\
&\leq \frac{1 - a_n}{a_n}\big((\mu_n)_t(k_t)\|x_n - z\|^2 - \|x_n - z\|^2\big) \\
&\leq \frac{((\mu_n)_t(k_t) - 1)_+}{a_n}\|x_n - z\|^2.
\end{aligned}$$

□

Lemma 5. *Each subsequence $\{x_{n_i}\}$ of $\{x_n\}$ contains a subsequence of $\{x_{n_i}\}$ converging strongly to an element of $F(\mathcal{S})$.*

Proof. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ and let ν be a Banach limit. Then there exists $z \in F(\mathcal{S})$ satisfying (4.1). By Lemma 4, we get $\nu_i\langle x_{n_i} - x, J(x_{n_i} - z)\rangle \leq (a)_+\nu_i\|x_{n_i} - z\|^2$. This inequality, $a < 1$ and [25, Lemma 1] yield

$$\nu_i\|x_{n_i} - z\|^2 \leq \frac{1}{1 - (a)_+}\nu_i\langle x_{n_i} - x, J(x_{n_i} - z)\rangle \leq 0.$$

Hence there exists a subsequence of $\{x_{n_i}\}$ converging strongly to z. □

Now we can prove Theorems 1 and 2.

Proof of Theorem 1. Assume that there is a left invariant mean μ on X. From $\inf_s \sup_t k_{st} \leq 1$, we know that $\mu_t(k_t) \leq 1$. Let x be an element of C and let $\{x_n\}$ be the sequence defined by

$$x_n = \frac{1}{n+1}x + \left(1 - \frac{1}{n+1}\right)T_\mu x_n \qquad \text{for each } n \in \mathbb{N}.$$

First we shall show that $\{x_n\}$ converges strongly to an element of $F(\mathcal{S})$. By Lemma 5, we know that each subsequence $\{x_{n_i}\}$ of $\{x_n\}$ contains a subsequence of $\{x_{n_i}\}$ converging strongly to an element of $F(\mathcal{S})$. Let $\{x_{n_i}\}$ and $\{x_{m_i}\}$ be subsequences of $\{x_n\}$ converging strongly to elements y and z of $F(\mathcal{S})$, respectively. From Lemma 4 and $\mu_t(k_t) \leq 1$, we have $\langle y - x, J(y - z)\rangle \leq 0$ and $\langle z - x, J(z - y)\rangle \leq 0$, which implies $y = z$. So $\{x_n\}$ converges strongly to an element of $F(\mathcal{S})$. Hence we can define a mapping P from C onto $F(\mathcal{S})$ by $Px = \lim_n x_n$. By the argument above, we have $\langle x - Px, J(z - Px)\rangle \leq 0$ for all $x \in C$ and $z \in F(\mathcal{S})$. Therefore P is the sunny, nonexpansive retraction from C onto $F(\mathcal{S})$. □

Proof of Theorem 2. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ converging strongly to an element y of $F(\mathcal{S})$. By Lemma 4, we have $\langle y - x, J(y - Px)\rangle \leq (a)_+\|y - Px\|^2$. Hence we obtain

$$(1 - (a)_+)\|y - Px\|^2 \leq \langle x - Px, J(y - Px)\rangle \leq 0$$

by (2.2). From $a < 1$, we have $y = Px$. Hence by Lemma 5, $\{x_n\}$ converges strongly to Px. □

Proof of Theorem 3. Since $\mathcal{S}$ is nonexpansive, we have $\overline{\lim}_n\big((\mu_n)_t(k_t)-1\big)/a_n = 0 < 1$. So we obtain the desired result by Theorem 2. $\square$

Next, we give the proofs of Theorems 4 and 5. Till the end of Lemma 8, we assume that C, E, S, $\mathcal{S}$, X, P, $\{\mu_n\}$, $\{b_n\}$, x and $\{y_n\}$ are as in Theorem 4.

By the standard measure theory argument, we have the following:

Lemma 6. *Let μ be a monotone convergent mean on X and let $\{f_n : n \in \mathbb{N}\}$ be a bounded sequence of X. Then $\overline{\lim}_n f_n \in X$ and*

$$\overline{\lim_{n\to\infty}}\, \mu_t(f_n(t)) \le \mu_t\big(\overline{\lim_{n\to\infty}}\, f_n(t)\big).$$

Since each μ_n is monotone convergent, the following holds:

Lemma 7. $\overline{\lim\limits_{m\to\infty}}\ \overline{\lim\limits_{n\to\infty}}\ \|T_{\mu_m}y_n - y_n\| = 0.$

Proof. Set $R = \sup\big(\{\|T_{\mu_n}y_n\| : n \in \mathbb{N}\} \cup \{\|T_t(T_{\mu_n}y_n)\| : t \in S, n \in \mathbb{N}\}\big)$. Let $\varepsilon > 0$. By Lemma 2, there exists $s_0 \in S$ such that $\overline{\lim}_n \|T_{s_0t}(T_{\mu_n}y_n) - T_{\mu_n}y_n\| \le \varepsilon$ for each $t \in S$. From the strong left regularity of $\{\mu_m\}$, there also exists $m_{s_0} \in \mathbb{N}$ such that $\|l^*_{s_0}\mu_m - \mu_m\| \le \varepsilon$ for all $m \ge m_{s_0}$. So by Lemma 6, we get

$$\begin{aligned}
\overline{\lim_{n\to\infty}}\|T_{\mu_m}y_n - y_n\|^2 &= \overline{\lim_{n\to\infty}}\,(\mu_m)_t\langle T_t(T_{\mu_n}y_n) - T_{\mu_n}y_n, J(T_{\mu_m}(T_{\mu_n}y_n) - T_{\mu_n}y_n)\rangle \\
&\le (\mu_m)_t\big(\overline{\lim_{n\to\infty}}\langle T_t(T_{\mu_n}y_n) - T_{\mu_n}y_n, J(T_{\mu_m}(T_{\mu_n}y_n) - T_{\mu_n}y_n)\rangle\big) \\
&\le (l^*_{s_0}\mu_m)_t\big(\overline{\lim_{n\to\infty}}\langle T_t(T_{\mu_n}y_n) - T_{\mu_n}y_n, J(T_{\mu_m}(T_{\mu_n}y_n) - T_{\mu_n}y_n)\rangle\big) + 4R^2\varepsilon \\
&= (\mu_m)_t\big(\overline{\lim_{n\to\infty}}\langle T_{s_0t}(T_{\mu_n}y_n) - T_{\mu_n}y_n, J(T_{\mu_m}(T_{\mu_n}y_n) - T_{\mu_n}y_n)\rangle\big) + 4R^2\varepsilon \\
&\le (2R + 4R^2)\varepsilon
\end{aligned}$$

for all $m \ge m_{s_0}$, and hence we have $\overline{\lim}_m\, \overline{\lim}_n\, \|T_{\mu_m}y_n - y_n\|^2 \le (2R + 4R^2)\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we obtain the conclusion. $\square$

The following is crucial to prove Theorem 4.

Lemma 8. $\overline{\lim\limits_{n\to\infty}}\langle x - Px, J(y_n - Px)\rangle \le 0.$

Proof. From $\inf_s \sup_t k_{st} \le 1$ and Lemma 7, we can choose a positive real sequence $\{\beta_m\}$ such that $\lim_m \beta_m = 0$, $\overline{\lim}_m\big((\mu_m)_t(k_t) - 1\big)/\beta_m < 1$, and $((\mu_m)_t(k_t))^2 \le 1 + \beta_m^2$ and $\overline{\lim}_n \|T_{\mu_m}y_n - y_n\| \le \beta_m^2$ for all $m \in \mathbb{N}$. Without loss of generality, we may assume $\beta_m \le 1/2$ for all $m \in \mathbb{N}$. By Remark 1, there exists a unique point x_m of C satisfying $x_m = \beta_m x + (1-\beta_m)T_{\mu_m}x_m$ for all sufficiently large m. Without loss of generality, we may also assume that x_m is defined for all $m \in \mathbb{N}$. We know that $\{x_m\}$ converges strongly to Px by Theorem 2. Set $R = \sup\big(\{\|T_{\mu_m}x_m\|\} \cup \{\|x_m\|\} \cup \{\|T_{\mu_m}y_n\|\} \cup \{\|y_n\|\}\big)$. From $(1-\beta_m)(T_{\mu_m}x_m - y_n) = (x_m - y_n) - \beta_m(x - y_n)$, we have

$$\begin{aligned}
(1-\beta_m)^2\|T_{\mu_m}x_m - y_n\|^2 &\ge \|x_m - y_n\|^2 - 2\beta_m\langle x - y_n, J(x_m - y_n)\rangle \\
&= (1 - 2\beta_m)\|x_m - y_n\|^2 + 2\beta_m\langle x - x_m, J(y_n - x_m)\rangle
\end{aligned}$$

for each $m, n \in \mathbb{N}$. Then we get

$$\begin{aligned}
&\langle x - x_m, J(y_n - x_m)\rangle \le \frac{1}{2\beta_m}\big((1-\beta_m)^2\|T_{\mu_m}x_m - y_n\|^2 - (1-2\beta_m)\|x_m - y_n\|^2\big)\\
&= \frac{1-2\beta_m}{2\beta_m}\big(\|T_{\mu_m}x_m - y_n\|^2 - \|x_m - y_n\|^2\big) + \frac{\beta_m}{2}\|T_{\mu_m}x_m - y_n\|^2\\
&\le \frac{1-2\beta_m}{2\beta_m}\big((\|T_{\mu_m}x_m - T_{\mu_m}y_n\| + \|T_{\mu_m}y_n - y_n\|)^2 - \|x_m - y_n\|^2\big) + 2R^2\beta_m\\
&\le \frac{1}{2\beta_m}\big(\beta_m^2\|x_m - y_n\|^2 + 6R\|T_{\mu_m}y_n - y_n\|\big) + 2R^2\beta_m\\
&\le 4R^2\beta_m + \frac{3R}{\beta_m}\|T_{\mu_m}y_n - y_n\|
\end{aligned}$$

for each $m, n \in \mathbb{N}$. So we have

$$\overline{\lim_{n\to\infty}}\langle x - x_m, J(y_n - x_m)\rangle \le (4R^2 + 3R)\beta_m$$

for each $m \in \mathbb{N}$. Since $\{x_m\}$ converges strongly to Px and the norm of E is uniformly Gâteaux differentiable, we obtain the conclusion. □

Now we can prove Theorem 4.

Proof of Theorem 4. Fix $\varepsilon > 0$. By Lemma 8, there exists $n \in \mathbb{N}$ such that $2\langle x - Px, J(y_m - Px)\rangle \le \varepsilon$ for each $m \ge n$. Since $(1-b_m)(T_{\mu_m}y_m - Px) = (y_{m+1} - Px) - b_m(x - Px)$, we have

$$(1-b_m)^2\|T_{\mu_m}y_m - Px\|^2 \ge \|y_{m+1} - Px\|^2 - 2b_m\langle x - Px, J(y_{m+1} - Px)\rangle$$

for each $m \in \mathbb{N}$. So we get

$$\|y_{m+1} - Px\|^2 \le b_m\varepsilon + (1-b_m)^2((\mu_m)_t(k_t))^2\|y_m - Px\|^2$$

for each $m \ge n$. Set $p_m = \|y_m - Px\|^2$, $c_m = ((\mu_m)_t(k_t))^2$ and $d_m = ((1-b_m)c_m - 1)_+$ for each $m \in \mathbb{N}$. We remark that $\sum_{m=0}^{\infty} d_m < \infty$ by (3.3). Then for each $m \in \mathbb{N}$, we have

$$\begin{aligned}
p_{n+m} \le\ &\big(b_{n+m-1} + (1-b_{n+m-1})^2 c_{n+m-1}b_{n+m-2} + \cdots\\
&\quad + (1-b_{n+m-1})^2 c_{n+m-1}(1-b_{n+m-2})^2 c_{n+m-2}\cdots(1-b_{n+1})^2 c_{n+1}b_n\big)\varepsilon\\
&\quad + (1-b_{n+m-1})^2 c_{n+m-1}(1-b_{n+m-2})^2 c_{n+m-2}\cdots(1-b_n)^2 c_n p_n\\
\le\ &(1+d_{n+m-1})(1+d_{n+m-2})\cdots(1+d_{n+1})\\
&\quad\cdot\big(b_{n+m-1} + (1-b_{n+m-1})b_{n+m-2} + \cdots + (1-b_{n+m-1})\cdots(1-b_{n+1})b_n\big)\varepsilon\\
&\quad + (1+d_{n+m-1})(1+d_{n+m-2})\cdots(1+d_n)\\
&\quad\cdot(1-b_{n+m-1})(1-b_{n+m-2})\cdots(1-b_n)p_n\\
\le\ &e^{\sum_{l=0}^{\infty} d_l}\big(\varepsilon + e^{-\sum_{l=n}^{n+m-1} b_l}p_n\big).
\end{aligned}$$

Hence we get

$$\overline{\lim_{m\to\infty}}\, p_m = \overline{\lim_{m\to\infty}}\, p_{n+m} \le e^{\sum_{l=0}^{\infty} d_l}\cdot\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\{y_n\}$ converges strongly to Px. □

Proof of Theorem 5. Since $\mathcal{S}$ is nonexpansive, we have $\sum_{n=0}^{\infty}\big((1-b_n)((\mu_n)_t(k_t))^2 - 1\big)_+ = 0 < \infty$. So we obtain the desired result by Theorem 5. □

5. Deduced theorems from main results

Throughout this section, we assume that C is a closed, convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable. Since we use a sequence of abstract means in our main results, we can deduce many theorems from them. We give the proofs for some theorems in this section. For others, see [8, 18, 17].

Theorem 6. *Let T be an asymptotically nonexpansive mapping from C into itself with Lipschitz constants $\{k_n : n \in \mathbb{N}\}$ such that $F(T) \neq \emptyset$ and let P be the sunny, nonexpansive retraction from C onto $F(T)$. Let $\{a_n\}$ be a real sequence such that $0 < a_n \leq 1$, $a_n \to 0$ and $\overline{\lim}_n\big(\sum_{j=0}^{n} k_j/(n+1) - 1\big)/a_n < 1$, and let $\{b_n\}$ be a real sequence such that $0 \leq b_n \leq 1$, $b_n \to 0$, $\sum_{n=0}^{\infty} b_n = \infty$ and $\sum_{n=0}^{\infty}\big((1-b_n)(\sum_{j=0}^{n} k_j/(n+1))^2 - 1\big)_+ < \infty$. Let x and y_0 be elements of C and let $\{x_n\}$ and $\{y_n\}$ be the sequences defined by*

$$x_n = a_n x + (1-a_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j x_n \qquad \text{for all sufficiently large } n,$$

and

$$y_{n+1} = b_n x + (1-b_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j y_n \qquad \text{for } n \in \mathbb{N},$$

respectively. Then both $\{x_n\}$ and $\{y_n\}$ converge strongly to Px.

Theorem 7. *Let T and U be asymptotically nonexpansive mappings from C into itself with Lipschitz constants $\{k_n : n \in \mathbb{N}\}$ and $\{\kappa_n : n \in \mathbb{N}\}$, respectively such that $TU = UT$ and $F(T) \cap F(U) \neq \emptyset$. Let P be the sunny, nonexpansive retraction from C onto $F(T) \cap F(U)$. Let $\{a_n\}$ be a real sequence such that $0 < a_n \leq 1$, $a_n \to 0$ and $\overline{\lim}_n\big(2\sum_{l=0}^{n}\sum_{i+j=l} k_i\kappa_j/(n+1)(n+2) - 1\big)/a_n < 1$, and let $\{b_n\}$ be a real sequence such that $0 \leq b_n \leq 1$, $b_n \to 0$, $\sum_{n=0}^{\infty} b_n = \infty$ and $\sum_{n=0}^{\infty}\big((1-b_n)(2\sum_{l=0}^{n}\sum_{i+j=l} k_i\kappa_j/(n+1)(n+2))^2 - 1\big)_+ < \infty$. Let x and y_0 be elements of C and let $\{x_n\}$ and $\{y_n\}$ be the sequences defined by*

$$x_n = a_n x + (1-a_n)\frac{2}{(n+1)(n+2)}\sum_{l=0}^{n}\sum_{i+j=l} T^i U^j x_n \qquad \text{for all sufficiently large } n,$$

and

$$y_{n+1} = b_n x + (1-b_n)\frac{2}{(n+1)(n+2)}\sum_{l=0}^{n}\sum_{i+j=l} T^i U^j y_n \qquad \text{for } n \in \mathbb{N},$$

respectively. Then both $\{x_n\}$ and $\{y_n\}$ converge strongly to Px.

The following is a generalization of Theorem 6. For simplicity, we state it for a nonexpansive mapping.

Theorem 8. *Let T be a nonexpansive mapping from C into itself such that $F(T) \neq \emptyset$ and let P be the sunny, nonexpansive retraction from C onto $F(T)$. Let $\{\alpha_{n,j} : n, j \in \mathbb{N}\}$ be a real sequence such that $\alpha_{n,j} \geq 0$, $\sum_{j=0}^{\infty} \alpha_{n,j} = 1$ and $\lim_n \sum_{j=0}^{\infty} |\alpha_{n,j+1} - \alpha_{n,j}| = 0$. Let $\{a_n\}$ be a real sequence such that $0 < a_n \leq 1$ and $a_n \to 0$ and let $\{b_n\}$ be a real sequence such that $0 \leq b_n \leq 1$, $b_n \to 0$ and $\sum_{n=0}^{\infty} b_n = \infty$. Let x and y_0 be elements of C and let $\{x_n\}$ and $\{y_n\}$ be the sequences defined by*

$$x_n = a_n x + (1 - a_n) \sum_{j=0}^{\infty} \alpha_{n,j} T^j x_n \qquad \textit{for } n \in \mathbb{N},$$

and

$$y_{n+1} = b_n x + (1 - b_n) \sum_{j=0}^{\infty} \alpha_{n,j} T^j y_n \qquad \textit{for } n \in \mathbb{N},$$

respectively. Then both $\{x_n\}$ and $\{y_n\}$ converge strongly to Px.

Proof. For each $n \in \mathbb{N}$, define a mean μ_n on $B(\mathbb{N})$ by $\mu_n(f) = \sum_{j=0}^{\infty} \alpha_{n,j} f_j$ for $f = (f_0, f_1, \cdots) \in B(\mathbb{N})$. Then $\{\mu_n\}$ is strongly regular; see [8]. We shall show that each μ_n is monotone convergent. Fix $n \in \mathbb{N}$. For $A \in 2^{\mathbb{N}}$, set $m(A) = \sum_{j \in A} \alpha_{n,j}$. Then $(\mathbb{N}, 2^{\mathbb{N}}, m)$ is a measure space. Let $f \in B(\mathbb{N})$ with $f \geq 0$ and let $\{f^i : i \in \mathbb{N}\}$ be a nonnegative, monotone increasing sequence of $B(\mathbb{N})$ such that $\lim_i f_j^i = f_j$ for each $j \in \mathbb{N}$. By the monotone convergence theorem, we have

$$\lim_{i \to \infty} \mu_n(f^i) = \lim_{i \to \infty} \int_{\mathbb{N}} f^i \, dm = \int_{\mathbb{N}} f \, dm = \mu_n(f).$$

So μ_n is monotone convergent. Hence by Theorems 3 and 5, we obtain the conclusion. $\square$

Theorem 9. *Let $\mathcal{S} = \{S(t) : t \geq 0\}$ be an asymptotically nonexpansive semigroup on C with Lipschitz constants $\{k(t) : t \geq 0\}$ such that $F(\mathcal{S}) \neq \emptyset$ and the mappings $t \mapsto k(t)$ and $t \mapsto \langle S(t)x, x^* \rangle$ are measurable for each $x \in C$ and $x^* \in E^*$, and let P be the sunny, nonexpansive retraction from C onto $F(\mathcal{S})$. Let $\{\gamma_n\}$ be a sequence of positive real numbers with $\gamma_n \to \infty$, let $\{a_n\}$ be a real sequence such that $0 < a_n \leq 1$, $a_n \to 0$ and $\overline{\lim}_n \big(\int_0^{\gamma_n} k(t)\, dt/\gamma_n - 1\big)/a_n < 1$, and let $\{b_n\}$ be a real sequence such that $0 \leq b_n \leq 1$, $b_n \to 0$, $\sum_{n=0}^{\infty} b_n = \infty$ and $\sum_{n=0}^{\infty} \big((1-b_n)(\int_0^{\gamma_n} k(t)\, dt/\gamma_n)^2 - 1\big)_+ < \infty$. Let x and y_0 be elements of C and let $\{x_n\}$ and $\{y_n\}$ be the sequences defined by*

$$x_n = a_n x + (1 - a_n) \frac{1}{\gamma_n} \int_0^{\gamma_n} S(t) x_n \, dt \qquad \textit{for all sufficiently large } n,$$

and

$$y_{n+1} = b_n x + (1 - b_n) \frac{1}{\gamma_n} \int_0^{\gamma_n} S(t) y_n \, dt \qquad \textit{for } n \in \mathbb{N},$$

respectively. Then both $\{x_n\}$ and $\{y_n\}$ converge strongly to Px.

Remark 3. Theorem 9 is also applicable to the case when the mappings $t \mapsto k(t)$ and $t \mapsto S(t)x$ are continuous for each $x \in C$. In this case, the corresponding result was obtained in [21].

Theorem 10. *Let $\mathcal{S}$ and P be as in Theorem 9. Let $\{\lambda_n\}$ be a sequence of positive real numbers with $\lambda_n \to 0$, let $\{a_n\}$ be a real sequence such that $0 < a_n \le 1$, $a_n \to 0$ and $\overline{\lim}_n \big(\lambda_n \int_0^\infty e^{-\lambda_n t} k(t)\, dt - 1\big)/a_n < 1$, and let $\{b_n\}$ be a real sequence such that $0 \le b_n \le 1$, $b_n \to 0$, $\sum_{n=0}^\infty b_n = \infty$ and $\sum_{n=0}^\infty \big((1-b_n)(\lambda_n \int_0^\infty e^{-\lambda_n t} k(t)\, dt)^2 - 1\big)_+ < \infty$. Let x and y_0 be elements of C and let $\{x_n\}$ and $\{y_n\}$ be the sequences defined by*

$$x_n = a_n x + (1-a_n)\lambda_n \int_0^\infty e^{-\lambda_n t} S(t) x_n \, dt \qquad \text{for all sufficiently large } n,$$

and

$$y_{n+1} = b_n x + (1-b_n)\lambda_n \int_0^\infty e^{-\lambda_n t} S(t) y_n \, dt \qquad \text{for } n \in \mathbb{N},$$

respectively. Then both $\{x_n\}$ and $\{y_n\}$ converge strongly to Px.

The following is a generalization of the above two theorems. For simplicity, we state it for a nonexpansive semigroup.

Theorem 11. *Let $\mathcal{S}$ and P be as in Theorem 9. Assume that $\mathcal{S}$ is nonexpansive. Let $\{\alpha_n\}$ be a sequence of measurable functions from $[0,\infty)$ into itself such that $\int_0^\infty \alpha_n(t)\, dt = 1$ for each $n \in \mathbb{N}$, $\lim_n \alpha_n(t) = 0$ for almost every $t \ge 0$, $\lim_n \int_0^\infty |\alpha_n(t+s) - \alpha_n(t)|\, dt = 0$ for each $s \ge 0$ and there exists $\beta \in L^1_{\mathrm{loc}}[0,\infty)$ such that $\sup_n \alpha_n(t) \le \beta(t)$ for almost every $t \ge 0$, where $\beta \in L^1_{\mathrm{loc}}[0,\infty)$ means the restriction of β on $[0,s]$ belongs to $L^1[0,s]$ for each $s > 0$. Let $\{a_n\}$ be a real sequence such that $0 < a_n \le 1$ and $a_n \to 0$ and let $\{b_n\}$ be a real sequence such that $0 \le b_n \le 1$, $b_n \to 0$ and $\sum_{n=0}^\infty b_n = \infty$. Let x and y_0 be elements of C and let $\{x_n\}$ and $\{y_n\}$ be the sequences defined by*

$$x_n = a_n x + (1-a_n) \int_0^\infty \alpha_n(t) S(t) x_n \, dt \qquad \text{for } n \in \mathbb{N},$$

and

$$y_{n+1} = b_n x + (1-b_n) \int_0^\infty \alpha_n(t) S(t) y_n \, dt \qquad \text{for } n \in \mathbb{N},$$

respectively. Then both $\{x_n\}$ and $\{y_n\}$ converge strongly to Px.

Proof. Let X be the subspace of $B([0,\infty))$ which consists of all bounded, measurable functions. We remark that an element f in X is not an equivalence class with the usual equivalence relation, where the usual equivalence relation $g \sim h$ means the Lebesgue measure of the set $\{t \in [0,\infty) : g(t) \ne h(t)\}$ is zero. The reason is that we consider that X is a subspace of $B([0,\infty))$ with the supremum norm. For each $n \in \mathbb{N}$, define a mean μ_n on X by $\mu_n(f) = \int_0^\infty \alpha_n(t) f(t)\, dt$ for each $f \in X$. Then for each $s \ge 0$, we have

$$\begin{aligned}
&\overline{\lim_{n\to\infty}} \|\mu_n - l_s^* \mu_n\| \\
&\quad = \overline{\lim_{n\to\infty}} \sup\left\{ \left| \int_0^\infty \alpha_n(t) f(t)\, dt - \int_0^\infty \alpha_n(t) f(s+t)\, dt \right| : f \in X, |f| \le 1 \right\} \\
&\quad = \overline{\lim_{n\to\infty}} \sup\left\{ \left| \int_0^s \alpha_n(t) f(t)\, dt + \int_s^\infty (\alpha_n(t) - \alpha_n(t-s)) f(t)\, dt \right| : f \in X, |f| \le 1 \right\}
\end{aligned}$$

$$\leq \varlimsup_{n\to\infty}\left(\int_0^s \alpha_n(t)\,dt + \int_s^\infty |\alpha_n(t)-\alpha_n(t-s)|\,dt\right) = 0.$$

So $\{\mu_n\}$ is strongly regular. Next, we shall show that each μ_n is monotone convergent. Fix $n \in \mathbb{N}$. Let $\mathcal{A}$ be the Lebesgue measurable field on $[0,\infty)$. For $A \in \mathcal{A}$, set $m(A) = \int_A \alpha_n(t)\,dt$. Then $([0,\infty),\mathcal{A},m)$ is a measure space. Let f be an element of X with $f \geq 0$ and let $\{f_j : j \in \mathbb{N}\}$ be a nonnegative, monotone increasing sequence of X such that $\lim_j f_j(t) = f(t)$ for each $t \geq 0$. Then by the monotone convergence theorem, we have

$$\lim_{j\to\infty} \mu_n(f_j) = \lim_{j\to\infty}\int_0^\infty \alpha_n(t)f_j(t)\,dt = \int_0^\infty \alpha_n(t)f(t)\,dt = \mu_n(f).$$

Hence by Theorems 3 and 5, we obtain the conclusion. □

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