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DUNFORD-PETTIS-TYPES THEOREM AND CONVERGENCES IN SET-VALUED INTEGRATION

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Abstract. We present new weak compactness results and convergences for bounded sequences of convex weakly compact valued integrably bounded multifunctions as well as we give applications to Multivalued biting-types lemma and Fatou-types lemma in Mathematical Economics.

1. Introduction

Let E be a Banach space and $L^1_E(\mu)$ the Banach space of (equivalence classes of) Lebesgue-Bochner integrable functions over a complete probability space $(\Omega, \mathcal{F}, \mu)$. Ülger-Diestel-Ruess-Shachermayer [26, 17] presented a characterization of weakly relatively compact subsets $\mathcal{H}$ of $L^1_E(\mu)$ as bounded uniformly integrable subsets for which :

(*) given any sequence (u_n) in $\mathcal{H}$, there exists a sequence (v_n), with $v_n \in co\{u_m : m \geq n\}$ such that $(v_n(\omega))$ is weakly convergent in E for almost all $\omega \in \Omega$.

In the present paper we state several results of convergence in the space $\mathcal{L}^1_{cwk(E)}(\mu)$ of convex weakly compact integrably bounded multifunctions [9, 10, 11, 12, 14] where $cwk(E)$ denotes the collection of all nonempty convex weakly compact subsets in a separable Banach space E. In Section 3 we provide some necessary conditions for which given a bounded sequence (X_n) in $\mathcal{L}^1_{cwk(E)}(\mu)$ there exists a sequence $(\widetilde{X}_n)$ with $\widetilde{X}_n \in co\{X_m : m \geq n\}$ (i.e. $\widetilde{X}_n$ has the form $\widetilde{X}_n = \Sigma_{i=n}^{\nu_n} \lambda_i^n X_i$ with $0 \leq \lambda_i^n \leq 1$, $\Sigma_{i=n}^{\nu_n} \lambda_i^n = 1$) such that $(\widetilde{X}_n)$ converges in the *linear topology* [4] to a multifunction $X \in \mathcal{L}^1_{cwk(E)}(\mu)$ for almost all $\omega \in \Omega$. These results (Theorem 3.2 and Theorem 3.3) provide the analogs of (*) in view of the *weak convergence* in $\mathcal{L}^1_{cwk(E)}(\mu)$: a sequence (X_n) in $\mathcal{L}^1_{cwk(E)}(\mu)$ *weakly converges* to $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$ if

$$\forall h \in L^\infty_{E^*}(\mu), \lim_{n\to\infty} \int_\Omega \delta^*(h, X_n)\, d\mu = \int_\Omega \delta^*(h, X_\infty)\, d\mu$$

where E^* is the dual of E, $x^* \mapsto \delta^*(x^*, K)$ denotes the support function of an element $K \in cwk(E)$. Afterwards we give (Theorem 3.4) two characterizations of weak convergence in $\mathcal{L}^1_{cwk(E)}(\mu)$ via the preceding mode of convergence (shortly Mazur τ_L-convergence). In section 4 we present two Komlós convergence-types theorem (Theorems 4.1-4.2) for bounded *weak tight* sequences in both $L^1_E(\mu)$ and $\mathcal{L}^1_{cwk(E)}(\mu)$. In section 5 we deal with Mazur convergence-types for bounded sequences in $\mathcal{L}^1_{cwk(E)}(\mu)$ (Theorem 5.1, Propositions 5.2-5.2′) and weak Komlós

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convergence in measure (Proposition 5.3). In section 6 we state a multivalued version of biting lemma for bounded sequences in $\mathcal{L}^1_{cwk(E)}(\mu)$ (Theorem 6.1) with application to Fatou-types lemma in Mathematical Economics (Theorems 6.2-6.3). Our results provide new Dunford-Pettis-types theorem in $\mathcal{L}^1_{cwk(E)}(\mu)$ via unusual modes of convergence, extending the classical weak compactness results in $L^1_E(\mu)$ (see, for example, [2, 5, 6, 8, 10, 11, 14, 17, 24, 26]).

2. Notations and terminology

Throughout E is a separable Banach space, E^* is the topological dual of E and $\overline{B}_{E^*}$ is the closed unit ball in E^*. We denote by s (resp. w) the topology defined by the norm of E (resp. the weak topology on E). By $cwk(E)$ (resp. $cbc(E)$) (resp. $\mathcal{L}wk(E)$) (resp. $\mathcal{R}wk(E)$) we denote the collection of all nonempty convex weakly compact subsets of E (resp. convex bounded closed subsets of E) (resp. closed convex weakly locally compact subsets of E which contain no lines) (resp. convex closed subsets of E such that their intersections with any closed ball are weakly compact, shortly *ball-wc*). The support function (resp. distance function) of a subset A in E is defined by

$$\delta^*(x^*, A) := \sup_{x\in A}\langle x^*, x\rangle \ (x^* \in E^*) \ \text{(resp. } d(x, A) := \inf_{y\in A} ||x - y|| \ (x \in E)).$$

On $cwk(E)$ we will consider the following limiting notions. A sequence (C_n) in $cwk(E)$ converges in the *Linear topology* [4] (shortly τ_L topology) to $C \in cwk(E)$ if the following two conditions are satisfied :

$$\forall x^* \in E^*, \ \lim_{n\to\infty} \delta^*(x^*, C_n) = \delta^*(x^*, C) \ \text{ and } \forall x \in E, \ \lim_{n\to\infty} d(x, C_n) = d(x, RC).$$

We also use the following limits

$$s\text{-}li\, C_n = \{x \in E : ||x_n - x|| \to 0;\ x_n \in C_n\}$$

and

$$w\text{-}ls\, C_n = \{x \in E : x = w\text{-}\lim_j x_{n_j};\ x_{n_j} \in C_{n_j}\}.$$

Let A and B be two convex weakly compact subsets of $E, H(A,B)$ denotes the Hausdorff distance between A and B. The gap [4] between A and B is denoted by $D(A,B) := \inf\{d(x,y) : x \in A; y \in B\}$.

Let $(\Omega, \mathcal{F}, \mu)$ be a complete probability space, $L^1_E(\Omega, \mathcal{F}, \mu)$ (shortly $L^1_E(\mu)$) the space (classes of equivalence) of Bochner integrable E-valued functions. A sequence (u_n) in $L^1_E(\mu)$ is $cwk(E)$-*tight* if, for every $\varepsilon > 0$, there is a $cwk(E)$-valued measurable multifunction $\Gamma_\varepsilon : \Omega \to E$ such that

$$\sup_n \mu(\{\omega \in \Omega : u_n(\omega) \notin \Gamma_\varepsilon(\omega)\}) \le \varepsilon.$$

A subset $\mathcal{H}$ of $L^1_E(\mu)$ has the weak Talagrand property (WTP) if, given any bounded sequence (u_n) in $\mathcal{H}$, there exist a sequence $(\widetilde{u}_n)$ with $\widetilde{u}_n \in co\{u_m : m \ge n\}$ and $u \in L^1_E(\mu)$ such that $(\widetilde{u}_n)$ weakly converges a.e to u. We refer to [5, 6, 13] for the study of WTP subsets of $L^1_E(\mu)$. Let us mention some useful results. Any bounded $cwk(E)$-*tight* sequence in $L^1_E(\mu)$ has the WTP. If (u_n) is a bounded WTP sequence in $L^1_E(\mu)$, then there exist a sequence $(\widetilde{u}_n)$ in $L^1_E(\mu)$ with $\widetilde{u}_n \in co\{u_m : m \ge n\}$ and

$u \in L^1_E(\mu)$ such that $(\widetilde{u}_n)$ converges a.e to u for the norm of E. Furthermore $L^1_E(\mu)$ has the WTP iff E reflexive.

Let us recall the following version of the biting lemma [21, 18, 23, 25, 13].

Biting lemma. *Suppose that (u_n) is a bounded sequence in $L^1_{\mathbb{R}}(\mu)$, then there exists a subsequence (u'_n) and an increasing sequence (A_n) in $\mathcal{F}$ with $\lim_n \mu(A_n) = 1$ such that the sequence $(1_{A_n} u'_n)$ is uniformly integrable.*

By $\mathcal{L}^1_{cwk(E)}(\mu)$ we denote the space of all nonempty convex weakly compact valued $\mathcal{F}$-measurable and integrably bounded multifunctions (see, for example, [10, 11, 14]). A sequence (X_n) in $\mathcal{L}^1_{cwk(E)}(\mu)$ is *bounded* (resp. *uniformly integrable*) if the sequence $(|X_n|)$ is bounded (resp. uniformly integrable) where

$$|X_n| : \omega \mapsto \sup_{x^* \in \overline{B}_{E^*}} |\delta^*(x^*, X_n(\omega))|.$$

If $X \in \mathcal{L}^1_{cwk(E)}(\mu)$ and $A \in \mathcal{F}$, the *integral of X over A* is defined by

$$E(1_A X) := \int_A X\, d\mu := \{\int_A u\, d\mu : u \in \mathcal{S}^1_X\}$$

where $\mathcal{S}^1_X$ denotes the set of all integrable selections [14] of X. Since $\mathcal{S}^1_X$ is convex weakly compact in $L^1_E(\mu)$ (see, for example, [2, 5, 6, 8, 10, 17, 24, 26]), $\int_A X\, d\mu$ is convex weakly compact in E. We refer also to [1] for the weak compactness in Pettis integration.

A sequence (X_n) in $\mathcal{L}^1_{cwk(E)}(\mu)$ *weakly converges* to $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$, if

$$\forall h \in L^\infty_{E^*}(\mu),\ \lim_{n\to\infty} \int_\Omega \delta^*(h(\omega), X_n(\omega))\, \mu(d\omega) = \int_\Omega \delta^*(h(\omega), X_\infty(\omega))\, \mu(d\omega).$$

3. Mazur convergence in $\mathcal{L}^1_{cwk(E)}(\mu)$

If (u_n) is a relatively weakly compact sequence in $L^1_E(\mu)$, by Eberlein-Smulian theorem and Mazur lemma, there is a sequence $(\widetilde{v}_n)$ with $\widetilde{v}_n \in co\{u_m : m \geq n\}$ which converges almost everywhere to a function $u \in L^1_E(\mu)$. We aim to present first two versions of this result in $\mathcal{L}^1_{cwk(E)}(\mu)$.

We begin with a technical lemma.

Lemma 3.1. *Suppose that $(u_{n,j})_{n,j\geq 1}$ is a sequence in $L^1_E(\mu)$ such that, for each j, the sequence $(u_{n,j})_n$ converges $\sigma(L^1, L^\infty)$ to $u_{\infty,j} \in L^1_E(\mu)$, then there exists a sequence $(v_{n,j})_{n,j\geq 1}$ with $v_{n,j} = \Sigma_{i=n}^{\nu_n} \lambda_i^n u_{i,j}$, where $\lambda_i^n \geq 0$ and $\Sigma_{i=n}^{\nu_n} \lambda_i^n = 1$ such that, for every j, $(v_{n,j})_n$ converges almost everywhere to $u_{\infty,j}$.*

Proof. Applying Mazur lemma to $(u_{n,1})_n$ provides a subsequence $(\widetilde{u}_{n,1})_n$ with $\widetilde{u}_{n,1} \in co\{u_{i,1} : i \geq n\}$ such that $(\widetilde{u}_{n,1})_n$ converges a.e to $u_{\infty,1}$. Each $\widetilde{u}_{n,1}$ has the form

$$\widetilde{u}_{n,1} = \Sigma_{n,1}^{\nu_{n,1}} \lambda_i^{n,1} u_{i,1}$$

with $\lambda_i^{n,1} \geq 0$ and $\Sigma_{n,1}^{\nu_{n,1}} \lambda_i^{n,1} = 1$. For every $(n, j) \in \mathbb{N}^* \times \mathbb{N}^*$, let us define

$$\widetilde{u}_{n,1,j} := \Sigma_{n,1}^{\nu_{n,1}} \lambda_i^{n,1} u_{i,j}.$$

Hence $(\widetilde{u}_{n,1,2})_n$ converges $\sigma(L^1, L^\infty)$ to $u_{\infty,2}$. Again from Mazur lemma there exists a sequence $(\widetilde{u}_{n,2})_n$ with $\widetilde{u}_{n,2} \in co\{\widetilde{u}_{m,1,2} : m \geq n, 2\}$ such that $(\widetilde{u}_{n,2})_n$ converges a.e to $u_{\infty,2}$. Each $\widetilde{u}_{n,2}$ has the form

$$\widetilde{u}_{n,2} = \Sigma_{n,2}^{\nu_{n,2}} \lambda_m^{n,2} \widetilde{u}_{m,1,2}$$

with $\lambda_m^{n,2} \geq 0$ and $\Sigma_{n,2}^{n,2} \lambda_m^{n,2} = 1$. For every $(n, j) \in \mathbb{N}^* \times \mathbb{N}^*$, let us define

$$\widetilde{u}_{n,2,j} := \Sigma_{n,2}^{\nu_{n,2}} \lambda_m^{n,2} \widetilde{u}_{m,1,j}.$$

Hence for $j = 1, 2$ $(\widetilde{u}_{n,2,j})_n$ converges a.e to $u_{\infty,j}$. By induction we find a sequence $(\widetilde{u}_{n,k,j})_n$ $((k, j) \in \mathbb{N}^* \times \mathbb{N}^*))$ where

$$\widetilde{u}_{n,k,j} = \Sigma_{n,k}^{\nu_{n,k}} \lambda_i^{n,k} \widetilde{u}_{i,k-1,j}$$

with $\lambda_i^{n,k} \geq 0$ and $\Sigma_{n,k}^{\nu_{n,k}} \lambda_i^{n,k} = 1$ such that for every $k \geq 2$ and every $j \leq k$, $(\widetilde{u}_{n,k,j})_n$ converges a.e to $u_{\infty,j}$. It is not difficult to check that each $\widetilde{u}_{n,k,j}$ has the form

$$\widetilde{u}_{n,k,j} = \Sigma_{n,k}^{\nu_{n,k}} \delta_i^{n,k} u_{i,j}$$

with $\delta_i^{n,k} \geq 0$ and $\Sigma_{n,k}^{\nu_{n,k}} \delta_i^{n,k} = 1$ and $\widetilde{u}_{n,k,j} \in co\{\widetilde{u}_{i,j,j} : i \geq n, k\}$ for $j < k$. Since for every j the sequence $(\widetilde{u}_{n,j,j})_n$ converges a.e to $u_{\infty,j}$, the diagonal sequence $(v_{n,j})_n = (\widetilde{u}_{n,n,j})_n$ converges a.e to $u_{\infty,j}$ and has the required properties. □

Remark. The same diagonal process used above allows us to prove the preceding lemma when we consider a finitely indexed sequence $(u_{n,j_1,j_2,\dots,j_N})$; for example, if $N = 2$ the diagonal process is based on applying successively Lemma 3.1 for each j_2.

Now we are able to state the following Mazur τ_L-convergence in $\mathcal{L}^1_{cwk(E)}(\mu)$.

Theorem 3.2. *Suppose that E^* is separable, E has the Radon-Nikodym property, (X_n) is a bounded sequence in $\mathcal{L}^1_{cwk(E)}(\mu)$ such that, for every $A \in \mathcal{F}$, $\bigcup_n \int_A X_n\, d\mu$ is relatively weakly compact in E, then there exist a sequence $\widetilde{X}_n$ in $\mathcal{L}^1_{cwk(E)}(\mu)$ with $\widetilde{X}_n \in co\{X_m : m \geq n\}$ and $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$ such that*

$$X_\infty = \tau_L\text{-}\lim_n \widetilde{X}_n \ a.e.$$

Proof. We will divide the proof in several steps.

Step 1. In view of the multivalued biting lemma in [12, Theorem 3.1], [9, Theorem 2.6], there exist an increasing sequence (A_p) in $\mathcal{F}$ with $\lim_{n\to\infty} \mu(A_p) = 1$, a subsequence (X'_n) of (X_n) such that the sequence $(1_{A_p} X'_n)$ is uniformly integrable for each p and $X'_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$ such that $\forall p, \forall A \in A_p \cap \mathcal{F}$ and $\forall x^* \in E^*$, the following holds :

$$\lim_{n\to\infty} \int_A \delta^*(x^*, X'_n)\, d\mu = \int_A \delta^*(x^*, X'_\infty)\, d\mu.$$

Now, since the sequence $(|X'_n|)$ is bounded there is a sequence (θ_n) in $L^1_{\mathbb{R}}(\mu)$ with $\theta_n \in co\,\{|X'_m| : m \geq n\}$ such that (θ_n) converges a.e to an integrable function θ. Each θ_n has the form $\theta_n = \Sigma_{i=n}^{k_n} \eta_i^n |X'_i|$ with $0 \leq \eta_i^n \leq 1$ and $\Sigma_{i=n}^{k_n} \eta_i^n = 1$.

Consequently the sequence $(S_n)_n = (\Sigma_{i=n}^{k_n} \eta_i^n X_i')_n$ is pointwise bounded almost everywhere, that is, there is a negligible set N_1 such that $\sup_n |S_n|(\omega) < \infty$ for $\omega \in \Omega \setminus N_1$.

Step 2. Let us consider a countable dense subset $D^* := (x_j^*)_{j\in\mathbb{N}^*}$ in $\overline{B}_{E^*}$. For every $j \in \mathbb{N}^*$ pick a maximal integrable selection $u_{n,j} \in \mathcal{S}^1_{S_n}$:

$$\langle x_j^*, u_{n,j}\rangle := \delta^*(x_j^*, S_n).$$

Then the sequence $(1_{A_p} u_{n,j})_n$ is uniformly integrable for every j and every p. By Dunford-Pettis theorem, $(1_{A_p} u_{n,j})_n$ is relatively sequentially weakly compact in $L^1_E(\mu)$. Therefore, by using an appropriate diagonal procedure, we may suppose for simplicity that, for every j and every p, $(1_{A_p} u_{n,j})_n$ converges $\sigma(L^1, L^\infty)$ to an integrable function $u_{\infty,j,p}$ so that we can apply the Remark of Lemma 3.1 to $(1_{A_p} u_{n,j})_{n,j,p\geq 1}$. Hence there exists a negligible set N_2, a sequence $(v_{n,j,p})_{n,j,p\geq 1}$ with $v_{n,j,p} = \Sigma_{i=n}^{\nu_n} \lambda_i^n 1_{A_p} u_{i,j}$, where $\lambda_i^n \geq 0$ and $\Sigma_{i=n}^{\nu_n} \lambda_i^n = 1$ such that, for every j and every p, $(v_{n,j,p})_n$ pointwise converges to $u_{\infty,j,p}$ for all $\omega \in \Omega \setminus N_2$. Since (A_p) is increasing, it is not difficult to check that, for every j, $u_{\infty,j,p} = u_{\infty,j,p+1}$ in $A_p \cap (\Omega \setminus N_2)$ for each p. Now set $u_{\infty,j}(\omega) = u_{\infty,j,p}$ if $\omega \in A_p \cap [\Omega \setminus N_2]$ and $u_{\infty,j}(\omega) = 0$ if $\omega \in N_3 := \cap_p (\Omega \setminus A_p) \cup N_2$. Then the sequence $(v_{n,j})_n = (\Sigma_{i=n}^{\nu_n} \lambda_i^n u_{i,j})_n$ pointwise converges to $u_{\infty,j}$ for all $\omega \in \Omega \setminus N_3$. Furthermore by Fatou lemma, we have

$$\int_\Omega |u_{\infty,j}|\, d\mu \leq \liminf_n \int_\Omega |v_{n,j}|\, d\mu \leq \sup_n \int_\Omega |u_{n,j}|\, d\mu$$

$$\leq \sup_n \int_\Omega |S_n|\, d\mu \leq \sup_n \int_\Omega |X_n|\, d\mu < \infty.$$

Whence we have $u_{\infty,j} \in L^1_E$.

Step 3. Let us consider the multifunctions

$$\widetilde{X}_n := \Sigma_{i=n}^{\nu_n} \lambda_i^n S_i, \;\; X_\infty := 1_{\Omega\setminus[N_1\cup N_3]}\, s\text{-}li\, \widetilde{X}_n.$$

Then $(\widetilde{X}_n)$ is pointwise bounded almost everywhere, X_∞ is $cbc(E)$-valued and $u_{\infty,j} \in X_\infty$ a.e for every j. Furthermore we have

$$\lim_{n\to\infty} \delta^*(x_j^*, \widetilde{X}_n) = \lim_{n\to\infty} \langle x_j^*, v_{n,j}\rangle = \langle x_j^*, u_{\infty,j}\rangle \leq \delta^*(x_j^*, X_\infty)\ a.e$$

for every j. On the other hand it is not difficult to see that

$$\delta^*(x_j^*, X_\infty) \leq \lim_{n\to\infty} \delta^*(x_j^*, \widetilde{X}_n)\ a.e$$

for every j. Hence we get

$$\delta^*(x_j^*, X_\infty) = \lim_{n\to\infty} \delta^*(x_j^*, \widetilde{X}_n)\ a.e \tag{3.2.1}$$

for every j. We claim that

$$\lim_{n\to\infty} \delta^*(x^*, \widetilde{X}_n) = \delta^*(x^*, X_\infty) \tag{3.2.2}$$

for all $x^* \in \overline{B}_{E^*}$ and for almost all $\omega \in \Omega$. We will use an argument in ([8], Lemma 3.2). We have

$$\begin{aligned} |\delta^*(x^*, \widetilde{X}_n) - \delta^*(x^*, X_\infty)| &\leq \max\{\delta^*(x^* - x_j^*, \widetilde{X}_n), \delta^*(x_j^* - x^*, \widetilde{X}_n)\} \\ &\quad + |\delta^*(x_j^*, \widetilde{X}_n) - \delta^*(x_j^*, X_\infty)| \\ &\quad + \max\{\delta^*(x^* - x_j^*, X_\infty), \delta^*(x_j^* - x^*, X_\infty)\} \end{aligned} \tag{3.2.3}$$

for all $x^* \in E^*$ and for all j. Now let $x^* \in \overline{B}_{E^*}$ and $\varepsilon > 0$. There is $x_j^* \in D^*$ such that $||x^* - x_j^*|| \leq \varepsilon$. Then we have

$$|\delta^*(x^*, \widetilde{X}_n) - \delta^*(x^*, X_\infty)| \leq \varepsilon \sup_n |\widetilde{X}_n| + |\delta^*(x_j^*, \widetilde{X}_n) - \delta^*(x_j^*, X_\infty)| + \varepsilon\, |X_\infty|. \tag{3.2.4}$$

Thus, by (3.2.1) and the pointwise boundness of $(\widetilde{X}_n)$ the claim follows. Now in view of the multivalued biting lemma given in Step 1 and (3.2.2), there is a negligible set N such that $X_\infty(\omega) = X'_\infty(\omega)$ for all $\omega \in \Omega \setminus N$. It suffices to set $X_\infty(\omega) = 0$ on N to get $X_\infty(\omega) \in cwk(E)$ for all $\omega \in \Omega$. Consequently, by (3.2.2) X_∞ is measurable and

$$|X_\infty| \leq \liminf_n |\widetilde{X}_n| \ a.e$$

By Fatou lemma

$$\int_\Omega |X_\infty|\, d\mu \leq \int_\Omega \liminf_n |\widetilde{X}_n|\, d\mu < \infty$$

because the sequence $(|X_n|)$ is bounded in $L^1_{\mathbb{R}}$.

Step 4. Now we claim that

$$\lim_{n\to\infty} d(x, \widetilde{X}_n) = d(x, X_\infty)$$

for all $x \in E$ and almost all $\omega \in \Omega$. Indeed we have

$$\begin{aligned} \liminf_n d(x, \widetilde{X}_n) &= \liminf_n \sup_{x^* \in \overline{B}_{E^*}} [\langle x^*, x\rangle - \delta^*(x^*, \widetilde{X}_n)] \\ &\geq \sup_{x^* \in \overline{B}_{E^*}} \lim_n [\langle x^*, x\rangle - \delta^*(x^*, \widetilde{X}_n)] \\ &= \sup_{x^* \in \overline{B}_{E^*}} [\langle x^*, x\rangle - \delta^*(x^*, X_\infty)] \\ &= d(x, X_\infty) \end{aligned}$$

for all $x \in E$ and almost all $\omega \in \Omega$. By definition of X_∞ we have

$$\limsup_n d(x, \widetilde{X}_n)) \leq d(x, X_\infty)$$

for all $x \in E$ and almost all $\omega \in \Omega$. So the claim follows. □

There is a useful corollary.

Corollary 3.2'. *Suppose that E^* is separable, E has the Radon-Nikodym property, (u_n) is a bounded sequence in $L^1_E(\mu)$ such that, for every $A \in \mathcal{F}$, $\bigcup_n \int_A u_n\, d\mu$ is relatively weakly compact in E, then there exist a sequence $(\widetilde{u}_n)$ in $L^1_E(\mu)$ with $\widetilde{u}_n \in co\{u_m : m \geq n\}$ and $u_\infty \in L^1_E(\mu)$ such that $||\widetilde{u}_n(\omega) - u_\infty(\omega)|| \to 0$ a.e.*

In other words, (u_n) has the WTP. It turns out that Corollary 3.2' provides new applications to minimization problems in the spirit of [5, 6, 13] dealing with closed convex bounded WTP sets which are closed in measure.

Remark. Actually the multifunction limit X_∞ in Theorem 3.2 satisfies the inclusion $X_\infty(\omega) \subset \cap_n \overline{co}\,[\cup_{m\geq n}\widetilde{X}_n(\omega)]$ *a.e* because τ_L-convergence implies Mosco convergence. See for instance [4].

Now we proceed to establish a significant variant of the preceding result.

Theorem 3.3. *Suppose that E is a separable Banach space, $L : \Omega \to E$ is a $\mathcal{R}wk(E)$-valued measurable multifunction and (X_n) is a bounded sequence in $\mathcal{L}^1_{cwk(E)}(\mu)$ such that $X_n(\omega) \subset L(\omega)$ for all $n \in \mathbb{N}^*$ and all $\omega \in \Omega$, then there exist a sequence $(\widetilde{X}_n)$ in $\mathcal{L}^1_{cwk(E)}(\mu)$ with $\widetilde{X}_n \in co\{X_m : m \geq n\}$ and $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$ such that*

$$X_\infty = \tau_L\text{-}\lim_n \widetilde{X}_n \ a.e.$$

Proof. We will use several arguments of the proof of Theorem 3.2 with appropriate modifications.

A. We begin to state the theorem in the particular case when (X_n) is uniformly integrable.

Step 1. Since the sequence (X_n) is bounded, there is a sequence (θ_n) in $L^1_{\mathbb{R}}(\mu)$ with $\theta_n \in co\{|X_m| : m \geq n\}$ such that (θ_n) converges a.e to an integrable function θ. Each θ_n has the form $\theta_n = \Sigma_{i=n}^{k_n}\eta_n^i|X_i|$ with $0 \leq \eta_n^i \leq 1$ and $\Sigma_{i=n}^{k_n}\eta_n^i = 1$. Let us consider the $cwk(E)$-valued multifunctions $S_n := \Sigma_{i=n}^{k_n}\eta_n^i X_i$ and $\Gamma(.) := L(.)\cap m(.)\overline{B}_E$ where $m := \sup_n |\theta_n|$. Then, $\forall n \in \mathbb{N}^*, S_n(\omega) \subset \Gamma(\omega)$ a.e.

Step 2. First we note that $\bigcup_n \mathcal{S}^1_{S_n}$ is relatively $\sigma(L^1, L^\infty)$ compact by [2, Théorème 6], [8, Théorème 4.1]. Let us consider a countable dense subset $D^* := (x_j^*)_{j\in\mathbb{N}^*}$ in $\overline{B}_{E^*}$ for the Mackey topology. For every $j \in \mathbb{N}^*$ pick a maximal integrable selection $u_{n,j} \in \mathcal{S}^1_{S_n}$. Then for every j, the sequence $(u_{n,j})_n$ is relatively $\sigma(L^1, L^\infty)$ sequentially compact. Therefore by using an appropriate diagonal procedure, we may suppose for simplicity that, for every j, $(u_{n,j})_n$ converges $\sigma(L^1, L^\infty)$ to an integrable function $u_{\infty,j}$ so that we can apply Lemma 3.1 to $(u_{n,j})_{n,j}$. Hence there exist a negligible set N, a sequence $(v_{n,j})_{n,j\geq 1}$ with $v_{n,j} = \Sigma_{i=n}^{\nu_n}\lambda_i^n u_{i,j}$, where $\lambda_i^n \geq 0$ and $\Sigma_{i=n}^{\nu_n}\lambda_i^n = 1$ such that, for every j, $(v_{n,j})_n$ pointwise converges to $u_{\infty,j}$ for all $\omega \in \Omega \setminus N$.

Step 3. Let us consider the multifunctions

$$\widetilde{X}_n := \Sigma_{i=n}^{\nu_n}\lambda_i^n S_i, \ \ X_\infty := 1_{\Omega\setminus N}\, s\text{-}li\, \widetilde{X}_n.$$

Then by Step 1, $u_{\infty,j} \in X_\infty$ a.e for every j and X_∞ is $cwk(E)$-valued. Furthermore we have

$$\lim_{n\to\infty} \delta^*(x_j^*, \widetilde{X}_n) = \lim_{n\to\infty}\langle x_j^*, v_{n,j}\rangle = \langle x_j^*, u_{\infty,j}\rangle \leq \delta^*(x_j^*, X_\infty) \ a.e$$

and

$$\delta^*(x_j^*, X_\infty) \leq \lim_{n\to\infty} \delta^*(x_j^*, \widetilde{X}_n) \ a.e$$

for every j. Hence we get

$$\delta^*(x_j^*, X_\infty) = \lim_{n\to\infty} \delta^*(x_j^*, \widetilde{X}_n) \ = a.e \tag{3.3.1}$$

for every j.

Step 3. We will use an argument in ([8], Lemma 3.2). Let $x^* \in \overline{B}_{E^*}$. We have

$$\begin{aligned}|\delta^*(x^*, \widetilde{X}_n) - \delta^*(x^*, X_\infty)| &\le \max\{\delta^*(x^* - x_j^*, \widetilde{X}_n), \delta^*(x_j^* - x^*, \widetilde{X}_n)\}\\ &\quad + |\delta^*(x_j^*, \widetilde{X}_n) - \delta^*(x_j^*, X_\infty)|\\ &\quad + \max\{\delta^*(x^* - x_j^*, X_\infty), \delta^*(x_j^* - x^*, X_\infty)\}\end{aligned}$$

or every j. Let $\omega \in \Omega \setminus N$ be fixed and $\varepsilon > 0$. Since $\Gamma(\omega)$ is weakly compact, there is $x_j^* \in D^*$ such that

$$\max\{\delta^*(x^* - x_j^*, \Gamma(\omega)), \delta^*(x_j^* - x^*, \Gamma(\omega))\} \le \varepsilon.$$

Since $\widetilde{X}_n(\omega) \subset \Gamma(\omega)$ for all $n \in \mathbb{N}^* \cup \{\infty\}$ and for almost all $\omega \in \Omega$, from (3.3.1) it follows that

$$|\delta^*(x^*, \widetilde{X}_n) - \delta^*(x^*, X_\infty)| \le |\delta^*(x_j^*, \widetilde{X}_n) - \delta^*(x_j^*, X_\infty)| + 2\varepsilon \tag{3.3.2}$$

showing that

$$\lim_{n\to\infty} \delta^*(x^*, \widetilde{X}_n) = \delta^*(x^*, X_\infty) \tag{3.3.3}$$

for all $x^* \in \overline{B}_{E^*}$ and almost all $\omega \in \Omega$. By (3.3.3) X_∞ is measurable and

$$|X_\infty| \le \liminf_n |\widetilde{X}_n| \ a.e.$$

By Fatou lemma we have

$$\int_\Omega |X_\infty|\, d\mu \le \int_\Omega \liminf_n |\widetilde{X}_n|\, d\mu < \infty$$

since the sequence $(|X_n|)$ is bounded in $L^1_{\mathbb{R}}(\mu)$.

Step 4. Now we claim that

$$\lim_{n\to\infty} d(x, \widetilde{X}_n) = d(x, X_\infty)$$

for all $x \in E$ and almost all $\omega \in \Omega$. Indeed we have

$$\begin{aligned}\liminf_n d(x, \widetilde{X}_n) &= \liminf_n \sup_{x^*\in\overline{B}_{E^*}} [\langle x^*, x\rangle - \delta^*(x^*, \widetilde{X}_n)]\\ &\ge \sup_{x^*\in\overline{B}_{E^*}} \lim_n [\langle x^*, x\rangle - \delta^*(x^*, \widetilde{X}_n)]\\ &= \sup_{x^*\in\overline{B}_{E^*}} [\langle x^*, x\rangle - \delta^*(x^*, X_\infty)]\\ &= \delta^*(x^*, X_\infty)\end{aligned}$$

for all $x \in E$ and almost all $\omega \in \Omega$. By definition of X_∞, we have

$$\limsup_n d(x, \widetilde{X}_n) \le d(x, X_\infty)$$

for all $x \in E$ and all $\omega \in \Omega$.

B. Now we pass to the general case, that is, $(|X_n|)$ is bounded. Using the biting lemma provides an increasing sequence (B_n) in $\mathcal{F}$ with $\lim_n \mu(B_n) = 1$ and a subsequence (X'_n) of (X_n) such that $X'_n = 1_{B_n}X'_n + 1_{\Omega\setminus B_n}X'_n$ where $(Y'_n) = (1_{B_n}X'_n)$ is uniformly integrable and $|Z'_n| \to 0$ a.e where $Z'_n = 1_{\Omega\setminus B_n}X'_n$. Consequently we may apply the result obtained in Step A to (Y'_n). This gives a sequence $(\widetilde{Y}_n)$ in $\mathcal{L}^1_{cwk(E)}(\mu)$ with $\widetilde{Y}_n \in co\{Y'_m : m \geq n\}$ and $Y_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$ such that $Y_\infty = \tau_L\text{-}\lim_n \widetilde{Y}_n$ a.e. Each $\widetilde{Y}_n$ has the form $\Sigma_{i=n}^{\nu_n}\lambda_i^n Y'_i$ with $\lambda_i^n \geq 0$ and $\Sigma_{i=n}^{\nu_n}\lambda_i^n = 1$. Since the sequence $(\Sigma_{i=n}^{\nu_n}\lambda_i^n|Z'_i|)$ converges to 0 a.e because $|Z'_n| \to 0$ a.e, the sequence $(\Sigma_{i=n}^{\nu_n}\lambda_i^n X'_i)$ τ_L-converges to Y_∞ a.e. □

Now we proceed to an application of the preceding theorems to weak compactness in the space $\mathcal{L}^1_{cwk(E)}(\mu)$.

Theorem 3.4. *Suppose that E is a separable Banach space, and (X_n) is a uniformly integrable sequence in $\mathcal{L}^1_{cwk(E)}(\mu)$ which satisfies one of the following two conditions :*

(a) *For any subsequence (Y_n), there is a sequence $(\widetilde{Y}_n)$ with $\widetilde{Y}_n \in co\{Y_m : m \geq n\}$ such that for almost all $\omega \in \Omega$, $\overline{co}[\cup_n \widetilde{Y}_n(\omega)]$ is ball-wc in E.*
(b) *E^* is separable, E has the RNP and for any subsequence (Y_n), there is a sequence $(\widetilde{Y}_n)$ with $\widetilde{Y}_n \in co\{Y_m : m \geq n\}$ such that $\cup_n \int_A \widetilde{Y}_n\, d\mu$ is relatively weakly compact in E for every $A \in \mathcal{F}$.*

Then there are a subsequence (X'_n) of (X_n) and $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$ such that

$$\forall u \in L^\infty_{E^*}(\mu)\,,\ \lim_{n\to\infty}\int_\Omega \delta^*(u, X'_n)\, d\mu = \int_\Omega \delta^*(u, X_\infty)\, d\mu.$$

Proof. We will divide the proof in three steps.

Step 1. Let $D^* := (e^*_p)$ be a dense sequence for the Mackey topology. As (X_n) is uniformly integrable, for each p, the sequence $(\delta^*(e^*_p, X_n))_n$ is relatively weakly compact in $L^1_{\mathbb{R}}(\mu)$. Using an appropriate diagonal procedure provides a sequence $(\varphi_p)_p$ of real-valued integrable functions and a subsequence (X'_n) of (X_n) such that $(\delta^*(e^*_p, X'_n))_n$ converges $\sigma(L^1, L^\infty)$ to φ_p for every p. Hence we have

$$\forall p,\ \forall A \in \mathcal{F},\ \lim_n \int_A \delta^*(e^*_p, X'_n)\, d\mu = \int_A \varphi_p\, d\mu. \tag{3.4.1}$$

Step 2. Let u be a fixed element in $L^\infty_{E^*}(\mu)$. Choose a subsequence (Y_n) of (X'_n) such that

$$\lim_{n\to\infty}\int_\Omega \delta^*(u, Y_n)\, d\mu = \limsup_{n\ \to\infty}\int_\Omega \delta^*(u, X'_n)\, d\mu \tag{3.4.2}$$

Now suppose that (a) holds. Let $(\widetilde{Y}_n)$ be the sequence associated to (Y_n) according to (a). By Theorem 3.3 there exist a multifunction $Z_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$ and a sequence (Z_n) with

$$Z_n \in co\{\widetilde{Y}_m : m \geq n\} \subset co\{Y_m : m \geq n\}$$

such that

$$Z_\infty = \tau_L\text{-}\lim_n Z_n \ a.e.$$

From Lebesgue-Vitali theorem, we have

$$\lim_{n\to\infty}\int_\Omega \delta^*(v, Z_n)\, d\mu = \int_\Omega \delta^*(v, Z_\infty)\, d\mu \tag{3.4.3}$$

for all $v \in L^\infty_{E^*}(\mu)$. Since each Z_n has the form

$$Z_n = \Sigma_{i=n}^{\nu_n}\lambda_i^n Y_i \ \text{ with } \ \lambda_i^n \geq 0 \ \text{ and } \ \Sigma_{i=n}^{\nu_n}\lambda_i^n = 1,$$

from (3.4.1), (3.4.2), (3.4.3) it follows that

$$\limsup_{n\to\infty}\int_\Omega \delta^*(u, X_n')\, d\mu = \int_\Omega \delta^*(u, Z_\infty)\, d\mu \tag{3.4.4}$$

and

$$\forall p,\ \forall A \in \mathcal{F},\ \lim_{n\to\infty}\int_A \delta^*(e_p^*, X_n')\, d\mu = \int_A \varphi_p\, d\mu = \int_A \delta^*(e_p^*, Z_\infty)\, d\mu. \tag{3.4.5}$$

Similarly, we obtain a multifunction $W_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$ which verifies :

$$\liminf_{n\to\infty}\int_\Omega \delta^*(u, X_n')\, d\mu = \int_\Omega \delta^*(u, W_\infty)\, d\mu \tag{3.4.6}$$

and

$$\forall p,\ \forall A \in \mathcal{F}\ \lim_{n\to\infty}\int_A \delta^*(e_p^*, X_n')\, d\mu = \int_A \varphi_p\, d\mu = \int_A \delta^*(e_p^*, W_\infty)\, d\mu. \tag{3.4.7}$$

From (3.4.5) and (3.4.7) we get

$$\delta^*(e_p^*, Z_\infty) = \delta^*(e_p^*, W_\infty)$$

for all p and for almost all $\omega \in \Omega$. Hence by [14, Prop. III.35], $Z_\infty = W_\infty$ a.e. Therefore, by (3.4.4) and (3.4.6) we deduce that

$$\lim_{n\to\infty}\int_\Omega \delta^*(u, X_n')\, d\mu = \int_\Omega \delta^*(u, Z_\infty)\, d\mu. \tag{3.4.8}$$

Step 3. Finally applying the results obtained in the preceding steps to any other element $v \in L^\infty_{E^*}(\mu)$ gives $Z_\infty' \in \mathcal{L}^1_{cwk(E)}(\mu)$ such that

$$\lim_{n\to\infty}\int_\Omega \delta^*(v, X_n')\, d\mu = \int_\Omega \delta^*(v, Z_\infty')\, d\mu$$

with

$$\forall p,\ \forall A \in \mathcal{F},\ \lim_{n\to\infty}\int_A \delta^*(e_p^*, X_n')\, d\mu = \int_A \varphi_p\, d\mu = \int_A = \delta^*(e_p^*, Z_\infty')\, d\mu. \tag{3.4.9}$$

Then from (3.4.5) and (3.4.9) we deduce that $Z_\infty = Z_\infty'$ a.e, thus completing the proof in case (a).

In case (b) the proof is similar. It is enough to apply Theorem 3.2 instead of Theorem 3.3. □

Let us focus our attention to a particular case of Theorem 3.4 that is a multivalued analog of Theorem 8 in [2] and Theorem 4.3 in [8].

Theorem 3.5. *Suppose that E is a separable Banach space, $L : \Omega \to E$ is a $\mathcal{L}wk(E)$-valued measurable multifunction and (X_n) is a uniformly integrable* *sequence in* $\mathcal{L}^1_{cwk(E)}(\mu)$ *such that* $X_n(\omega) \subset L(\omega)$ *for all* $n \in \mathbb{N}^*$ *and all* $\omega \in \Omega$, *then there exist a subsequence* (X'_n) *of* (X_n) *and* $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$ *such that*

$$(a) \qquad \forall u \in L^\infty_{E^*}(\mu)\,,\ \lim_{n\to\infty}\int_\Omega \delta^*(u, X'_n)\,d\mu = \int_\Omega \delta^*(u, X_\infty)\,d\mu$$

and

$$(b) \qquad X_\infty(\omega) \subset \overline{co}\,[w\text{-}\,ls\,X'_n(\omega)]\ a.e.$$

Proof. (a) follows from Theorem 3.4(a) because "$L(\omega) \in \mathcal{L}wk(E) \Longrightarrow L(\omega)$ ball-wc". Now let us prove (b). We will produce the arguments in [9, page 66-67]. Suppose by contradiction that (b) does not hold. As $X_n(\omega) \subset L(\omega)$ for all $n \in \mathbb{N}^*$ and all $\omega \in \Omega$, the multifunction $\overline{co}\,[w\text{-}\,ls\,X'_n(\omega)]$ is measurable and $\mathcal{L}wk(E)$-valued. Using [14, lemma III.34](**) provides $x^* \in E^*$ and a $\mathcal{F}$-measurable set A with $\mu(A) > 0$ such that $\delta^*(x^*, X_\infty(\omega)) > \delta^*(x^*, \overline{co}\,[w\text{-}\,ls\,X'_n(\omega)])$ for all $\omega \in A$. For each n, let s_n be a maximal integrable selection of X'_n associated to x^*, namely, $\langle x^*, s_n(\omega)\rangle = \delta^*(x^*, X'_n(\omega)), \forall \omega \in \Omega$. It is obvious that (s_n) satisfies the conditions of Theorem 8 in [2]. Then there exists a subsequence still denoted by (s_n) which converges weakly in $L^1_E(\mu)$ to a function s_∞ such that

$$s_\infty(\omega) \in \overline{co}\,[w\text{-}\,ls\,s_n(\omega)] \subset \overline{co}\,[w\text{-}\,ls\,X'_n(\omega)]\ a.e.$$

As $s_n \to s_\infty$ weakly, we get

$$\lim_{n\to\infty}\int_A \langle x^*, s_n(\omega)\rangle\,\mu(d\omega = \int_A \langle x^*, s_\infty(\omega)\rangle\,\mu(d\omega).$$

By (a) we have

$$\lim_{n\to\infty}\int_A \delta^*(x^*, X'_n(\omega))\,\mu(d\omega) = \int_A \delta^*(x^*, X_\infty(\omega))\,\mu(d\omega).$$

It follows that

$$\int_A \langle x^*, s_\infty(\omega)\rangle\,\mu(d\omega) = \int_A \delta^*(x^*, X_\infty(\omega))\,\mu(d\omega).$$

As $s_\infty(\omega) \in \overline{co}\,[w\text{-}\,ls\,X'_n(\omega)]$ $a.e$ by integrating on A we deduce that

$$\int_A \delta^*(x^*, \overline{co}\,[w\text{-}\,ls\,X'_n(\omega)])\,\mu(d\omega) \geq \int_A \delta^*(x^*, X_\infty(\omega))\,\mu(d\omega)$$

that contradicts the inequality

$$\int_A \delta^*(x^*, \overline{co}\,[w\text{-}\,ls\,X'_n(\omega)])\,\mu(d\omega) < \int_A \delta^*(x^*, X_\infty(\omega))\,\mu(d\omega)). \quad \square$$

There is a useful application. See also [8, Lemma 3.3].

Corollary 3.6. *Suppose that* E *is a separable Banach space,* $D^* = (e^*_p)$ *be a dense sequence for the Mackey topology. Let* $L : \Omega \to E$ *is a* $cwk(E)$*-valued measurable multifunction and* (X_n) *is a uniformly integrable sequence in* $\mathcal{L}^1_{cwk(E)}(\mu)$ *such that* $X_n(\omega) \subset L(\omega)$ *for all* $n \in \mathbb{N}^*$ *and all* $\omega \in \Omega$. *Suppose further that, for all* p *and*

(**)That is the reason for which L is assumed to be $\mathcal{L}wk(E)$-valued ; for further generalizations different techniques are necessary.

for almost all $\omega \in \Omega$, *the sequence* $(\delta^*(e_p^*, X_n(\omega))_n$ *converges in* $\mathbb{R}$, *then there exist* $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$ *such that*

$$(a) \qquad \lim_{n\to\infty} \delta^*(x^*, X_n(\omega)) = \delta^*(x^*, X_\infty(\omega))$$

for all $x^* \in E^*$ *and for almost all* $\omega \in \Omega$ *and*

$$(b) \qquad X_\infty(\omega) = \overline{co}\,[w\text{-}ls\, X_n'(\omega)] \ a.e.$$

Proof. In view of Theorem 3.5, there exist a subsequence (X_n') of (X_n) and $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$ such that

$$\forall u \in L^\infty_{E^*}(\mu)\,,\ \lim_{n\to\infty} \int_\Omega \delta^*(u, X_n')\, d\mu = \int_\Omega \delta^*(u, X_\infty)\, d\mu$$

and

$$X_\infty(\omega) \subset \overline{co}\,[w\text{-}ls\, X_n'(\omega)] \ a.e.$$

By our assumption and Lebesgue-Vitali theorem we deduce that

$$\int_A \lim_{n\to\infty} \delta^*(e_p^*, X_n)\, d\mu = \lim_{n\to\infty} \int_A \delta^*(e_p^*, X_n)\, d\mu = \int_A \delta^*(e_p^*, X_\infty)\, d\mu$$

for all p and for all $A \in \mathcal{F}$. Then we have

$$\lim_{n\to\infty} \delta^*(e_p^*, X_n(\omega)) = \delta^*(e_p^*, X_\infty(\omega))$$

for all p and for almost all $\omega \in \Omega$. Since L is $cwk(E)$-valued, we can finished the proof by a routine density argument. □

The following is a combined application of the biting lemma in $L^1_{\mathbb{R}}(\mu)$ and Theorem 3.5 via the arguments given in the proof of the biting lemma given in [12, Theorem 3.1].

Theorem 3.7. *Suppose that* E *is a separable Banach space,* $L : \Omega \to E$ *is a* $\mathcal{L}wk(E)$*-valued measurable multifunction and* (X_n) *is a bounded sequence in* $\mathcal{L}^1_{cwk(E)}(\mu)$ *such that* $X_n(\omega) \subset L(\omega)$ *for all* $n \in \mathbb{N}^*$ *and all* $\omega \in \Omega$, *then there exist an increasing sequence* (A_p) *in* $\mathcal{F}$ *with* $\lim_{n\to\infty} \mu(A_p) = 1$ *and a subsequence* (X_n') *of* (X_n) *such that the sequence* $(1_{A_p} X_n')$ *is uniformly integrable for each* p *and* $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$ *such that* $\forall p, \forall A \in A_p \cap \mathcal{F}$ *and* $\forall u \in L^\infty_{E^*}(A_p, A_p \cap \mathcal{F}, \mu)$, *the following holds :*

$$(a) \qquad \lim_{n\to\infty} \int_A \delta^*(u, X_n')\, d\mu = \int_A \delta^*(u, X_\infty)\, d\mu.$$

and

$$(b) \qquad X_\infty(\omega) \subset \overline{co}\,[w\text{-}ls\, X_n'(\omega)] \ a.e.$$

Proof. (a) follows by repeating mutatis mutandis the arguments of the proof of the biting lemma in [12, Theorem 3.1] via the *sequential* weak convergence stated in Theorem 3.5 for the sequences $(1_{A_p} X_n')_n$. Whereas (b) follows from the second part of Theorem 3.5 and the biting lemma. So we omit the details. □

Further generalizations of Theorems 3.4-3.5-3.7 are given in the next sections using some heavy techniques whereas the above-mentioned proofs are more direct. Yet, in Theorems 3.5-3.7, they allow only to treat the case when the dominated multifunction L is $\mathcal{L}wk(E)$-valued. In $L^1_E(\mu)$ case, a seminal version of Theorem 3.7 appears in [9, Theorem 2.1] using approximation and truncation techniques via a suitable tightness condition and a weak compactness analog of Theorem 3.5 in $L^1_E(\mu)$. Compare with Theorem 6.2 given in Section 6.

4. Komlós theorem in $L^1_E(\mu)$ and $\mathcal{L}^1_{cwk(E)}(\mu)$ under tightness condition

Theorem 3.4 generalizes some weak compactness results in $L^1_E(\mu)$ (see, for example, [5, 6, 8, 10, 14, 17, 26]) and also in $\mathcal{L}^1_{cwk(E)}(\mu)$ (see, for example, [9, 11]) and provides an alternative proof via the Mazur τ_L-convergence. Now we proceed to the *weak Komlós convergence* in $L^1_E(\mu)$ and $\mathcal{L}^1_{cwk(E)}(\mu)$. See [3, 13, 24] for other related results involving this mode of convergence.

Let us mention first the following result [24] showing that any bounded $cwk(E)$-tight sequence in $L^1_E(\mu)$ " weakly Komlós converges in measure", a property which captures the recent weak compactness results in [2, 17]. We will give the details of proof since this leads to a multivalued version of Komlós theorem (see Theorem 4.2 below).

Theorem 4.1. *If (u_n) is a bounded $cwk(E)$-tight sequence in $L^1_E(\mu)$, then there exist a subsequence (v_m) of (u_n) and $u_\infty \in L^1_E(\mu)$ such that for all $h \in L^\infty_{E^*}(\mu)$ and for each further subsequence (w_l) of (v_m), the following holds :*

$$\frac{1}{n}\Sigma_{i=1}^n \langle h, w_i\rangle \to \langle h, u_\infty\rangle \text{ in measure.}$$

Proof. Step 1. Using the biting lemma provides an increasing sequence (B_n) in $\mathcal{F}$ with $\lim_{n\to\infty}\mu(B_n) = 1$ and a subsequence (u'_n) such that $u'_n = 1_{B_n}u'_n + 1_{\Omega\setminus B_n}u'_n$ where the sequence $(v'_n) = (1_{B_n}u'_n)$ is uniformly integrable and the sequence $(w'_n) = (1_{\Omega\setminus B_n}u'_n)$ converges to 0 almost everywhere in E. In particular, if (u_n) is a bounded $cwk(E)$-tight sequence in $L^1_E(\mu)$, the corresponding subsequence $(v'_n) = (1_{B_n}u'_n)$ given by the above decomposition is uniformly integrable and $cwk(E)$-tight. Consequently it is enough to suppose that (u_n) is uniformly integrable $cwk(E)$-tight sequence. From what has been said, it remains to prove that there exist a subsequence (v_m) of (u_n) and $u_\infty \in L^1_E(\mu)$ such that for all $h \in L^\infty_{E^*}(\mu)$ and for each further subsequence (w_l) of (v_m), the following holds :

$$\left|\frac{1}{n}\Sigma_{i=1}^n \langle h, w_i\rangle - \langle h, u_\infty\rangle\right|_1 \to 0.$$

By [2, Theorem 6] the sequence (u_n) is relatively weakly compact in $L^1_E(\mu)$. Now let $D^* = (e^*_p)$ be a dense sequence in $\overline{B}_{E^*}$ for the Mackey topology. Using Komlós theorem [22] via an appropriate diagonal procedure and the weak compactness of (u_n), we find a subsequence (v_m) of (u_n) and $u_\infty \in L^1_E(\mu)$ such that for all p and for each further subsequence (w_l) of (v_m)

$$\frac{1}{n}\Sigma_{i=1}^n \langle e^*_p, w_i\rangle \to \langle e^*_p, u_\infty\rangle$$

for almost all $\omega \in \Omega$.

Step 2. We suppose that there exists a $cwk(E)$-valued measurable multifunction $\Gamma : \Omega \to E$ such that $u_n(\omega) \in \Gamma(\omega)$ for all n and all $\omega \in \Omega$.

Since $\frac{1}{n}\Sigma_{i=1}^n w_i(\omega) \in \Gamma(\omega)$ a.e, by using a routine density argument we deduce that

$$\frac{1}{n}\Sigma_{i=1}^n \langle x^*, w_i\rangle \to \langle x^*, u_\infty\rangle$$

for all $x^* \in E^*$ and for almost all $\omega \in \Omega$.

Step 3. Now we pass to the general case. By tightness assumption, for every $q \in \mathbb{N}^*$ there exists a $cwk(E)$-valued measurable multifunction $\Gamma_{\frac{1}{q}} : \Omega \to E$ such that

$$(*) \qquad \forall n,\ \mu(\{\omega \in \Omega : u_n(\omega) \notin \Gamma_{\frac{1}{q}}(\omega)\}) \le \frac{1}{q}.$$

W.l.o.g we may suppose that $0 \in \Gamma_{\frac{1}{q}}(\omega)$ for all $\omega \in \Omega$. Let us set $A_{n,q} := \{\omega \in \Omega : u_n(\omega) \in \Gamma_{\frac{1}{q}}(\omega)\}$. By construction $1_{A_{n,q}} u_n \in \mathcal{S}^1_{\Gamma_{\frac{1}{q}}}$ for all n and all q so that we may apply the results in the foregoing steps via an appropriate diagonal procedure. There are a subsequence (v'_m) and $u^q_\infty \in L^1_E(\mu)$ such that for all $x^* \in E^*$, for all p and for each further subsequence (w'_l), the following hold :

$$(4.1.1) \qquad \frac{1}{n}\Sigma_{j=1}^n \langle x^*, 1_{A'_{j,q}} w'_j\rangle \to \langle x^*, u^q_\infty\rangle$$

for almost all $\omega \in \Omega$ with $A'_{j,q} = \{\omega \in \Omega : w'_j(\omega) \in \Gamma_{\frac{1}{q}}(\omega)\}$ and

$$(4.1.2) \qquad \frac{1}{n}\Sigma_{j=1}^n \langle e^*_p, w'_j\rangle \to \langle e^*_p, u_\infty\rangle$$

for almost all $\omega \in \Omega$.

Let $h \in L^\infty_{E^*}(\mu)$ with $||h||_\infty \le 1$. By (4.1.1) and Lebesgue-Vitali theorem, we have

$$(4.1.3) \qquad |\frac{1}{n}\Sigma_{j=1}^n \langle h, 1_{A'_{j,q}} w'_j\rangle - \langle h, u^q_\infty\rangle|_1 \to 0.$$

Using $(*)$, and the uniform integrability assumption we have

$$(4.1.4) \qquad \lim_{q\to\infty} \sup_n \int_{\Omega\setminus A'_{n,q}} |w'_n|\, d\mu = 0.$$

Moreover an easy computation gives

$$(4.1.5) \qquad \begin{aligned} |\frac{1}{n}\Sigma_{j=1}^n w'_j - \frac{1}{n}\Sigma_{j=1}^n 1_{A'_{j,q}} w'_j|_1 &\le \frac{1}{n}\Sigma_{j=1}^n \int_{\Omega\setminus A'_{j,q}} |w'_j|\, d\mu \\ &\le \sup_n \int_{\Omega\setminus A'_{n,q}} |w'_n|\, d\mu. \end{aligned}$$

From (4.1.1) and (4.1.2) it follows that

$$\frac{1}{n}\Sigma_{j=1}^n \langle e^*_p, w'_j\rangle - \frac{1}{n}\Sigma_{j=1}^n \langle e^*_p, 1_{A'_{j,q}} w'_j\rangle \to \langle e^*_p, u_\infty - u^q_\infty\rangle$$

for all p and almost all ω. Since D^* is Mackey dense in $\overline{B}_{E^*}$, we deduce that

$$(4.1.6) \qquad |u_\infty - u^q_\infty| \le \liminf_n |\frac{1}{n}\Sigma_{j=1}^n w'_j - \frac{1}{n}\Sigma_{j=1}^n 1_{A'_{j,q}} w'_j|$$

for almost all ω. Using (4.1.5), (4.1.6) and Fatou Lemma we get

$$\begin{aligned}\int_\Omega |u_\infty - u_\infty^q|\, d\mu &\le \int_\Omega \liminf_n |\frac{1}{n}\Sigma_{j=1}^n w'_j - \frac{1}{n}\Sigma_{j=1}^n 1_{A'_{j,q}} w'_j|\, d\mu \\ &\le \liminf_n \int_\Omega |\frac{1}{n}\Sigma_{j=1}^n w'_j - \frac{1}{n}\Sigma_{j=1}^n 1_{A'_{j,q}} w'_j|\, d\mu \\ &\le \liminf_n \frac{1}{n}\Sigma_{j=1}^n \int_{\Omega\setminus A'_{j,q}} |w'_j|\, d\mu \\ &\le \sup_n \int_{\Omega\setminus A'_{n,q}} w'_n d\mu. \end{aligned} \tag{4.1.7}$$

Now using (4.1.5) and (4.1.7) it is not difficult to check that

$$\begin{aligned}|\langle h, \frac{1}{n}\Sigma_{j=1}^n w'_j\rangle - \langle h, u_\infty\rangle|_1 &\le |\langle h, \frac{1}{n}\Sigma_{j=1}^n w'_j - \frac{1}{n}\Sigma_{j=1}^n 1_{A'_{j,q}} w'_j\rangle|_1 \\ &\quad + |\langle h, \frac{1}{n}\Sigma_{j=1}^n 1_{A'_{j,q}} w'_j\rangle - \langle h, u_\infty^q\rangle|_1 \\ &\quad + |\langle h, u_\infty^q - u_\infty\rangle|_1 \\ &\le 2\sup_n \int_{\Omega\setminus A'_{n,q}} |w'_n|\, d\mu \\ &\quad + |\langle h, \frac{1}{n}\Sigma_{j=1}^n 1_{A'_{j,q}} w'_j\rangle - \langle h, u_\infty^q\rangle|_1. \end{aligned} \tag{4.1.8}$$

From (4.1.3), (4.1.4) and (4.1.8) it follows that

$$|\langle h, \frac{1}{n}\Sigma_{j=1}^n w'_j\rangle - \langle h, u_\infty\rangle|_1 \to 0. \quad \square$$

In the light of the preceding result it is worth to have an analog in the space $\mathcal{L}^1_{cwk(E)}(\mu)$. For this purpose let us introduce the following terminology. A sequence (X_n) in $\mathcal{L}^1_{cwk(E)}(\mu)$ is *$cwk(E)$-tight* if, for every $\varepsilon > 0$, there is a $cwk(E)$-valued measurable multifunction $\Gamma_\varepsilon : \Omega \to E$ such that

$$\forall n,\ \ \mu(\Omega \setminus \{\omega \in \Omega : X_n(\omega) \subset \Gamma_\varepsilon(\omega)\}) \le \varepsilon.$$

As an example, one can easily check that the sequence (X_n) given in Theorem 3.3 is $cwk(E)$-tight. Let $D^* := (e^*_p)$ be a dense sequence in $\overline{B}_{E^*}$ for the Mackey topology. A sequence (X_n) in $\mathcal{L}^1_{cwk(E)}(\mu)$ *$\sigma(E, D^*)$-Komlós converges* to $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$ if there exists a subsequence (Y_n) of (X_n) such that for all p and for each further subsequence (Z_n) of (Y_n),

$$\lim_n \frac{1}{n}\Sigma_{j=1}^n \delta^*(e^*_p, Z_j) = \delta^*(e^*_p, X_\infty)$$

for almost all $\omega \in \Omega$.

Theorem 4.2. *Suppose that $D^* := (e^*_p)$ is a dense sequence in $\overline{B}_{E^*}$ for the Mackey topology, (X_n) is a bounded $cwk(E)$-tight sequence in $\mathcal{L}^1_{cwk(E)}(\mu)$ which $\sigma(E, D^*)$-Komlós converges to $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$, then there exists a subsequence (Y_n) of*

(X_n) such that for all $h \in L^\infty_{E^*}(\mu)$ and for each further subsequence (Z_n) of (Y_n), the following holds :

$$\delta^*(h, \frac{1}{n}\Sigma_{i=1}^n Z_i) \to \delta^*(h, X_\infty) \text{ in measure.}$$

Proof. Step 1. Suppose that (X_n) is uniformly integrable.

By tightness assumption, for every $q \in \mathbb{N}^*$ there exists a $cwk(E)$-valued measurable multifunction $\Gamma_{\frac{1}{q}} : \Omega \to E$ such that

$$(*) \qquad \forall n,\ \mu(\Omega \setminus \{\omega \in \Omega : X_n(\omega) \subset \Gamma_{\frac{1}{q}}(\omega)\}) \le \frac{1}{q}.$$

We may suppose that $0 \in \Gamma_{\frac{1}{q}}(\omega)$ for all ω. Let us set $A_{n,q} := \{\omega \in \Omega : X_n(\omega) \subset \Gamma_{\frac{1}{q}}(\omega)\}$. Applying Komlós theorem to the sequence $(\delta^*(e_p^*, 1_{A_{n,q}} X_n))_n$ via an appropriate diagonal process provides a subsequence (Y_n) of (X_n) and a sequence $(\varphi_{p,q})$ in $L^1_{\mathbb{R}+}$ such that

$$(4.2.1) \qquad \lim_n \frac{1}{n}\Sigma_{j=1}^n \delta^*(e_p^*, 1_{A'_{j,q}} Z_j) = \varphi_{p,q} \ a.e$$

with $A'_{j,q} = \{\omega \in \Omega : Z_j(\omega) \subset \Gamma_{\frac{1}{q}}(\omega)\}$ for each further subsequence (Z_n) of (Y_n) and for every $p \in \mathbb{N}^*$ and every $q \in \mathbb{N}^*$. By hypothesis we have

$$(4.2.2) \qquad \lim_n \frac{1}{n}\Sigma_{j=1}^n \delta^*(e_p^*, Z_j) = \delta =^* (e_p^*, X_\infty) \ a.e$$

for each subsequence (Z_n) of (Y_n) and for every $p \in \mathbb{N}^*$. For simplicity let us set

$$S_n := \frac{1}{n}\Sigma_{j=1}^n Z_j,\ \ S_{n,q} := \frac{1}{n}\Sigma_{j=1}^n 1_{A'_{j,q}} Z_j$$

for all $n \in \mathbb{N}^*$ and for all $q \in \mathbb{N}^*$. Now we proceed as in the vector valued case, so we don't want to go into details too much, but only show the difference. By construction we may apply Theorem 3.3 to $(S_{n,q})_n$. There are sequences $(\widetilde{S}_{n,q})_n$ with $\widetilde{S}_{n,q} \in co\{S_{i,q} : i \ge n\}$ and $X_{\infty,q} \in \mathcal{L}^1_{cwk(E)}(\mu)$ such that $\tau_L\text{-}\lim_n \widetilde{S}_{n,q} = X_{\infty,q}$ a.e. In particular, we have

$$\forall x^* \in E^*,\ \lim_n \delta^*(x^*, \widetilde{S}_{n,q}) = \delta^*(x^*, X_{\infty,q}))$$

a.e, and by (4.2.1)

$$(4.2.3) \qquad \forall x^* \in E^*,\ \lim_n \delta^*(x^*, S_{n,q}) = \delta^*(x^*, X_{\infty,q}))$$

a.e because $S_{n,q}(\omega) \subset \Gamma_{\frac{1}{q}}(\omega)$. By (4.2.2) and (4.2.3) we get

$$\delta^*(e_p^*, S_n) - \delta^*(e_p^*, S_{n,q}) \to \delta^*(e_p^*, X_\infty) - \delta^*(e_p^*, X_{\infty,q}) \ a.e$$

for every $p \in \mathbb{N}^*$ and every $q \in \mathbb{N}^*$. Consequently we deduce that

$$\begin{aligned} H(X_\infty, X_{\infty,q}) &= \sup_{x^* \in \overline{B}_{E^*}} |\delta^*(x^*, X_\infty) - \delta^*(x^*, X_{\infty,q})| \\ &= \sup_{x^* \in D^*} |\delta^*(x^*, X_\infty) - \delta^*(x^*, X_{\infty,q})| \\ &= \sup_{x^* \in D^*} \lim_n |\delta^*(x^*, S_n) - \delta^*(x^*, S_{n,q})| \\ &\le \liminf_n \frac{1}{n} \Sigma_{j=1}^n 1_{\Omega \setminus A'_{j,q}} |Z_i| \ a.e \end{aligned}$$

where the second identity in the second member follows from the Mackey density of D^* and the Mackey continuity of the support functions of X_∞ and $X_{\infty,q}$.

Now let $h \in L^\infty_{E^*}(\mu)$ with $||h||_\infty \le 1$. Using the preceding inequality and Fatou lemma, we obtain the estimate

$$\begin{aligned} \int_\Omega |\delta^*(h, S_n) - \delta^*(h, X_\infty)| \, d\mu &\le \int_\Omega |\delta^*(h, S_n) - \delta^*(h, S_{n,q})| \, d\mu \\ &+ \int_\Omega |\delta^*(h, S_{n,q}) - \delta^*(h, X_{\infty,q})| \, d\mu \\ &+ \int_\Omega |\delta^*(h, X_\infty) - \delta^*(h, X_{\infty,q})| \, d\mu \\ &\le 2 \sup_n \frac{1}{n} \Sigma_{j=1}^n \int_{\Omega \setminus A'_{j,q}} |Z_j| \, d\mu \\ &+ \int_\Omega |\delta^*(h, S_{n,q}) - \delta^*(h, X_{\infty,q})| \, d\mu \\ &\le 2 \sup_n \int_{\Omega \setminus A'_{n,q}} |Z_n| \, d\mu \\ &+ \int_\Omega |\delta^*(h, S_{n,q}) - \delta^*(h, X_{\infty,q})| \, d\mu. \end{aligned}$$

Since (X_n) is uniformly integrable, by (4.2.3) and Lebesgue-Vitali theorem

$$\int_\Omega |\delta^*(h, S_{n,q}) - \delta^*(h, X_{\infty,q})| \, d\mu \to 0$$

when $n \to \infty$ and

$$\limsup_q \sup_n \int_{\Omega \setminus A'_{n,q}} |Z_n| \, d\mu = 0.$$

Whence we get

$$\lim_{n \to \infty} \int_\Omega |\delta^*(h, S_n) - \delta^*(h, X_\infty)| \, d\mu = 0.$$

General case. Since (X_n) is bounded, by the biting lemma there exist an increasing sequence (B_n) in $\mathcal{F}$ with $\lim_n \mu(B_n) = 1$ and a subsequence of (X_n) still denoted (X_n) such that $X_n = 1_{B_n} X_n + 1_{\Omega \setminus B_n} X_n$ where $(X_n^1) := (1_{B_n} X_n)$ is a uniformly integrable and $|X_n^2| \to 0$ a.e where $X_n^2 = 1_{\Omega \setminus B_n} X_n$. It is obvious that the uniformly integrable sequence (X_n^1) is $cwk(E)$-tight. By Step 1, there exist a subsequence $(X_{n_k}^1)$ of (X_n^1) and $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$ such that for all $h \in L^\infty_{E^*}(\mu)$ and for each further subsequence (Z_n) of $(X^1_{n_k})$, the sequence $(\delta^*(h, \frac{1}{n}\Sigma_{i=1}^n Z_i))$ converges in measure to $\delta^*(h, X_\infty)$. Since the sequence $(\frac{1}{n}\Sigma_{i=1}^n |Z_i|)$ converges to 0 almost everywhere, for every further subsequence (Z_n) of $(X^2_{n_k})$, because $|X^2_{n_k}| \to 0$ a.e, it is easy to check that X_∞ and the sequence $(Y_k) = (X_{n_k})$ have the required property.

The following is a direct application of Theorem 4.2 to the SLLN (strong law of large numbers) for convex weakly compact valued integrably bounded multifunctions (alias convex weakly compact valued random sets) and is a multivalued version of a result due to Daffer-Taylor ([16], Theorem 2.3). A sequence (X_n) in $\mathcal{L}^1_{cwk(E)}(\mu)$ is *independent* if for every $x^* \in E^*$, the sequence of real-valued integrable random variables $(\delta^*(x^*, X_n))$ is independent.

Proposition 4.3. *Suppose that $(X_n)_{n\geq 1}$ is a uniformly integrable $cwk(E)$-tight independent sequence in $\mathcal{L}^1_{cwk(E)}(\mu)$ and $D^* := (e^*_k)_{k\geq 1}$ is a dense sequence in $\overline{B}_{E^*}$ for the Mackey topology satisfying :*

(i) $\forall n \geq 1,\ EX_n = EX_1$.
(ii) $\forall k \geq 1,\ \Sigma_{n=1}^\infty n^{-p} E|\delta^*(e^*_k, X_n)|^p < \infty$, *for some* $1 \leq p \leq 2$.

Then there is a subsequence (Y_n) of (X_n) such that for all $h \in L^\infty_{E^*}(\mu)$ and for each further subsequence (Z_n) of (Y_n), the following holds :

$$\lim_{n\to\infty} \int_\Omega |\delta^*(h, \frac{1}{n}\Sigma_{i=1}^n Z_i) - \delta^*(h, EX_1)|\ d\mu = 0.$$

Proof. For simplicity we set, for all n and for all k, $Z^k_n := \delta^*(e^*_k, X_n)$. By an easy computation, we have $E|Z^k_n - EZ^k_n|^p \leq 2^p E|Z^k_n|^p$.

From (ii) it follows that

$$\Sigma_{n=1}^\infty n^{-p} E|Z^k_n - EZ^k_n|^p < \infty.$$

Since the sequence $(Z^k_n)_n$ is independent, using (i) and K.L. Chung [15] we have

$$\forall k,\ \lim_{n\to\infty} \frac{1}{n}\Sigma_{j=1}^n \delta^*(e^*_k, X_j) = \delta^*(e^*_k, EX_1)$$

for almost all $\omega \in \Omega$. Evidently, this holds for every further subsequence of (X_n) for almost all $\omega \in \Omega$. Thus $(X_n)\, \sigma(E, D^*)$-Komlós converges to EX_1. To complete the proof it is enough to apply Theorem 4.2 to the uniformly integrable $cwk(E)$-tight sequence (X_n). □

5. Mazur's convergence under tightness condition

If (u_n) is a bounded $cwk(E)$-tight sequence in $L^1_E(\mu)$, then (u_n) has the weak Talagrand property (see, for example, [5, Theorem 2.8]) and there exists a sequence (v_n) with $v_n \in co\{u_m : m \geq n\}$ such that (v_n) converges almost everywhere to an integrable function $u \in L^1_E(\mu)$. As Theorem 3.3 provides an analoguous property for the space $\mathcal{L}^1_{cwk(E)}(\mu)$, it is worth to pose the question of the validity of Theorem 3.3 in the case when (X_n) is bounded and $cwk(E)$-tight. The following result provides a satisfactory answer but its proof is a bit more technical.

Theorem 5.1. *Suppose that E is a separable Banach space, (X_n) is a bounded $cwk(E)$-tight sequence in $\mathcal{L}^1_{cwk(E)}(\mu)$, then there exist $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$ and a sequence $(\widetilde{X}_n)_n$ with $\widetilde{X}_n \in co\{X_i : i \geq n\}$ such that*

(a) $\forall x \in E, \quad \lim_{n\to\infty} d(x, \widetilde{X}_n(\omega)) = d(x, X_\infty(\omega))$ *a.e.*

(b) $\forall h \in L^\infty_{E^*}(\mu), \quad \delta^*(h, \widetilde{X}_n) \to \delta^*(h, X_\infty)$ *in measure.*

Proof. We will divide the proof in several steps.

A. We begin to state the theorem in the particular case when (X_n) is uniformly integrable

Step 1. Let $D^* := (x^*_k)_{k\in\mathbb{N}^*}$ be a countable dense subset in $\overline{B}_{E^*}$ for the Mackey topology. First we note that $\bigcup_n S^1_{X_n}$ is relatively $\sigma(L^1, L^\infty)$ compact in view of [2, Theorem 6], [8, Theorem 4.1]. For every $k \in \mathbb{N}^*$ pick a maximal integrable selection $u_{n,k} \in S^1_{X_n}$:

$$\langle x^*_k, u_{n,k}\rangle := \delta^*(x^*_k, X_n).$$

Then $(u_{n,k})_n$ is relatively $\sigma(L^1, L^\infty)$ sequentially compact for every k. Therefore by using an appropriate diagonal procedure, we may suppose for simplicity that, for every k, $(u_{n,k})_n$ converges $\sigma(L^1, L^\infty)$ to an integrable function $u_{\infty,k}$ so that we can apply Lemma 3.1 to $(u_{n,k})_{n,k}$. Hence there exists a negligible set N_0 and a sequence $(v_{n,k})_{n,k}$ with

$$v_{n,k} = \sum_{i\in I_n} \lambda^n_i u_{i+n,k}, \ \lambda^n_i \geq 0 \text{ and } \sum_{i\in I_n} \lambda^n_i = 1$$

such that, for every k, $(v_{n,k}(\omega))_n$ s-converges a.e. to $u_{\infty,k}(\omega)$, for all $\omega \in \Omega \setminus N_0$.

Step 2. By tightness assumption, for every $q \in \mathbb{N}^*$, there exists a $cwk(E)$-valued measurable multifonction $\Gamma_{\frac{1}{q}}$ such that

$$\forall n, \ \mu(\Omega \setminus A_{n,q}) \leq \frac{1}{q}, \tag{5.1.1}$$

where

$$A_{n,q} := \{\omega \in \Omega : X_n(\omega) \subset \Gamma_{\frac{1}{q}}(\omega)\}.$$

W.l.o.g we may suppose that $0 \in \Gamma_{\frac{1}{q}}(\omega)$ for all $\omega \in \Omega$. Let us define

$$v_{n,k,q} := \sum_{i\in I_n} \lambda^n_i 1_{A_{i+n,q}} u_{i+n,k}, \quad (n, q, k \in \mathbb{N}^*).$$

Using again [2, Theorem 6], [8, Theorem 4.1] and the diagonal method we may suppose that, for each k and each $q \in \mathbb{N}^*$, the sequence $(v_{n,k,q})_n$ converges $\sigma(L^1, L^\infty)$ to an integrable function $u_{\infty,k,q}$. Then an appeal to the Remark of Lemma 3.1 produces a negligible set $N_1 \supset N_0$ and a sequence $(\widetilde{v}_{n,k,q})_{n,k,q}$ with

$$\widetilde{v}_{n,k,q} = \sum_{j\in J_n} \mu^n_j v_{j+n,k,q}, \ \mu^n_j \geq 0 \text{ and } \sum_{j\in J_n} \mu^n_j = 1$$

such that $(\widetilde{v}_{n,k,q})_n$ s-converges a.e. to $u_{\infty,k,q}$, for every k and every $q \in N^*$. Since $(v_{n,k})_n$ s-converges to $u_{\infty,k}$ for every k, so is the sequence $(\widetilde{v}_{n,k})_n$ defined by

$$\widetilde{v}_{n,k}(\omega) := \sum_{j\in=J_n} \mu_j^n v_{j+n,k}(\omega).$$

Step 3. Now let us consider the multifunctions

$$\tilde{X}_n = \sum_{j\in J_n} \mu_j^n \sum_{i\in I_{j+n}} \lambda_i^{j+n} X_{i+j+n}, \quad X_\infty = 1_{\Omega\setminus N_1} s\text{-}li\tilde{X}_n$$

and

$$\tilde{X}_{n,q} = \sum_{j\in J_n} \mu_j^n \sum_{i\in I_{j+n}} \lambda_i^{j+n} 1_{A_{i+j+n,q}} X_{i+j+n}, \quad X_{\infty,q} = 1_{\Omega\setminus N_1} s\text{-}li\tilde{X}_{n,q}.$$

Fix $k \in \mathbb{N}^*$, $q \in \mathbb{N}^*$ and $\omega \in \Omega \setminus N_1$. Then, by Step 2, $u_{\infty,k}(\omega) \in X_\infty(\omega)$ and $u_{\infty,k,q}(\omega) \in X_{\infty,q}(\omega)$. Furthermore we have

$$\begin{aligned}\lim_{n\to+\infty} \delta^*(x_k^*, \widetilde{X}_n(\omega)) &= \lim_n \langle x_k^*, \widetilde{v}_{n,k}(\omega)\rangle \\ &= \langle x_k^*, u_{\infty,k}(\omega)\rangle \\ &\le \delta^*(x_k^*, X_\infty(\omega)).\end{aligned}$$

On the other hand it easy to see that

$$\delta^*(x_k^*, X_\infty(\omega)) \le \lim_{n\to\infty} \delta^*(x_k^*, \widetilde{X}_n(\omega)).$$

Whence we get

$$\lim_{n\to\infty} \delta^*(x_k^*, \widetilde{X}_n(\omega)) = \delta^*(x_k^*, X_\infty(\omega)). \tag{5.1.2}$$

Similarly, we obtain

$$\lim_{n\to\infty} \delta^*(x_k^*, \widetilde{X}_{n,q}(\omega)) = \delta^*(x_k^*, X_{\infty,q}(\omega))$$

and then

$$\lim_{n\to\infty} \delta^*(x^*, \widetilde{X}_{n,q}(\omega)) = \delta^*(x^*, X_{\infty,q}(\omega)) \tag{5.1.3}$$

for every $x^* \in E^*$, since $X_{n,q}(\omega)$ is included in the weakly compact set $\Gamma_{\frac{1}{q}}(\omega)$ for all $n \in \mathbb{N}^*$.

Step 4. Let $q \in \mathbb{N}^*$, $x^* \in \overline{B}_{E^*}$ and $\omega \in \Omega \setminus N_1$ be fixed. For simplicity set

$$\phi_q(\omega) := \liminf_n \sum_{j\in J_n} \mu_j^n \sum_{i\in I_{j+n}} \lambda_i^{j+n} 1_{\Omega\setminus A_{i+j+n,q}} |X_{i+j+n}|(\omega).$$

We claim that

$$|\delta^*(x^*, X_\infty(\omega)) - \delta^*(x^*, X_{\infty,q}(\omega))| \le \phi_q(\omega). \tag{5.1.4}$$

Indeed

$$\begin{aligned}
&\delta^*(x^*, X_\infty(\omega)) - \delta^*(x^*, X_{\infty,q}(\omega)) \\
&\qquad \le \liminf_n \delta^*(x^*, \widetilde{X}_n(\omega)) - \lim_n \delta^*(x^*, \widetilde{X}_{n,q}(\omega)) \\
&\qquad = \liminf_n [\delta^*(x^*, \widetilde{X}_n(\omega)) - \delta^*(x^*, \widetilde{X}_{n,q}(\omega))] \\
&\qquad = \liminf_n [\delta^*(x^*, \sum_{j\in J_n} \mu_j^n \sum_{i\in I_{j+n}} \lambda_i^{j+n} 1_{\Omega\setminus A_{i+j+n,q}} X_{i+j+n}(\omega))] \\
&\qquad \le \phi_q(\omega).
\end{aligned}$$

Let us prove the converse inequality. Let $x_k^* \in D^*$ such that

$$\max\{\delta^*(x^* - x_k^*, X_{\infty,q}(\omega)), \delta^*(x_k^* - x^*, X_{\infty,q}(\omega))\} < \frac{\epsilon}{2}.$$

Then

$$\begin{aligned}
&\delta^*(x^*, X_\infty(\omega)) - \delta^*(x^*, X_{\infty,q}(\omega)) \\
&\qquad \ge \delta^*(x^*, X_\infty(\omega)) - \delta^*(x_k^*, X_{\infty,q}(\omega)) \\
&\qquad - \max\{\delta^*(x^* - x_k^*, X_{\infty,q}(\omega)), \delta^*(x_k^* - x^*, X_{\infty,q}(\omega))\} \\
&\qquad \ge \delta^*(x^*, X_\infty(\omega)) - \delta^*(x_k^*, X_{\infty,q}(\omega)) - \frac{\epsilon}{2} \\
&\qquad \ge \langle x^*, u_{\infty,k}(\omega)\rangle - \delta^*(x_k^*, X_{\infty,q}(\omega)) - \frac{\epsilon}{2} \\
&\qquad = \langle x^*, u_{\infty,k}(\omega)\rangle - \langle x_k^*, u_{\infty,k,q}(\omega)\rangle - \frac{\epsilon}{2} \\
&\qquad = \langle x^*, u_{\infty,k}(\omega) - u_{\infty,k,q}(\omega)\rangle + \langle x^* - x_k^*, u_{\infty,k,q}(\omega)\rangle - \frac{\epsilon}{2} \\
&\qquad \ge \langle x^*, u_{\infty,k}(\omega) - u_{\infty,k,q}(\omega)\rangle - \delta^*(x_k^* - x^*, X_{\infty,q}(\omega)) - \frac{\epsilon}{2} \\
&\qquad \ge \langle x^*, u_{\infty,k}(\omega) - u_{\infty,k,q}(\omega)\rangle - \frac{\epsilon}{2} - \frac{\epsilon}{2} \\
&\qquad = \lim_n \langle x^*, \widetilde{v}_{n,k}(\omega) - \widetilde{v}_{n,k,q}(\omega)\rangle - \epsilon \\
&\qquad \ge -\phi_q(\omega) - \epsilon.
\end{aligned}$$

Hence

$$\delta^*(x^*, X_\infty(\omega)) - \delta^*(x^*, X_{\infty,q}(\omega)) \ge -\phi_q(\omega)$$

So Claim (5.1.4) follows.

Step 5. Now we are going to prove the main fact in this step, namely we show that it suffices to change X_∞ on a negligible set to get $X_\infty(\omega) \in cwk(X)$ for all ω. First, there exists a negligible set $N_2 \supset N_1$ such that for every $\omega \in \Omega\backslash N_2$, $X_\infty(\omega) \in ccb(E)$. Indeed, (5.1.2) implies

$$\forall \omega \in \Omega \setminus N_1, |X_\infty|(\omega) \le \liminf_n |\widetilde{X}_n|(\omega) \tag{5.1.5}$$

and it follows from Fatou lemma and boundedness of $(X_n)_n$ that the function $\liminf_n |\widetilde{X}_n| \in L^1_{\mathbb{R}^+}$. Then (5.1.4) yields

$$H(X_\infty(\omega), X_{\infty,q}(\omega)) = \sup_{x^*\in \overline{B}_{E^*}} |\delta^*(x^*, X_\infty(\omega)) - \delta^*(x^*, X_{\infty,q}(\omega))| \le \phi_q(\omega) \tag{5.1.6}$$

for all $\omega \in \Omega \setminus N_2$. By Fatou lemma we get

$$\begin{aligned}\int_\Omega \phi_q d\mu &\le \liminf_n \sum_{j\in J_n} \mu_j^n \sum_{i \in I_{j+n}} \lambda_i^{j+n} \int_{\Omega \setminus A_{i+j+n,q}} |X_{i+j+n}| d\mu \\ &\le \sup_n \int_{\Omega\setminus A_{n,q}} |X_n| d\mu.\end{aligned} \tag{5.1.7}$$

As (X_n) is uniformly integrable, by (5.1.1) we get

$$\lim_{q\to\infty} \sup_n \int_{\Omega\setminus A_{n,q}} |X_n| d\mu = 0, \tag{5.1.8}$$

Whence using (5.1.8) we deduce that

$$\lim_{q\to\infty} \int_\Omega \phi_q d\mu = 0.$$

Therefore, there exist a stricly increasing sequence $(\alpha(q))_q$ and a negligible set $N \supset N_2$ such that for all $\omega \in \Omega \setminus N, \phi_{\alpha(q)}(\omega) \to 0$. Hence, by (5.1.6) we have that $X_\infty(\omega) \in cwk(E)$, for all $\omega \in \Omega \setminus N$, because $X_{\infty,\alpha(q)}(\omega) \in cwk(E)$ and the space $(cwk(E), H)$ is complete [8] using Grothendieck lemma [19, p.296]. Then it suffices to suppose $X_\infty \equiv 0$ on N to get $X_\infty(\omega) \in cwk(E)$ for all $\omega \in \Omega$. Consequently, by (5.1.2) and (5.1.5), X_∞ is measurable and $|X_\infty| \in L^1_{\mathbb{R}^+}$. Hence $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$.

Step 6. Let us prove (a). Let $x \in E$ and $\omega \in \Omega \setminus N$ be fixed. Using (5.1.2) and weak compactness of $X_\infty(\omega)$ and $\widetilde{X}_n(\omega), (n \in \mathbb{N}^*)$, we obtain

$$\begin{aligned}\liminf_n d(x, \tilde{X}_n(\omega)) &= \liminf_n \sup_{x^* \in \overline{B}_{E^*}} [\langle x^*, x\rangle - \delta^*(x^*, \widetilde{X}_n(\omega))] \\ &= \liminf_n \sup_k [\langle x_k^*, x\rangle - \delta^*(x_k^*, \widetilde{X}_n(\omega))] \\ &\ge \sup_k \lim_n [\langle x_k^*, x\rangle - \delta^*(x_k^*, \widetilde{X}_n(\omega))] \\ &= \sup_k [\langle x_k^*, x\rangle - \delta^*(x_k^*, X_\infty(\omega))] \\ &= d(x, X_\infty(\omega)).\end{aligned}$$

On the other hand, using the definition of X_∞, it is easily seen that

$$\limsup_n d(x, \widetilde{X}_n(\omega)) \le d(x, X_\infty(\omega)).$$

So the desired conclusion follows.

Step 7. Let us prove (b). Fix $h \in L^\infty_{E^*}(\mu)$ with $||h||_\infty \leq 1$. Using (5.1.6) and (5.1.7) we obtain for every $n \in \mathbb{N}^*$ and every $q \in \mathbb{N}^*$

$$\begin{aligned}
&\int_\Omega |\delta^*(h, \widetilde{X}_n) - \delta^*(h, X_\infty)| d\mu \\
&\qquad \leq \int_\Omega |\delta^*(h, \widetilde{X}_n) - \delta^*(h, \widetilde{X}_{n,q})| d\mu + \int_\Omega |\delta^*(h, \widetilde{X}_{n,q}) - \delta^*(h, X_{\infty,q})| d\mu \\
&\qquad + \int_\Omega |\delta^*(h, X_\infty) - \delta^*(h, X_{\infty,q})| d\mu \\
&\qquad \leq \sup_n \sum_{j \in J_n} \mu^n_j \sum_{i \in I_{j+n}} \lambda^{j+n}_i \int_{\Omega \setminus A_{i+j+n,q}} |X_{i+j+n}| d\mu \\
&\qquad + \int_\Omega |\delta^*(h, \widetilde{X}_{n,q}) - \delta^*(h, X_{\infty,q})| d\mu + \sup_n \int_{\Omega \setminus A_{n,q}} |X_n| d\mu \\
&\qquad \leq 2 \sup_n \int_{\Omega \setminus A_{n,q}} |X_n| d\mu + \int_\Omega |\delta^*(h, \widetilde{X}_{n,q}) - \delta^*(h, X_{\infty,q})| d\mu.
\end{aligned}$$

As $(\delta^*(h, \widetilde{X}_{n,q}))_n$ is uniformly integrable, from (5.1.3) and Lebesgue-Vitali theorem, it follows that

$$\int_\Omega | \ \delta^*(h, \widetilde{X}_{n,q}) - \delta^*(h, X_{\infty,q}) \ | \ d\mu \to 0.$$

Therefore using (5.1.8) we get

$$\int_\Omega | \ \delta^*(h, \widetilde{X}_n) - \delta^*(h, X_\infty) \ | \ d\mu \to 0.$$

B. Now we pass to the general case, i.e (X_n) is bounded.

Thank to the biting lemma it is now easy to reduce to assumption A. There are an increasing sequence (B_n) in $\mathcal{F}$ with $\lim_n \mu(B_n) = 1$ and a subsequence (X'_n) of (X_n) such that $X'_n = 1_{B_n} X'_n + 1_{\Omega \setminus B_n} X'_n$ where $(Y'_n)_n = (1_{B_n} X'_n)_n$ is a uniformly integrable and $|Z'_n| \to 0$ a.e where $Z'_n = 1_{\Omega \setminus B_n} X'_n$. By the above decomposition, it is obvious that (Y'_n) is $cwk(E)$-tight. Consequently we may apply the results obtained in Step A to (Y'_n). This gives a sequence $(\widetilde{Y}_n)$ in $\mathcal{L}^1_{cwk(E)}(\mu)$ with $\widetilde{Y}_n \in co\{Y'_m : m \geq n\}$ and $Y_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$ such that

(a) $\forall x \in E, \quad \lim_{n\to\infty} d(x, \widetilde{Y}_n(\omega)) = d(x, Y_\infty(\omega))$ a.e.
(b) $\forall u \in L^\infty_{E^*}(\mu), \quad \delta^*(u, \widetilde{Y}_n) \to \delta^*(u, Y_\infty)$ in measure.

Each $\widetilde{Y}_n$ has the form $\Sigma^{\nu_n}_{i=n} \lambda^n_i Y'_i$ with $\lambda^n_i \geq 0$ and $\Sigma^{\nu_n}_{i=n} \lambda^n_i = 1$. Since the sequence $(\Sigma^{\nu_n}_{i=n} \lambda^n_i |Z'_i|)$ converges to 0 a.e because $|Z'_n| \to 0$ a.e, the sequence $(\widetilde{X}_n) = (\Sigma^{\nu_n}_{i=n} \lambda^n_i X'_i)$ has the required properties. □

Before going further we give some useful properties of the multifunction-limit which occurs in Theorems 3.2–3.3–5.1.

Proposition 5.2. *Let (X_n) be a bounded sequence in $\mathcal{L}^1_{cwk(E)}(\mu)$ satisfying the hypotheses and notations of Theorems 3.2 and 3.3 respectively, then there exist*

$X_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$ and a sequence $(\widetilde{X}_n)$ with $\widetilde{X}_n \in co\{X_m : m \geq n\}$ such that the following holds :

$$\lim_{n\to\infty} D(B, \widetilde{X}_n) = D(B, X_\infty)$$

for every bounded closed convex subset B in E and for almost all $\omega \in \Omega$.

Proof. It is enough to consider the case of Theorems 3.2 since in the other case the proof quite similar. By the theorem under consideration there exist $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$ and a sequence $(\widetilde{X}_n)$ with $\widetilde{X}_n \in co\{X_m : m \geq n\}$ such that $X_\infty = \tau_L\text{-}\lim_n \widetilde{X}_n$ a.e.

Step 1. Claim : $D(B, X_\infty) \leq \liminf_{n\to\infty} D(B, \widetilde{X}_n)$ for every bounded closed convex subset B in E and for almost all $\omega \in \Omega$. We have

$$\begin{aligned}
\liminf_{n\to\infty} D(B, \widetilde{X}_n) &= \liminf_{n\to\infty} \sup_{x^*\in \overline{B}_{E^*}} \{-\delta^*(x^*, \widetilde{X}_n) - \delta^*(-x^*, B)\} \\
&\geq \sup_{x^*\in \overline{B}_{E^*}} \{-\lim_{n\to\infty} \delta^*(x^*, \widetilde{X}_n) - \delta^*(-x^*, B)\} \\
&= \sup_{x^*\in \overline{B}_{E^*}} \{-\delta^*(x^*, X_\infty) - \delta^*(-x^*, B)\} \\
&= D(B, X_\infty)
\end{aligned}$$

for every bounded closed convex subset B in E and for almost all $\omega \in \Omega$, thus proving the *liminf part*. Let us prove now the *limsup part*.

Step 2. Claim : $\limsup_{n\to\infty} D(B, \widetilde{X}_n) \leq D(B, X_\infty)$ for all bounded closed convex subset B in E and almost surely $\omega \in \Omega$.

Now let B be a bounded closed convex subset of E, then

$$\begin{aligned}
\limsup_{n\to\infty} D(B, \widetilde{X}_n(\omega)) &= \limsup_{n\to\infty} \inf_{x\in B} d(x, \widetilde{X}_n(\omega)) \\
&\leq \inf_{x\in B} \lim_{n\to\infty} d(x, \widetilde{X}_n(\omega)) \\
&\leq \inf_{x\in B} d(x, X_\infty(\omega)) \\
&= D(B, X_\infty(\omega))
\end{aligned}$$

for almost all $\omega \in \Omega$. □

Proposition 5.2'. *Suppose that (X_n) be a bounded $cwk(E)$-tight sequence in $\mathcal{L}^1_{cwk(E)}(\mu)$, then there exist $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$ and a sequence $(\widetilde{X}_n)$ with $\widetilde{X}_n \in co\{X_m : m \geq n\}$ such that the following holds :*

$$\lim_{n\to\infty} D(K, \widetilde{X}_n) = D(K, X_\infty)$$

for every K in $cwk(E)$ and for almost all $\omega \in \Omega$.

Proof. Let X_∞ and $(\widetilde{X}_n)$ as given in Theorem 5.1. A careful look of the proof of this theorem shows that

$$\lim_{n\to\infty} \delta^*(x_k^*, \widetilde{X}_n(\omega)) = \delta^*(x_k^*, X_\infty(\omega))$$

a.e where $(x_k^*)_k$ is a countable dense for the Mackey topology in $\overline{B}_{E^*}$. So, the *liminf part* : $D(K, X_\infty) \leq \liminf_{n\to\infty} D(K, \widetilde{X}_n)$ for every K in $cwk(E)$ and for almost all $\omega \in \Omega$ follows from Step 1 of Proposition 5.2 while the *limsup part* follows from the arguments of Step 2 of this proposition. □

The following result is an application of Theorem 4.2 and Theorem 5.1 illustrating the combined use of Mazur convergence and Komlós convergence. Let us mention that Theorem 3.4(a) is valid if we replace condition (a) by the following tightness condition :

(T_a) for any subsequence (Y_n), there is a $cwk(E)$-tight sequence $(\widetilde{Y}_n)$ with $\widetilde{Y}_n \in co\{Y_m : m \geq n\}$.

Indeed it is enough to use Theorem 5.1 instead of Theorem 3.3 in the proof of Theorem 3.4(a).

Proposition 5.3. *Suppose that E is a separable Banach space, (X_n) is a bounded sequence in $\mathcal{L}^1_{cwk(E)}(\mu)$ which satisfies the following condition :*

(T_a) for any subsequence (Y_n), there is a $cwk(E)$-tight sequence $(\widetilde{Y}_n)$ with $\widetilde{Y}_n \in co\{Y_m : m \geq n\}$.

Then there exist a multifunction $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$ and a subsequence (Y_n) of (X_n) such that (Y_n) $\sigma(E, D^*)$-Komlós converges to X_∞. In particular, if (X_n) is $cwk(E)$-tight, then for all $h \in L^\infty_{E^*}(\mu)$ and for each further sequence (Z_n) of (Y_n) the following holds :

$$\delta^*(h, \frac{1}{n}\Sigma_{i=1}^n Z_i) \to \delta^*(h, X_\infty) \text{ in measure.}$$

Proof. Using the biting lemma, it suffices to consider the case where (X_n) is uniformly integrable. In view of the preceding remark concerning the validity of Theorem 3.4(a) under the tightness condition (T_a), there exist a subsequence still denoted by (X_n) and $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$ such that

$$(5.3.1) \qquad \forall h \in L^\infty_{E^*}(\mu)\,,\ \lim_{n\to\infty} \int_\Omega \delta^*(h, X_n)\, d\mu = \int_\Omega \delta^*(h, X_\infty)\, d\mu.$$

Let (x_k^*) be a dense sequence in $\overline{B}_{E^*}$ for the Mackey topology. Applying Komlós theorem [22] to the sequences $(\delta^*(x_k^*, X_n))_n, (k \in \mathbb{N}^*)$ via a diagonal procedure, provides a subsequence (Y_n) of (X_n) and a sequence $(\varphi_k), (k \in \mathbb{N}^*)$ in $L^1_{\mathbb{R}}(\mu)$ such that

$$(5.3.2) \qquad \forall k,\ \lim_n \frac{1}{n}\Sigma_{i=1}^n \delta^*(x_k^*, Z_i) = \varphi_k$$

almost everywhere, for every further subsequence (Z_m) of (Y_n), so that, by (5.3.2) and Lebesgue-Vitali theorem, we get

$$(5.3.3) \qquad \forall k,\ \forall A \in \mathcal{F},\ \lim_n \int_A \delta^*(x_k^*, Y_n)\, d\mu = \int_A \varphi_k\, d\mu.$$

From (5.3.1) and (5.3.3) it follows that

$$\varphi_k = \delta^*(x_k^*, X_\infty) \ a.e.$$

Returning to (5.3.2) we conclude that (Y_n) $\sigma(E, D^*)$-Komlós converges to X_∞. Now it suffices to apply Theorem 4.2 to complete the proof. □

Remark. It is worth to mention that Proposition 5.3 shows that it is now possible to remove the assumption : " X_n $\sigma(E, D^*)$-Komlós converges to $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$" in Theorem 4.2.

The following is a variant of Proposion 5.3.

Proposition 5.4. *Suppose that E^* is separable and E has the RNP, (X_n) is a bounded sequence in $\mathcal{L}^1_{cwk(E)}(\mu)$ satisfying :*

(T_b) *for any sequence (Y_n) there is $(\widetilde{Y}_n)$ with $\widetilde{Y}_n \in co\{X_m : m \geq n\}$ such that, for each $A \in \mathcal{F}$, $\cup_n \int_A \widetilde{Y}_n \, d\mu$ is relatively weakly compact in E,*

then there exist a subsequence (Y_n) of (X_n) and $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$ such that for all $x^* \in E^*$ and for each further subsequence (Z_n) of (Y_n), the following holds :

$$\delta^*(x^*, \frac{1}{n}\Sigma_{i=1}^n Z_i) \to \delta^*(x^*, X_\infty) \ a.e.$$

Proof. Step 1. We suppose first that (X_n) is uniformly integrable.

In view of Theorem 3.4(b) there exist a subsequence still denoted by (X_n) and $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$ such that

$$\forall h \in L^\infty_{E^*}(\mu)\,, \ \lim_{n\to\infty} \int_\Omega \delta^*(h, X_n)\, d\mu = \int_\Omega \delta^*(h, X_\infty)\, d\mu. \tag{5.4.1}$$

Let (x_k^*) be a dense sequence in E^*. Applying Komlós theorem to the sequences $(\delta^*(x_k^*, X_n))_n, (k \in \mathbb{N}^*)$ and $(|X_n|)$ via a diagonal procedure, provides a subsequence (Y_n) of (X_n) and a sequence $(\varphi_k), (k \in \mathbb{N}^*)$ in $L^1_{\mathbb{R}}(\mu)$ and $\theta \in L^1_{\mathbb{R}^+}(\mu)$ such that

$$\forall k, \ \lim_n \frac{1}{n}\Sigma_{i=1}^n \delta^*(x_k^*, Z_i) = \varphi_k \tag{5.4.2}$$

and

$$\lim_n \frac{1}{n}\Sigma_{i=1}^n |Z_i| = \theta \tag{5.4.3}$$

almost everywhere, for every further subsequence (Z_n) of (X_n). By (5.4.3) the sequence $(\frac{1}{n}\Sigma_{i=1}^n |Z_i|)_n$ is pointwise bounded almost everywhere. By (5.4.1) and (5.4.2) it is immediate that

$$\forall k, \ \varphi_k = \delta^*(x_k^*, X_\infty). \tag{5.4.4}$$

Whence using the separability of E^* and the pointwise boundness of $(\frac{1}{n}\Sigma_{i=1}^n |Z_i|)_n$ we get by a routine argument

$$\lim_n \frac{1}{n}\Sigma_{i=1}^n \delta^*(x^*, Z_i) = \delta^*(x^*, X_\infty) \tag{5.4.5}$$

for all $x^* \in E^*$ and almost everywhere.

B. *Step 2.* Now we pass to the general case. In view of the biting lemma there exist an increasing sequence (A_p) in $\mathcal{F}$ with $\lim_{p\to\infty} \mu(A_p) = 1$ and a subsequence of (X_n) still denoted by (X_n) such that $(X_n|_{A_p})_n$ is uniformly integrable for each p. Applying the result obtained in Step 1 via a diagonal procedure to the sequences

$(1_{A_p}X_n)_n(p \in \mathbb{N}^*)$ provides multifunctions $X^p_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$, $(p \in \mathbb{N}^*)$ and a subsequence (Y_n) of (X_n) such that

$$\forall p \in \mathbb{N}^*,\ \forall x^* \in E^*,\ \lim_n \frac{1}{n}\Sigma_{i=1}^n \delta^*(x^*, 1_{A_p}Z_i) = \delta^*(x^*, X^p_\infty)$$

almost everywhere, for every subsequence (Z_n) of (Y_n). As (A_p) is increasing, there exists a negligible set N such that for all p and for all $\omega \in A_p \setminus N$, $X^p_\infty(\omega) = X^{p+1}_\infty(\omega)$. Set $X_\infty(\omega) = X^p_\infty(\omega)$ if $\omega \in A_p \setminus N$ and $X_\infty(\omega) = 0$ if $\omega \in \cap_p(\Omega \setminus A_p) \cup N$. Then it is easy to check that (Y_n) and X_∞ have the required properties. □

Comments. (1) Theorems 3.4-4.1-4.2 are natural extensions of Dunford-Pettis-types theorem in $L^1_E(\mu)$ [see, for instance, 2, 5, 6, 8, 10, 11, 14, 17, 26]. In this context we obtain sharp results of convergence which cannot be demonstrated by routine arguments. In particular we mention the validity of Theorem 3.4(a) when we replace condition (a) by tightness condition (T_a). When comparing with the $L^1_E(\mu)$ case, this tightness condition is even weaker than the weak compactness condition (*) (alias Mazur convergence in our terminology) in [17, 26]. Yet, in $L^1_E(\mu)$ case, Theorem 8 in [24] is the analogue of Theorem 3.4 with tightness condition (T_a). Let us mention that most convergence results for the space $\mathcal{L}^1_{cwk(E)}$ we present here need a careful look in constrast to the $L^1_E(\mu)$ case. For example, in the case when the strong dual of E is separable, one can see that, if (u_n) is a uniformly integrable sequence in $L^1_E(\mu)$, which pointwise converges to $u_\infty \in L^1_E(\mu)$ on $L^\infty(\mu) \otimes E^*$, then (u_n) weakly converges to u_∞ by using a general fact [7] : "on bounded subsets of $L^\infty_{E^*}$ convergence in measure coincide with uniform convergence on uniformly integrable subsets of $L^1_E(\mu)$ (see [19] for the case $L^1_{\mathbb{R}}(\mu)$)" and usual arguments. Indeed any $h \in L^\infty_{E^*}(\mu)$ is limit of an almost everywhere convergent sequence $(h_p)_p$ of step functions satisfying $\forall p$, $\|h_p(\omega)\| \le \|h\|_\infty$. As $\forall n$, $\forall p$, we have

$$|\langle h, u_n - u_\infty\rangle| \le \sup_n |\langle h - h_p, u_n\rangle| + |\langle h - h_p, u_\infty\rangle| + |\langle h_p, u_n - u_\infty\rangle|$$

u_n converges $\sigma(L^1_E, L^\infty_{E^*})$ to u_∞. It would be interesting to know whether the preceding property still holds in the problem of weak convergence for uniformly integrable sequences in $\mathcal{L}^1_{cwk(E)}(\mu)$.

(2) In the context of Proposition 5.4, Komlós arguments in (5.4.2) and (5.4.3) are first employed in [3]. Apart from theses facts, our proof is different since it is based upon Theorem 3.2 and Theorem 3.4(b) providing Mazur τ_l-convergence of the sequences under consideration instead of Komlós convergence. We refer to [3] for details and other related results.

6. Applications : Biting-types lemma and Fatou-types lemma in $\mathcal{L}^1_{cwk(E)}(\mu)$

We need first the following version of biting lemma which is a direct consequence of Proposition 5.3 and Proposition 5.4 respectively. See also [9, 12].

Theorem 6.1. *Suppose that E is a separable Banach space, (X_n) is a bounded sequence in $\mathcal{L}^1_{cwk(E)}(\mu)$ satisfying one of the following conditions :*

(a) (X_n) *is* $cwk(E)$*-tight.*

(b) E^* *is separable,* E *has the R.N.P and for each* $A \in \mathcal{F}$, $\cup_n \int_A X_n\, d\mu$ *is relatively weakly compact in* E.

Then there exist an increasing sequence (A_p) *in* $\mathcal{F}$ *such that* $\lim_{p\to\infty} \mu(A_p) = 1$, *a subsequence* (X'_n) *of* (X_n) *and* $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$ *such that, for each* p *and for each* $v \in L^\infty_{E^*}(A_p, A_p \cap \mathcal{F}, \mu|_{A_p})$, *the following holds :*

$$\lim_{n\to\infty} \int_{A_p} \delta^*(v, X'_n)\, d\mu = \int_{A_p} \delta^*(v, X_\infty)\, d\mu.$$

Proof. The biting lemma provides an increasing sequence (A_p) in $\mathcal{F}$ with $\lim_{p\to\infty} \mu(A_p) = 1$ and a subsequence (X'_n) of (X_n) such that $(X'_n|_{A_p})_n$ is uniformly integrable.

(a) Assume that (X_n) is $cwk(E)$-tight. By Proposition 5.3 there exists a multifunction $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$ and a subsequence of (X'_n) still denoted by (X'_n) such that for all $v \in L^\infty_{E^*}(\mu)$ and for each further subsequence (Y_n) of (X'_n), $\delta^*(v, \frac{1}{n}\Sigma_{i=1}^n Y_i) \to \delta^*(v, X_\infty)$ in measure. As (X'_n) is uniformly integrable on each A_p, by Lebesgue-Vitali theorem, we get

$$\forall p,\ \forall v \in L^\infty_{E^*}(\mu),\ \int_{A_p} \delta^*(v, \frac{1}{n}\Sigma_{i=1}^n Y_i)\, d\mu \to \int_{A_p} \delta^*(v, X_\infty)\, d\mu$$

for every further subsequence (Y_n) of (X'_n). This is equivalent to

$$\forall p,\ \forall v \in L^\infty_{E^*}(\mu),\ \int_{A_p} \delta^*(v, X'_n)\, d\mu \to \int_{A_p} \delta^*(v, X_\infty)\, d\mu.$$

Under assumption (b) the proof is similar by using Theorem 5.4. □

Now we give an alternative proof of Theorem 6.1 via Mazur convergence in $\mathcal{L}^1_{cwk(E)}(\mu)$ (cf. Theorem 3.2 and Theorem 5.1). We want to mention that some results given in this proof will be used in the next Fatou-types lemma.

Let $D := (e^*_k)$ be a dense sequence in $\overline{B}_{E^*}$ for the Mackey topology and let (A_p) and (X'_n) as given in the begining of proof of Theorem 6.1. Using Dundord-Pettis theorem in $L^1_{\mathbb{R}}(\mu)$ and a diagonal procedure we may suppose that

$$\forall k, \forall p, \forall A \in A_p \cap \mathcal{F},\ \lim_{n\to\infty} \int_A \delta^*(e^*_k, X'_n)\, d\mu \text{ exists.} \tag{6.1.1}$$

We will divide the proof in two steps.

Step 1. Claim : Suppose that (a) holds, then there exists $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$ which satisfies :

Given any subsequence (Y_n) of (X'_n), there exists a sequence $(\widetilde{Y}_n)$ with $\widetilde{Y}_n \in co\{Y_m : m \geq n\}$ such that

$$\forall x \in E,\ \lim_{n\to\infty} d(x, \widetilde{Y}_n) = d(x, X_\infty)\ a.e. \tag{6.1.2}$$

$$\forall v \in L^\infty_{E^*}(\mu),\ \delta^*(v, \widetilde{Y}_n) \to \delta^*(v, X_\infty) \text{ in measure.} \tag{6.1.3}$$

Indeed, let (Y_n) be a subsequence of (X'_n). By Theorem 5.1, there exist $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$ and $(\widetilde{Y}_n)$ with $\widetilde{Y}_n \in co\{Y_m : m \geq n\}$ satisfying (6.1.2) and (6.1.3). Using (6.1.1), (6.1.3), the uniform integrability of $(\widetilde{Y}_n)$ on each A_p and Lebesgue-Vitali theorem, we get

$$\text{(6.1.4)} \qquad \forall k,\ \forall p,\ \forall A \in A_p \cap \mathcal{F},\ \lim_n \int_A \delta^*(e_k^*, X'_n)\, d\mu = \int_A \delta^*(e_k^*, X_\infty)\, d\mu.$$

Now replacing (Y_n) by any other subsequence (Z_n) of (X'_n) and applying again Theorem 5.1 provides a multifunction $Y_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$ and a sequence $(\widetilde{Z}_n)$ with $\widetilde{Z}_n \in co\{Z_m : m \geq n\}$ satisfying :

$$\text{(6.1.2)}' \qquad \forall x \in E,\ \lim_{n\to\infty} d(x, \widetilde{Z}_n) = d(x, Y_\infty)\ a.e.$$

$$\text{(6.1.3)}' \qquad \forall v \in L^\infty_{E^*}(\mu),\ \delta^*(v, \widetilde{Z}_n) \to \delta^*(v, Y_\infty) \text{ in measure.}$$

$$\text{(6.1.4)}' \qquad \forall k,\ \forall p,\ \forall A \in A_p \cap \mathcal{F},\ \lim_n \int_A \delta^*(e_k^*, X'_n)\, d\mu = \int_A \delta^*(e_k^*, Y_\infty)\, d\mu.$$

Therefore, by (6.1.4) and (6.1.4)$'$ we get

$$\forall k,\ \forall p,\ \delta^*(e_k^*, X_\infty) = \delta^*(e_k^*, Y_\infty)\ \ a.e\, \omega \in A_p$$

Hence we deduce that $X_\infty = Y_\infty$ a.e, thus proving the claim.

Step 2. Let $p \in \mathbb{N}^*$ and $v \in L^\infty_{E^*}(\mu)$ be fixed. Choose a subsequence (Y_n) of (X'_n) such that

$$\text{(6.1.5)} \qquad \lim_n \int_{A_p} \delta^*(v, Y_n)\, d\mu = \limsup_n \int_{A_p} \delta^*(v, X'_n)\, d\mu.$$

By Step 1, there exists $(\widetilde{Y}_n)$ with $\widetilde{Y}_n \in co\{Y_m : m \geq n\}$ such that $\delta^*(v, \widetilde{Y}_n) \to \delta^*(v, X_\infty)$ in measure so that

$$\int_{A_p} \delta^*(v, \widetilde{Y}_n)\, d\mu \to \int_{A_p} \delta^*(v, X_\infty)\, d\mu.$$

From (6.1.5) it follows that

$$\limsup_n \int_{A_p} \delta^*(v, X'_n)\, d\mu = \int_{A_p} \delta^*(v, X_\infty)\, d\mu.$$

Similarly we have

$$\liminf_n \int_{A_p} \delta^*(v, X'_n)\, d\mu = \int_{A_p} \delta^*(v, X_\infty)\, d\mu.$$

Hence

$$\lim_n \int_{A_p} \delta^*(v, X'_n)\, d\mu = \int_{A_p} \delta^*(v, X_\infty)\, d\mu.$$

The proof is therefore complete when (X_n) is $cwk(E)$-tight. In case (b) the proof is similar by applying Theorem 3.2.

Taking into account Theorem 6.1, it is convenient to say that "(X'_n) biting weakly converges to X_∞". Now we proceed to multivalued Fatou-types lemma. See also [3, 9, 11, 12] for other related results.

Theorem 6.2. *Suppose that E is a separable Banach space, (X_n) is a bounded $cwk(E)$-tight sequence in $\mathcal{L}^1_{cwk(E)}(\mu)$, then there exist a multifunction X_∞ and a subsequence (X'_n) of (X_n) such that (X'_n) biting weakly converges to X_∞. Furthermore the following properties hold :*

(1) $X_\infty(\omega) \subset \overline{co}\,[w\text{-}\,ls\, X'_n(\omega)]$ *a.e.*

(2) *If $(v_n)_{n\in\mathbb{N}\cup\{\infty\}}$ is a sequence of scalarly measurable mappings from Ω to $\overline{B}_{E^*}$ such that $|v_n - v_\infty| \to 0$ in measure and that $(\delta^*(v_n, X_n)^-)_n$ is uniformly integrable, then $\liminf_{n\to\infty} \int_\Omega \delta^*(v_n, X'_n)\, d\mu \geq \int_\Omega \delta^*(v_\infty, X_\infty)\, d\mu$.*

(3) *For each measurable multifunction $K : \Omega \to cwk(E)$, one has*

$$\liminf_n \int_\Omega D(K, X'_n)\, d\mu \geq \int_\Omega D(K, X_\infty)\, d\mu.$$

Proof. Let $(A_p), (X'_n)$ and X_∞ be defined as in the second proof of Theorem 6.1.

(1) We will divide this main part of the proof in several steps.

Step 1. According to Step 1 of the second proof of Theorem 6.1, there exists a sequence $(\widetilde{Y}_n)$ of the form $\widetilde{Y}_n = \Sigma_{i\in I_n} \lambda_i^n X'_{i+n}$ with $\lambda_i^n \geq 0$ and $\Sigma_{i\in I_n}\lambda_i^n = 1$ such that, for every $v \in L^\infty_{E^*}(\mu)$, $(\delta^*(v, \widetilde{Y}_n))$ converges in measure to $\delta^*(v, X_\infty)$. Let $D^* := (e_k^*)$ be a countable dense sequence in $\overline{B}_{E^*}$ for the Mackey topology. By what has been said, we may suppose that

$$\forall k,\ \lim_n \delta^*(e_k^*, \widetilde{Y}_n) = \delta^*(e_k^*, X_\infty) \text{ a.e.} \tag{6.2.1}$$

Step 2. By tightness assumption, for every $q \in \mathbb{N}^*$ there exists a $cwk(E)$-valued measurable multifunction $\Gamma_{\frac{1}{q}}$ such that

$$\forall n,\ \mu(\Omega \setminus A_{n,q}) \leq \frac{1}{q}, \tag{6.2.2}$$

where

$$A_{n,q} := \{\omega \in \Omega : X_n(\omega) \subset \Gamma_{\frac{1}{q}}(\omega)\}.$$

Let x be any fixed element in E. Let us define

$$\widetilde{Y}^x_{n,q} = \Sigma_{i\in I_n} \lambda_i^n 1_{A'_{i+n,q}}(X'_{i+n} - x)$$

where for each $q \in \mathbb{N}^*, (A'_{n,q})$ is the subsequence of $(A_{n,q})$ corresponding to (X'_n). Let $q \in \mathbb{N}^*$. As $\widetilde{Y}^x_{n,q}(\omega) \subset \overline{co}(\Gamma_{\frac{1}{q}}(\omega) - x \cup \{0\})$ for all $n \in \mathbb{N}^*$ and for all $\omega \in \Omega$, there exist by Theorem 3.3 a sequence $(\widetilde{Z}^x_{n,q})$ of the form $\widetilde{Z}^x_{n,q} = \Sigma_{j\in J_{n,q}} \mu^{n,x}_{j,q} \widetilde{Y}^x_{j+n,q}$ with $\mu^{n,x}_{j,q} \geq 0$ and $\Sigma_{j\in J_{n,q}} \mu^{n,x}_{j,q} = 1$ and $X^x_{\infty,q} \in \mathcal{L}^1_{cwk(E)}(\mu)$ such that

$$X^x_{\infty,q} = \tau_L\text{-}\lim_n \widetilde{Z}^x_{n,q} \quad a.e. \tag{6.2.3}$$

In particular we have

$$\forall x^* \in E^*, \lim_n \delta^*(x^*, \widetilde{Z}^x_{n,q}) = \delta^*(x^*, X^x_{\infty,q}) \quad a.e. \tag{6.2.4}$$

Step 3. For simplicity let us set

$$\Phi_q^x(.) := \liminf_n \Sigma_{j\in J_{n,q}}\mu_{j,q}^{n,x}\,\Sigma_{i\in I_{j+n}}\lambda_i^{j+n}1_{\Omega\setminus A'_{i+j+n,q}}|X'_{i+j+n}-x|,\ (q\in\mathbb{N}^*).$$

We claim that

$$\forall q,\ H(X_\infty(\omega)-x, X_{\infty,q}^x(\omega)) \le \Phi_q^x(\omega)\ a.e. \tag{6.2.5}$$

and

$$\forall p,\ \lim_q \int_{A_p}\Phi_q^x\,d\mu = 0. \tag{6.2.6}$$

Indeed, from (6.2.1) and (6.2.4) it follows that

$$\delta^*(e_k^*, \widetilde{W}_{n,q}^x) - \delta^*(e_k^*, \widetilde{Z}_{n,q}^x) \to \delta^*(e_k^*, X_\infty - x) - \delta^*(e_k^*, X_{\infty,q}^x)\ \ a.e.$$

for every k and every q, where $\widetilde{W}_{n,q}^x = \Sigma_{j\in J_{n,q}}\mu_{j,q}^{n,x}(\widetilde{Y}_{j+n}-x)$. Consequenly we deduce that

$$\begin{aligned} H(X_\infty - x, X_{\infty,q}^x) &= \sup_{x^*\in \overline{B}_{E^*}} |\delta^*(x^*, X_\infty - x) - \delta^*(x^*, X_{\infty,q}^x)| \\ &= \sup_{x^*\in D^*} |\delta^*(x^*, X_\infty - x) - \delta^*(x^*, X_{\infty,q}^x)| \\ &= \sup_{x^*\in D^*}\lim_n |\delta^*(x^*, \widetilde{W}_{n,q}^x) - \delta^*(x^*, \widetilde{Z}_{n,q}^x)| \\ &\le \Phi_q^x\ \ a.e. \end{aligned}$$

thus proving (6.2.5). By Fatou lemma we get

$$\int_{A_p}\Phi_q^x\,d\mu \le \sup_n \int_{A_p\cap[\Omega\setminus A'_{n,q}]} |X'_n - x|\,d\mu \tag{6.2.7}$$

for every $p\in\mathbb{N}^*$ and every $q\in\mathbb{N}^*$. As (X'_n) is uniformly integrable on each A_p, by (6.2.2) we get

$$\forall p,\ \lim_q\sup_n \int_{A_p\cap[\Omega\setminus A'_{n,q}]} |X'_n - x|\,d\mu = 0. \tag{6.2.8}$$

So (6.2.6) follows from (6.2.7) and (6.2.8). *Step 4.* Now we are able to prove the inclusion

$$X_\infty(\omega) \subset \overline{co}\,[w\text{-}ls\,X'_n(\omega)]\ a.e.$$

By (6.2.3) we have

$$X_{\infty,q}^x(\omega) \subset \bigcap_n \overline{co}\{\widetilde{Z}_{m,q}^x : m\ge n\} \subset \bigcap_n \overline{co}\{1_{A'_{m,q}}(X'_m(\omega)-x) : m\ge n\}$$

for almost all $\omega\in\Omega$. As $1_{A'_{n,q}}(X'_m(\omega)-x) \subset \overline{co}(\Gamma_{\frac{1}{q}}(\omega) - x\cup\{0\})$ for all $n\in\mathbb{N}^*$ and for all $\omega\in\Omega$, from [2, lemma 2'] it follows that

$$X_{\infty,q}^x(\omega) \subset \overline{co}[w\text{-}ls\,1_{A'_{n,q}}(X'_n(\omega)-x)] \subset \overline{co}[w\text{-}ls(X'_n(\omega)-x)\cup\{0\}]$$

for almost all $\omega \in \Omega$. By (6.2.5) we have, $\forall q$,

$$\begin{aligned}\sup_{y \in X_\infty(\omega)-x} d(y, \overline{co}[w\text{-}ls(X'_n(\omega)-x) \cup \{0\}] &\le \sup_{y \in X_\infty(\omega)-x} d(y, X^x_{\infty,q}(\omega)) \\ &\le H(X_\infty(\omega)-x, X^x_{\infty,q}(\omega)) \\ &\le \Phi^x_q(\omega) \ a.e.\end{aligned} \tag{6.2.9}$$

As $\mu(A_p) \uparrow 1$, in view of (6.2.6) there is a subsequence of (Φ^x_q) converging to 0 a.e.
From (6.2.9) it follows that

$$X_\infty(\omega) - x \subset \overline{co}\,[w\text{-}\,ls\,(X'_n(\omega)-x) \cup \{0\})] \ \ a.e. \tag{6.2.10}$$

Whence we have

$$X_\infty(\omega) \subset \overline{co}\,[w\text{-}\,ls\,X'_n(\omega) \cup \{x\}] \ \ a.e. \tag{6.2.11}$$

Now we use the following fact [2, page 177]. *Let C be a subset in E and let $(x_m)_{m\in\mathbb{N}}$ be a dense sequence in E. Then the following hold :*

$$\overline{co}[C] = \bigcap_{m\in\mathbb{N}} \overline{co}[C \cup \{x_m\}].$$

Applying the preceding fact and (6.2.11) yields immediately :

$$X_\infty(\omega) \subset \bigcap_{m\in\mathbb{N}} \overline{co}\,[w\text{-}\,ls\,X'_n(\omega) \cup \{x_m\}] = \overline{co}\,[w\text{-}\,ls\,X'_n(\omega)] \ \ a.e.$$

(2) Let $\varepsilon > 0$ be given. Pick $N \in \mathbb{N}$ such that

$$\int_{A_N} \delta^*(v_\infty, X_\infty)\, d\mu \ge \int_\Omega \delta^*(v_\infty, X_\infty)\, d\mu - \varepsilon$$

and that

$$\limsup_{n\to\infty} \int_{\Omega\setminus A_N} \delta^*(v_n, X'_n)^-\, d\mu \le \varepsilon$$

because $(\delta^*(v_n, X'_n)^-)_n$ is uniformly integrable by hypothesis. As $|v_n - v_\infty| \to 0$ in measure, in view of [7] $|v_n - v_\infty| \to 0$ uniformly on uniformly integrable subsets of $L^1_{\mathbb{R}}(\Omega, \mathcal{F}, \mu)$. It follows that that

$$\lim_{n\to\infty} \int_{A_N} |v_n - v_\infty||X'_n|\, d\mu = 0.$$

Whence

$$\lim_{n\to\infty} \Big[\int_{A_N} \delta^*(v_n, X'_n)\, d\mu - \int_{A_N} \delta^*(v_\infty, X'_n)\, d\mu \Big] = 0.$$

Let $a := \liminf_{n\to\infty} \int \delta^*(v_n, X'_n)\, d\mu$. We may suppose that

$$a := \lim_{n\to\infty} \int \delta^*(v_n, X'_n)\, d\mu \in \mathbb{R}.$$

An easy computation gives

$$a \geq \lim_{n\to\infty} \int_{A_N} \delta^*(v_n, X'_n)\, d\mu - \limsup_{n\to\infty} \int_{\Omega\setminus A_N} \delta^*(v_n, X'_n)^-\, d\mu$$
$$\geq \lim_{n\to\infty} \int_{A_N} \delta^*(v_n, X'_n)\, d\mu - \varepsilon.$$

Finally we get

$$\begin{aligned} a &\geq \lim_{n\to\infty} \int_{A_N} \delta^*(v_n, X'_n)\, d\mu - \varepsilon \\ &= \lim_{n\to\infty} \int_{A_N} \delta^*(v_\infty, X'_n)\, d\mu - \varepsilon \\ &= \int_{A_N} \delta^*(v_\infty, X_\infty)\, d\mu - \varepsilon \\ &\geq \int_{\Omega} \delta^*(v_\infty, X_\infty)\, d\mu = -2\varepsilon \end{aligned}$$

thus proving (2).

(3) Let $K : \Omega \to cwk(E)$ be a measurable multifunction. Choose a subsequence (Y_n) of (X'_n) such that

$$\lim_n \int_\Omega D(K, Y_n)\, d\mu = \liminf_n \int_\Omega D(K, X'_n)\, d\mu.$$

Applying Step 1 of the second proof of Theorem 6.1 to (Y_n) provides a sequence $(\widetilde{Y}_n)$ with $\widetilde{Y}_n \in co\{Y_m : m \geq n\}$ satisfying (6.1.2) and (6.1.3). Reasoning as in the proof of Proposition 5.2' we get

$$D(K, \widetilde{Y}_n) \to D(K, X_\infty)\ a.e.$$

Therefore by Fatou lemma,

$$\liminf_n \int_\Omega D(K, \widetilde{Y}_n)\, d\mu \geq \int_\Omega D(K, X_\infty)\, d\mu.$$

Since $\widetilde{Y}_n = \Sigma_{i=n}^{\nu_n} \lambda_i^n Y_i$ with $\lambda_i^n \geq 0$ and $\Sigma_{i=n}^{\nu_n} \lambda_i^n = 1$ and the function $D(K(.), .)$ is convex on $cwk(E)$ (using the gap functional in terms of the support functions) it follows that

$$\lim_n \Sigma_{i=n}^{\nu_n} \lambda_i^n \int_\Omega D(K, Y_i)\, d\mu \geq \int_\Omega D(K, X_\infty)\, d\mu.$$

The proof is therefore complete. □

Remarks. It seems that the properties given in Theorem 6.2 are optimal for bounded $cwk(E)$-tight sequences. Theorem 6.2 is a multivalued analog of Theorem 3.5 in [9] dealing with Fatou lemma for bounded $cwk(E)$-tight sequences in $L^1_E(\mu)$. Also it is worth to mention that the multifunction X_∞ which occurs in Theorems 3.2-3.3-5.1 is almost everywhere equal to the Mosco-limit (see e.g. [4]) of the sequence $(\widetilde{X}_n)$ with $\widetilde{X}_n \in co\{X_m : m \geq n\}$. This result involves the Mosco-convergence of $E^{\mathcal{F}_n}\widetilde{X}_n$ towards $E^{\mathcal{F}_\infty}X_\infty$ under suitable domination condition on (X_n) where $(\mathcal{F}_n)_n$

is an increasing (resp. decreasing) sequence of sub σ-algebras of $\mathcal{F}$ with $\mathcal{F}_\infty =\uparrow \mathcal{F}_n$ (resp. $\mathcal{F}_\infty =\downarrow \mathcal{F}_n$). For shortness we don't emphasize this fact.

The following is an other variant of Theorem 6.2.

Theorem 6.3. *Suppose that E^* is separable, E has the RNP, (X_n) is a bounded sequence in $\mathcal{L}^1_{cwk(E)}(\mu)$ such that, for each $A \in \mathcal{F}$, $\cup_n \int_A X_n\, d\mu$ is relatively weakly compact in E, then there exist $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$ and a subsequence (X'_n) of (X_n) such that (X'_n) biting weakly converges to X_∞. Furthermore the following properties hold :*

(1) *There exists a sequence $(\widetilde{X}'_n)$ with $\widetilde{X}'_n \in co\{X'_m : m \geq n\}$ such that $X_\infty = \tau_L\text{-}\lim_n \widetilde{X}'_n$ a.e.*
(2) *If $(v_n)_{n\in\mathbb{N}\cup\{\infty\}}$ is a sequence of scalarly measurable mappings from Ω to $\overline{B}_{E^*}$ such that $|v_n - v_\infty| \to 0$ in measure and that $(\delta^*(v_n, X_n)^-)_n$ is uniformly integrable, then $\liminf_{n\to\infty} \int_\Omega \delta^*(v_n, X'_n)\, d\mu \geq \int_\Omega \delta^*(v_\infty, X_\infty)\, d\mu$.*
(3) *For each measurable multifunction $K : \Omega \to cbc(E)$, $\liminf_n \int_\Omega D(K, X'_n)\, d\mu \geq \int_\Omega D(K, X_\infty)\, d\mu$.*

Proof. Using Theorem 3.2 instead of Theorem 5.1, Step 1 of the second proof of Theorem 6.1 becomes

Step 1′. There exists $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$ satisfying : Given any sequence (Y_n) of (X'_n), there exists a sequence $(\widetilde{Y}_n)$ with $\widetilde{Y}_n \in co\{Y_m : m \geq n\}$ such that

$$X_\infty = \tau_L\text{-}\lim_n \widetilde{Y}_n.$$

Then (1) is consequence of Step 1' whereas (2) follows from the arguments of the proof of Theorem 6.2(2) via Theorem 6.1(b). The proof of (3) is similar to the one given in Theorem 6.2(3) by using Step 1' and the arguments of the proof of Proposition 5.2. □

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