Preface

This book has been written as a part of my program of teaching nonlinear and convex analysis. Nonlinear and convex analysis is an area of mathematics which has suddenly grown up over the past few decades, influenced by nonlinear problems posed in physics, mechanics, operations research and economics. It is no longer a subsidiary of linear functional analysis. Its applicable field is far and away wider than that of the linear case because most of problems arising in natural sciences or social sciences are nonlinear.

The main purpose of this book is to present the theory of nonlinear and convex analysis in a Hilbert space with nonlinear mappings, fixed point theorems and convex functions as essential ingredients and then to give applications of the theory to proximal point algorithms, variational inequalities and minimax problems. With this in mind we have divided the book into eight chapters.

In Chapter 1, we state elementary results related to real numbers. In particular, we deal with elementary properties of superior limits and inferior limits. To prove many results in the book, we shall very often use superior limits and inferior limits. In Chapter 2, we study metric spaces. Analysis is primarily concerned with limit processes and continuity. Our purpose in this chapter is to develop in a systematic manner the main elementary facts about metric spaces. In Chapter 3, we deal with finite dimensional spaces and function spaces. Before studying Banach spaces and Hilbert spaces in Chapters 4 and 5, we study Euclidean spaces, unitary spaces and function spaces. In Chapter 4, we study elementary facts about Banach spaces. These facts are important for their own sake, and also for the sake of the motivation they provide for our later work on Hilbert spaces. In Chapter 5, we deal with Hilbert spaces. Our purpose in this chapter is to present enough of the elementary theory of Hilbert spaces and their nonlinear operators to provide an adequate foundation for the deeper theory discussed in these later chapters. The definition of a Hilbert space in this chapter is interesting. The definition is different from the usual definition of a Hilbert space. Our definition will be useful to pure and applied mathematicians who are not familiar with Hilbert spaces. In Chapter 6, we study nonlinear mappings which are connected with convex analysis. The nonlinear mappings are nonexpansive mappings and nonlinear monotone mappings. We first prove a fixed point theorem for nonexpansive mappings in a Hilbert space. Further, we discuss iterative methods for approximation of fixed points of nonexpansive mappings. Finally, we study fundamental properties of nonlinear monotone mappings in a Hilbert space. This chapter is the most interesting and understandable part in the book. In Chapter 7, we study fundamental results in convex analysis. Convex analysis is an area of mathematics which has suddenly grow up over the past few decades, influenced by convex problems in operations research and economics. Subdifferentials of convex functions are very useful and have interesting properties. Finally, in Chapter 8, we discuss some applications. We first deal with proximal point algorithms which are due to Rockafellar, and Kamimura and Takahashi. That is, we prove strong and weak convergence theorems for resolvents of maximal monotone operators in a Hilbert space. Using one of convergence theorems, we further consider the problem of finding a saddle point of a two variable function.

The primary aim of the book is to assist students in learning fundamental ideas and theorems about nonlinear and convex analysis. While the book is principally addressed to students, it is also intended to be useful to mathematicians, both pure and applied, who need a simple and direct presentation of the fundamentals of the theory of nonlinear and convex analysis.

Finally, I appreciare the very skillful assistance of Professors H. Iiduka, F. Kohsaka, and S. Matsushita in the course of preparation of the book. Also my thanks go to Dr. S. Iemoto and Mr. S. Takahashi for careful reading the manuscript.

I dedicate this book to my wife Yoko who provided environment for me in which devotion to mathematical work was possible.

Wataru Takahashi
March, 2009

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