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ASYMPTOTIC BEHAVIOR OF DYNAMICAL SYSTEMS WITH A CONVEX LYAPUNOV FUNCTION

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ABSTRACT. We consider a complete metric space of sequences of mappings acting on a bounded closed convex subset K of a Banach space which share a common convex Lyapunov function f. In a previous paper we introduced the concept of normality and showed that a generic element taken from this space is normal. The sequence of values of the Lyapunov uniformly continuous function f along any (unrestricted) trajectory of such an element tends to the infimum of f on K. In the present paper we first establish a convergence result for perturbations of such trajectories. We then show that if f is Lipschitzian, then the complement of the set of normal sequences is σ-porous.

1. Normality and porosity

Assume that $(X, ||\cdot||)$ is a Banach space with norm $||\cdot||$, $K \subset X$ is a nonempty bounded closed convex subset of X, and $f : K \to R^1$ is a convex uniformly continuous function. Observe that the function f is bounded because K is bounded and f is uniformly continuous. Set

$$\inf(f) = \inf\{f(x) : \ x \in K\} \text{ and } \sup(f) = \sup\{f(x) : \ x \in K\}.$$

We consider the topological subspace $K \subset X$ with the relative topology. Denote by $\mathfrak{A}$ the set of all self-mappings $A : K \to K$ such that

$$f(Ax) \leq f(x) \text{ for all } x \in K \tag{1.1}$$

and by $\mathfrak{A}_c$ the set of all continuous mappings $A \in \mathfrak{A}$. In [12, Section 4] we constructed many mappings belonging to $\mathfrak{A}_c$.

For the set $\mathfrak{A}$ we define a metric $\rho : \mathfrak{A} \times \mathfrak{A} \to R^1$ by

$$\rho(A, B) = \sup\{||Ax - Bx|| : \ x \in K\}, \ A, B \in \mathfrak{A}. \tag{1.2}$$

Clearly the metric space $\mathfrak{A}$ is complete and $\mathfrak{A}_c$ is a closed subset of $\mathfrak{A}$. In the sequel we will study the metric space $(\mathfrak{A}_c, \rho)$. Denote by $\mathfrak{M}$ the set of all sequences $\{A_t\}_{t=1}^\infty \subset \mathfrak{A}$ and by $\mathfrak{M}_c$ the set of all sequences $\{A_t\}_{t=1}^\infty \subset \mathfrak{A}_c$. For the set $\mathfrak{M}$ we define a metric $\rho_{\mathfrak{M}} : \mathfrak{M} \times \mathfrak{M} \to R^1$ by

(1.3)

$$\rho_{\mathfrak{M}}(\{A_t\}_{t=1}^\infty, \{B_t\}_{t=1}^\infty) = \sup\{\rho(A_t, B_t) : \ t = 1, 2, \dots\}, \ \{A_t\}_{t=1}^\infty, \{B_t\}_{t=1}^\infty \in \mathfrak{M}.$$

Clearly the metric space $\mathfrak{M}$ is complete and $\mathfrak{M}_c$ is a closed subset of $\mathfrak{M}$. In the sequel we will also study the metric space $(\mathfrak{M}_c, \rho_{\mathfrak{M}})$.

From the point of view of the theory of dynamical systems each element of $\mathfrak{M}$ describes a nonstationary dynamical system with a Lyapunov function f. Also,

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some optimization procedures in Hilbert and Banach spaces can be represented by elements of $\mathfrak{M}$ (see [9, 10, 12]). For recent studies of the minimization of convex functionals on abstract spaces see, for example, [1], [8] and [13].

In [12], instead of considering a certain convergence property for a single sequence of continuous operators, we investigated it for the space $\mathfrak{M}_c$ of all such sequences, and showed that this property holds for most of them. More precisely, we showed there that for a generic sequence taken from the space $\mathfrak{M}_c$, the sequence of values of the Lyapunov function f along any trajectory tends to the infimum of f.

This approach has already been successfully applied in global analysis and the theory of dynamical systems ([4], [11]), approximation theory [5], as well as in optimization theory and the calculus of variations (see [3], [6], [8], [12], [13], [15] and [16]).

The following definition was given in [7].

A mapping $A \in \mathfrak{A}$ is called normal if given $\epsilon > 0$, there is $\delta(\epsilon) > 0$ such that for each $x \in K$ satisfying $f(x) \geq \inf(f) + \epsilon$, the inequality

$$f(Ax) \leq f(x) - \delta(\epsilon)$$

is true.

A sequence $\{A_t\}_{t=1}^{\infty} \in \mathfrak{M}$ is called normal if given $\epsilon > 0$, there is $\delta(\epsilon) > 0$ such that for each $x \in K$ satisfying $f(x) \geq \inf(f) + \epsilon$ and each integer $t \geq 1$, the inequality

$$f(A_t x) \leq f(x) - \delta(\epsilon)$$

holds.

In [7] we showed that a generic element taken from the spaces $\mathfrak{A}$, $\mathfrak{A}_c$, $\mathfrak{M}$ and $\mathfrak{M}_c$ is normal. This is important because it turns out that the sequence of values of the Lyapunov function f along any (unrestricted) trajectory of such an element tends to the infimum of f on K (see [7, Theorems 1.1 and 1.2]).

In the present paper we will prove two theorems. The first one extends Theorem 1.1 in [7] to perturbed trajectories of a normal sequence. The study of such trajectories is obviously of considerable practical significance [9, 10].

Theorem 1. *Let $\{A_t\}_{t=1}^{\infty} \in \mathfrak{M}$ be normal and let ϵ be positive. Then there exist a natural number n_0 and a number $\gamma > 0$ such that for each integer $n \geq n_0$, each mapping $r : \{1, \dots, n\} \to \{1, 2, \dots\}$ and each sequence $\{x_i\}_{i=0}^{n} \subset K$ which satisfies*

$$||x_{i+1} - A_{r(i+1)} x_i|| \leq \gamma, \; i = 0, \dots, n-1,$$

the inequality $f(x_i) \leq \inf(f) + \epsilon$ holds for $i = n_0, \dots, n$.

Our second result improves upon Theorems 1.3 and 1.4 in [7]. For each of the spaces $\mathfrak{M}$, $\mathfrak{M}_c$, $\mathfrak{A}$ and $\mathfrak{A}_c$ these theorems establish the existence of an everywhere dense G_δ subset such that each one of its elements is normal. In the present paper we will show that if the function f is Lipschitzian, then for each of the spaces mentioned above, the complement of the subset of all normal elements is not only of the first category, but also a σ-porous set.

Before stating our second theorem we recall the concept of porosity [2, 5, 14].

Let (Y, d) be a complete metric space. We denote by $B(y, r)$ the closed ball of center $y \in Y$ and radius $r > 0$. A subset $E \subset Y$ is called porous if there exist

$\alpha \in (0,1)$ and $r_0 > 0$ such that for each $r \in (0, r_0]$ and each $y \in Y$ there exists a point $z \in Y$ for which

$$B(z, \alpha r) \subset B(y, r) \setminus E.$$

A subset of the space Y is called σ-porous if it is a countable union of porous subsets of Y.

Since porous sets are nowhere dense, all σ-porous sets are of the first category. If Y is a finite dimensional Euclidean space, then σ-porous sets are of Lebesgue measure 0. In fact, the class of σ-porous sets in such a space is much smaller than the class of sets which have measure 0 and are of the first category. Also, every Banach space contains a set of the first category which is not σ-porous.

To point out the difference between porous and nowhere dense sets note that if $E \subset Y$ is nowhere dense, $y \in Y$ and $r > 0$, then there is a point $z \in Y$ and a number $s > 0$ such that $B(z, s) \subset B(y, r) \setminus E$. If, however, E is also porous, then for small enough r we can choose $s = \alpha r$, where $\alpha \in (0,1)$ is a constant which depends only on E.

Theorem 2. *Let $\mathcal{F}$ be the set of all normal sequences in the space $\mathfrak{M}$ and let*

$$F = \{A \in \mathfrak{A} : \{A_t\}_{t=1}^{\infty} \in \mathcal{F} \text{ where } A_t = A,\ t = 1, 2, \dots\}.$$

Assume that the function f is Lipschitzian. Then the complement of the set $\mathcal{F}$ is a σ-porous subset of $\mathfrak{M}$ and the complement of the set $\mathcal{F} \cap \mathfrak{M}_c$ is a σ-porous subset of $\mathfrak{M}_c$. Moreover, the complement of the set F is a σ-porous subset of $\mathfrak{A}$ and the complement of the set $F \cap \mathfrak{A}_c$ is a σ-porous subset of $\mathfrak{A}_c$.

2. Proof of Theorem 1

We may assume that $\epsilon < 1$. Since $\{A_t\}_{t=1}^{\infty}$ is normal, there exists a function $\delta : (0, \infty) \to (0, \infty)$ such that for each $s > 0$, each $x \in K$ satisfying $f(x) \geq \inf(f) + s$ and each integer $t \geq 1$,

$$f(A_t x) \leq f(x) - \delta(s). \tag{2.1}$$

We may assume that $\delta(s) < s$, $s \in (0, \infty)$. Choose a natural number

$$n_0 > 4(1 + \sup(f) - \inf(f))\delta(8^{-1}\epsilon)^{-1}. \tag{2.2}$$

Since f is uniformly continuous there exists a number $\gamma > 0$ such that for each $y_1, y_2 \in K$ satisfying $||y_1 - y_2|| \leq \gamma$, the following inequality holds:

$$|f(y_1) - f(y_2)| \leq \delta(8^{-1}\epsilon)8^{-1}(n_0 + 1)^{-1}. \tag{2.3}$$

We claim that the following assertion is true:

(A) Suppose that

$$\{x_i\}_{i=0}^{n_0} \in K,\ r : \{1, \dots, n_0\} \to \{1, 2, \dots\},\ ||x_{i+1} - A_{r(i+1)} x_i|| \leq \gamma, \quad i = 0, \dots, n_0 - 1. \tag{2.4}$$

Then there exists an integer $n_1 \in \{1, \dots, n_0\}$ such that

$$f(x_{n_1}) \leq \inf(f) + \epsilon/8. \tag{2.5}$$

Assume the contrary. Then

$$f(x_i) > \inf(f) + \epsilon/8,\ i = 1, \dots, n_0. \tag{2.6}$$

By (2.6) and the definition of $\delta : (0, \infty) \to (0, \infty)$ (see (2.1)), we have, for each $i = 1, \ldots, n_0 - 1$,

$$f(A_{r(i+1)} x_i) \le f(x_i) - \delta(8^{-1}\epsilon). \tag{2.7}$$

It follows from (2.4) and the definition of γ (see (2.3)) that for $i = 1, \ldots, n_0 - 1$,

$$|f(x_{i+1}) - f(A_{r(i+1)} x_i)| \le \delta(8^{-1}\epsilon) 8^{-1} (n_0 + 1)^{-1}.$$

When combined with (2.7) this inequality implies that for $i = 1, \ldots, n_0 - 1$,

$$f(x_{i+1}) - f(x_i) \le f(x_{i+1}) - f(A_{r(i+1)} x_i) + f(A_{r(i+1)} x_i)$$

$$-f(x_i) \le \delta(8^{-1}\epsilon) 8^{-1} (n_0 + 1)^{-1} - \delta(8^{-1}\epsilon) \le (-1/2)\delta(8^{-1}\epsilon).$$

This, in turn, implies that

$$\inf(f) - \sup(f) \le f(x_{n_0}) - f(x_1) \le (n_0 - 1)(-1/2)\delta(8^{-1}\epsilon),$$

a contradiction (see (2.2)). Thus there exists an integer $n_1 \in \{1, \ldots, n_0\}$ such that (2.5) is true. Therefore assertion (A) is valid, as claimed.

Assume now that we are given an integer $n \ge n_0$, a mapping

$$r : \{1, \ldots, n\} \to \{1, 2, \ldots\} \tag{2.8}$$

and a finite sequence

$$\{x_i\}_{i=0}^n \subset K \text{ such that } ||x_{i+1} - A_{r(i+1)} x_i|| \le \gamma, \ i = 0, \ldots, n-1. \tag{2.9}$$

It follows from assertion (A) that there exists a finite sequence of natural numbers $\{j_p\}_{p=1}^q$ such that

$$1 \le j_1 \le n_0, \ 1 \le j_{p+1} - j_p \le n_0 \text{ if } 1 \le p \le q - 1, \tag{2.10}$$

$$n - j_q < n_0, \ f(x_{j_p}) \le \inf(f) + \epsilon/8, \ p = 1, \ldots, q.$$

Let $i \in \{n_0, \ldots, n\}$. We will show that $f(x_i) \le \inf(f) + \epsilon/2$. There exists $p \in \{1, \ldots, q\}$ such that

$$0 \le i - j_p \le n_0.$$

If $i = j_p$, then by (2.10), $f(x_i) = f(x_{j_p}) \le \inf(f) + \epsilon/8$. Thus we may assume that $i > j_p$. For all integers $j_p \le s < i$, it follows from (1.1), (2.9) and the definition of γ (see (2.3)) that

$$f(A_{r(s+1)} x_s) \le f(x_s),$$

$$|f(x_{s+1}) - f(A_{r(s+1)} x_s)| \le \delta(8^{-1}\epsilon) 8^{-1} (n_0 + 1)^{-1}$$

and

$$f(x_{s+1}) \le f(A_{r(s+1)} x_s) + \delta(8^{-1}\epsilon) 8^{-1} (n_0 + 1)^{-1} \le f(x_s) + \delta(8^{-1}\epsilon) 8^{-1} (n_0 + 1)^{-1}.$$

Thus

$$f(x_{s+1}) - f(x_s) \le \delta(8^{-1}\epsilon) 8^{-1} (n_0 + 1)^{-1}, \ j_p \le s < i.$$

This implies that

$$f(x_i) \le f(x_{j_p}) + \delta(8^{-1}\epsilon) 8^{-1} (n_0 + 1)^{-1} (n_0 + 1) \le$$

$$\inf(f) + \epsilon/8 + 8^{-1}\delta(8^{-1}\epsilon) \le \inf(f) + \epsilon/2.$$

Therefore $f(x_i) \le \inf(f) + \epsilon/2$ for all integers $i \in [n_0, n]$ and Theorem 1 is proved.

3. Proof of Theorem 2

Since $f : K \to R^1$ is assumed to be Lipschitzian, there exists a constant $L(f) > 0$ such that

$$|f(x) - f(y)| \leq L(f)||x - y|| \text{ for all } x, y \in K. \tag{3.1}$$

By Proposition 2.1 in [7] there exist a normal continuous mapping $A_* : K \to K$ and a function $\phi : (0, \infty) \to (0, \infty)$ such that for each $\epsilon > 0$ and each $x \in K$ satisfying $f(x) \geq \inf(f) + \epsilon$, the inequality $f(A_* x) \leq f(x) - \phi(\epsilon)$ holds.

Let $\epsilon > 0$ be given. We will say that a sequence $\{A_t\}_{t=1}^{\infty} \in \mathfrak{M}$ is (ϵ)-quasinormal if there exists $\delta > 0$ such that if $x \in K$ satisfies $f(x) \geq \inf(f) + \epsilon$, then $f(A_t x) \leq f(x) - \delta$ for all integers $t \geq 1$.

Recall that $\mathcal{F}$ is defined to be the set of all normal sequences in $\mathfrak{M}$. For each integer $n \geq 1$ denote by $\mathcal{F}_n$ the set of all (n^{-1})-quasinormal sequences in $\mathfrak{M}$. Clearly

$$\mathcal{F} = \cap_{n=1}^{\infty} \mathcal{F}_n. \tag{3.2}$$

Set

$$d(K) = \sup\{||z|| : \ z \in K\} \tag{3.3}$$

and let $n \geq 1$ be an integer. Choose $\alpha \in (0, 1)$ such that

$$2L(f)\alpha < (1 - \alpha)\phi(n^{-1})8^{-1}(d(K) + 1)^{-1}. \tag{3.4}$$

Assume that $0 < r \leq 1$ and that $\{A_t\}_{t=1}^{\infty} \in \mathfrak{M}$. Set

$$\gamma = (1 - \alpha)r8^{-1}(d(K) + 1)^{-1} \tag{3.5}$$

and define for all integers $t \geq 1$ the mapping $A_{t\gamma} : K \to K$ by

$$A_{t\gamma} x = (1 - \gamma)A_t x + \gamma A_* x, \ x \in K. \tag{3.6}$$

Clearly $\{A_{t\gamma}\}_{t=1}^{\infty} \in \mathfrak{M}$ and

$$\rho_{\mathfrak{M}}(\{A_t\}_{t=1}^{\infty}, \{A_{t\gamma}\}_{t=1}^{\infty}) \leq 2\gamma \sup\{||z|| : z \in K\} = 2\gamma d(K). \tag{3.7}$$

Note that $\{A_{t\gamma}\}_{t=1}^{\infty} \in \mathfrak{M}_c$ if $\{A_t\}_{t=1}^{\infty} \in \mathfrak{M}_c$ and that $A_{t\gamma} = A_{1\gamma}$, $t = 1, 2, \ldots$, if $A_t = A_1$, $t = 1, 2, \ldots$.

Assume that

$$\{C_t\}_{t=1}^{\infty} \in \mathfrak{M} \text{ and } \rho_{\mathfrak{M}}(\{A_{t\gamma}\}_{t=1}^{\infty}, \{C_t\}_{t=1}^{\infty}) \leq \alpha r. \tag{3.8}$$

Then by (3.8), (3.7) and (3.5),

$$\begin{aligned} \rho_{\mathfrak{M}}(\{A_t\}_{t=1}^{\infty}, \{C_t\}_{t=1}^{\infty}) &\leq \alpha r + 2\gamma d(K) \leq \alpha r + (1 - \alpha)r/2 \\ &= r(1 + \alpha)/2 < r. \end{aligned} \tag{3.9}$$

Assume that $x \in K$ satisfies

$$f(x) \geq \inf(f) + n^{-1} \tag{3.10}$$

and that $t \geq 1$ is an integer. By (3.10), the properties of A_* and ϕ, (3.6) and (1.1),

$$\begin{aligned} f(A_* x) \leq f(x) - \phi(n^{-1}), \ f(A_{t\gamma} x) &\leq (1 - \gamma)f(A_t x) + \gamma f(A_* x) \leq \\ (1 - \gamma)f(x) + \gamma(f(x) - \phi(n^{-1})) &= f(x) - \gamma\phi(n^{-1}). \end{aligned} \tag{3.11}$$

By (3.8), $||C_t x - A_{t\gamma} x|| \leq \alpha r$. Together with (3.1) this inequality yields

$$|f(C_t x) - f(A_{t\gamma} x)| \leq L(f)\alpha r.$$

By the latter inequality, (3.11), (3.5) and (3.4),

$$f(C_t x) \le f(A_{t\gamma} x) + L(f)\alpha r \le$$
$$L(f)\alpha r + f(x) - \gamma\phi(n^{-1}) \le$$
$$f(x) - \phi(n^{-1})(1-\alpha)r8^{-1}(d(K)+1)^{-1} + L(f)\alpha r \le$$
$$f(x) - L(f)\alpha r.$$

Thus for each $\{C_t\}_{t=1}^{\infty} \in \mathfrak{M}$ satisfying (3.8), the inequalities (3.9) hold and $\{C_t\}_{t=1}^{\infty} \in \mathcal{F}_n$. We have shown that for each integer $n \ge 1$, $\mathfrak{M} \setminus \mathcal{F}_n$ is porous in $\mathfrak{M}$, $\mathfrak{M}_c \setminus \mathcal{F}_n$ is porous in $\mathfrak{M}_c$, the complement of the set

$$\{A \in \mathfrak{A} : \ \{A_t\}_{t=1}^{\infty} \in \mathcal{F}_n \text{ with } A_t = A \text{ for all integers } t \ge 1\}$$

is porous in $\mathfrak{A}$ and the complement of the set

$$\{A \in \mathfrak{A}_c : \ \{A_t\}_{t=1}^{\infty} \in \mathcal{F}_n \text{ with } A_t = A \text{ for all integers } t \ge 1\}$$

is porous in $\mathfrak{A}_c$.

Combining these facts with (3.2) we conclude that $\mathfrak{M} \setminus \mathcal{F}$ is σ-porous in $\mathfrak{M}$, $\mathfrak{M}_c \setminus \mathcal{F}$ is σ-porous in $\mathfrak{M}_c$, $\mathfrak{A} \setminus F$ is σ-porous in $\mathfrak{A}$ and $\mathfrak{A}_c \setminus F$ is σ-porous in $\mathfrak{A}_c$. This completes the proof of Theorem 2.

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