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# APPROXIMATION OF FIXED POINTS IN COMPLETE GEODESIC SPACES AND COEFFICIENT CONDITIONS

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Abstract. In this paper, we show a strong convergence theorem for the Halpern iteration procedure in a complete CAT(1) space with two quasinonexpansive $\Delta$-demiclosed mappings and one nonexpansive mapping. We consider the assumptions for the coefficient of convex combination in detail.

## 1. Introduction

Halpern type method is a technique to approximate a fixed point of a nonlinear mapping and has been studied by many mathematicians in various spaces. In 1992, Halpern type iteration with a nonexpansive mapping was obtained by Wittmann [12] in a Hilbert space. In 2010, Saejung [9] proved a convergence theorem in a complete CAT(0) space. In 2013, Kimura and Satô [8] proved a convergence theorem with a single mapping and a single anchor point in a complete CAT(1) space.

In 2016 Wada [11] proved a strong convergence theorem for two kinds of mappings in a complete CAT(0) space:

**Theorem 1.1.** *Let $X$ be a Hadamard space. Assume that $R\colon X \to X$ is a nonexpansive mapping, $S, T\colon X \to X$ are quasinonexpansive and $\Delta$-demiclosed mappings, and $F = F(R) \cap F(S) \cap F(T) \neq \emptyset$. Let $u \in X$. We define an iterative scheme $\{x_n\}$ by*

$$\begin{cases} s_n = \gamma_n u \oplus (1-\gamma_n) S x_n \\ t_n = \gamma_n u \oplus (1-\gamma_n) T x_n \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) R(\beta_n s_n \oplus (1-\beta_n) t_n) \end{cases}$$

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*for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $]0,1[$ such that*

$$\alpha_n \to 0, \sum_{n=1}^{\infty} \alpha_n = \infty,$$

*and $\{\beta_n\}, \{\gamma_n\}$ belong to $[a,b] \subset ]0,1[$. Then, $\{x_n\}$ converges to the nearest point of $F$ to $u$.*

Further, Kimura *et al.* [6] proved this result in the setting of complete CAT(1) spaces:

**Theorem 1.2.** *Let $X$ be a complete* CAT(1) *space with $M = \operatorname{diam} X < \pi/2$. Let $R\colon X \to X$ be a nonexpansive mapping and $S, T\colon X \to X$ strongly quasinonexpansive and $\Delta$-demiclosed mappings from $X$ into itself $F = F(R) \cap F(S) \cap F(T) \neq \emptyset$. For $r \in ]0, 1/2[$, let $\{\alpha_n\} \subset [r, 1-r] \subset ]0,1[, \{\beta_n\}, \{\gamma_n\} \subset ]0,1[$ be real sequences satisfying $\beta_n \to \beta \in ]0,1[, \gamma_n \to 0$, and $\sum_{n=1}^{\infty} \gamma_n = \infty$. Define $\{x_n\} \subset X$ by $x_1, u \in X$ and*

$$\begin{cases} s_n = \gamma_n u \oplus (1-\gamma_n) S x_n \\ t_n = \gamma_n u \oplus (1-\gamma_n) T x_n \\ y_n = \beta_n s_n \oplus (1-\beta_n) t_n \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) R y_n \end{cases}$$

*for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges to $P_F u \in F$.*

In this paper, we consider the case that the coefficient $\{\beta_n\}$ in the Halpern type iterative scheme for three mappings in Theorem 1.2 converges to 0 or 1, and we prove that $\{x_n\}$ converges to $P_{F(T)\cap F(R)} u$ if $\beta_n \to 0$ and converges to $P_{F(S)\cap F(R)} u$ if $\beta_n \to 1$.

## 2. Preliminaries

Let $X$ be a geodesic metric space. For $x, y \in X$, a mapping $\gamma_{x,y} : [0, l] \to X$ is called a geodesic with endpoints $x, y$ if $\gamma_{x,y}$ satisfies $\gamma_{x,y}(0) = x, \gamma_{x,y}(l) = y$ and $d(\gamma_{x,y}(s), \gamma_{x,y}(t)) = |s - t|$ for all $s, t \in [0, l]$. The image $[x, y]$ of $\gamma_{x,y}$ is called a geodesic segment joining $x$ and $y$. If a geodesic with endpoints $x, y$ exists for any $x, y \in X$, then we call $X$ a geodesic metric space. Moreover, if a geodesic exists uniquely for each $x, y \in X$, then we call $X$ a uniquely geodesic space.

Let $X$ be a uniquely geodesic metric space such that $d(v, v') < \pi/2$ for all $v, v' \in X$. A geodesic triangle is defined by $\triangle(x, y, z) = [x, y] \cup [y, z] \cup [z, x]$. Let $\mathbb{S}^2$ be the two-dimensional unit sphere in $\mathbb{R}^3$. For $\bar{x}, \bar{y}, \bar{z} \in \mathbb{S}^2$, a triangle $\overline{\triangle}(\bar{x}, \bar{y}, \bar{z})$ in $\mathbb{S}^2$ is called a comparison triangle for $\triangle(x, y, z)$ if $d_{\mathbb{S}^2}(\bar{x}, \bar{y}) = d(x, y), d_{\mathbb{S}^2}(\bar{y}, \bar{z}) = d(y, z), d_{\mathbb{S}^2}(\bar{z}, \bar{x}) = d(z, x)$. A point $\bar{p} \in [\bar{y}, \bar{z}]$ is called a comparison point for $p \in [y, z]$ if $d(y, p) = d(\bar{y}, \bar{p})$. If, for any $p, q \in \triangle(x, y, z)$ and their comparison points $\bar{p}, \bar{q} \in \overline{\triangle}(\bar{x}, \bar{y}, \bar{z})$, the inequality $d(p, q) \leq d_{\mathbb{S}^2}(\bar{p}, \bar{q})$ is satisfied for all triangles in $X$, then $X$ is called a CAT(1) space.

Let $X$ be a geodesic metric space and $\{x_n\} \subset X$ a bounded sequence. For $x \in X$, we put $r(x, \{x_n\}) = \overline{\lim}_{n\to\infty} d(x, x_n)$ and $r(\{x_n\}) = \inf_{x\in X} r(x, \{x_n\})$. If there exists $x \in X$ such that $r(x, \{x_n\}) = r(\{x_n\})$, we call $x$ an asymptotic center of $\{x_n\}$. The set of all asymptotic centers of $\{x_n\}$ is denoted by $\mathrm{AC}\{x_n\}$. If $x_0$ is a unique asymptotic center of all subsequences of $\{x_n\}$, then we say that $\{x_n\}$ is $\Delta$-converges to $x_0$ which is denoted by $x_n \xrightarrow{\Delta} x_0$. Let $X$ be a CAT(1) space and $T$ a mapping from $X$ into itself. If $x_n \xrightarrow{\Delta} x_0$ and $\lim_{n\to\infty} d(Tx_n, x_n) = 0$ imply $x_0 \in F(T)$, then $T$ is said to be $\Delta$-demiclosed.

Let $X$ be a metric space. A mapping $T\colon X \to X$ is nonexpansive if the inequality $d(Tx, Ty) \leq d(x, y)$ is satisfied for all $x, y \in X$. The set of fixed points of $T$ is denoted by $F(T) = \{z \in X : Tz = z\}$. Further, a mapping $T\colon X \to X$ with $F(T) \neq \emptyset$ is said to be stongly quasinonexpansive if, for any $x \in X$ and $z \in F(T)$, $d(Tx, z) \leq d(x, z)$ and if, for any bounded sequence $\{x_n\} \subset X$ and $z \in F(T)$, $\lim_{n\to\infty} \cos d(x_n, z)/\cos d(Tx_n, z) = 1$, we have $d(x_n, Tx_n) \to 0$.

## 3. Tools for the main result

In this section, we introduce some tools for the main theorem.

**Lemma 3.1** ([1], [10])**.** *Let $\{\alpha_n\} \subset [0, \infty[, \{d_n\} \subset \mathbb{R}$ and $\{\gamma_n\} \subset ]0, 1[$ such that $\sum_{n=1}^{\infty} \gamma_n = \infty$. Define a set $\Phi = \{\varphi : \mathbb{N} \to \mathbb{N}$, nondecreasing and $\lim_{i\to\infty} \varphi(i) = \infty\}$. Suppose that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n d_n$$

*for any $n \in \mathbb{N}$. If $\overline{\lim}_{i\to\infty} d_{\varphi(i)} \leq 0$ for any $\varphi \in \Phi$ satisfying $\underline{\lim}_{i\to\infty}(a_{\varphi(i+1)} - a_{\varphi(i)}) \geq 0$, then $\lim_{n\to\infty} a_n = 0$.*

**Lemma 3.2** (Kimura and Satô [8])**.** *Let $X$ be a complete CAT(1) space such that $d(v, v') < \pi/2$ for every $v, v' \in X$. Let $\alpha \in [0, 1]$ and $u, y, z \in X$. Then*

$$1 - \cos d(\alpha u \oplus (1 - \alpha)y, z)$$
$$\leq (1 - \beta)(1 - \cos d(y, z)) + \beta \left(1 - \frac{\cos d(u, z)}{\sin d(u, y) \tan((\frac{\alpha}{2})d(u, y)) + \cos d(u, y)}\right),$$

*where*

$$\beta = \begin{cases} 1 - \dfrac{\sin((1 - \alpha)d(u, y))}{\sin d(u, y)} & (u \neq y), \\ \alpha & (u = y). \end{cases}$$

**Lemma 3.3** (Kimura, Nakagawa and Wada [6]). *Let* $\{s_n\}$ *and* $\{t_n\} \subset ]-\infty, 0]$. *Suppose that* $\lim_{n\to\infty}(s_n + t_n) = 0$. *Then* $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = 0$.

**Lemma 3.4** (He, Fang, Lopez, and Li [4]). *Let* $X$ *be a complete* CAT(1) *space and* $u \in X$. *If a sequence* $\{x_n\}$ *in* $X$ *satisfies that* $\overline{\lim}_{n\to\infty} d(u, x_n) < \pi/2$ *and* $x_n \xrightarrow{\Delta} x \in X$, *then*

$$\underline{\lim}_{n\to\infty} d(u, x_n) \geq d(u, x).$$

**Lemma 3.5** (Hirono and Kimura [5]). *Let* $X$ *be a complete* CAT(1) *space such that* $M = \operatorname{diam} X < \pi/2$. *Let* $\{d_n\} \subset [0, \pi/2[$, $\{\alpha_n\} \subset ]0, 1[$ *and* $\{\beta_n\} \subset ]0, 1[$ *satisfying* $\sum_{n=1}^{\infty} \beta_n = \infty$. *Then*

$$\sum_{n=1}^{\infty}(\alpha_n\sigma_n + (1 - \alpha_n)\tau_n) = \infty,$$

*where*

$$\sigma_n = \begin{cases} 1 - \dfrac{\sin(1 - \beta_n)d_n}{\sin d_n} & (d_n \neq 0), \\ \beta_n & (d_n = 0), \end{cases}$$

$$\tau_n = \begin{cases} 1 - \dfrac{\sin(1 - \beta_n)d'_n}{\sin d'_n} & (d'_n \neq 0), \\ \beta_n & (d'_n = 0). \end{cases}$$

*Proof.* Let $n \in \mathbb{N}$. If $d_n = 0$, then

$$\sigma_n = \beta_n \geq \beta_n \cos M.$$

If $d_n \neq 0$, then

$$\begin{aligned} \sigma_n &= \frac{\sin d_n - \sin(1 - \beta_n)d_n}{\sin d_n} \\ &= \frac{2}{\sin d_n} \cos\left(\left(1 - \frac{\beta_n}{2}\right) d_n\right) \sin\frac{\beta_n}{2} d_n \\ &\geq \frac{2}{\sin d_n} \cdot \frac{\beta_n}{2} \cos\left(\left(1 - \frac{\beta_n}{2}\right) d_n\right) \sin d_n \\ &= \beta_n \cos\left(1 - \frac{\beta_n}{2}\right) d_n \\ &\geq \beta_n \cos d_n \\ &\geq \beta_n \cos M. \end{aligned}$$

Similarly, we also have $\tau_n \geq \beta_n \cos M$ for any $n \in \mathbb{N}$. Hence we get

$$\alpha_n \sigma_n + (1-\alpha_n)\tau_n \geq \alpha_n \beta_n \cos M + (1-\alpha_n)\beta_n \cos M = \beta_n \cos M$$

for any $n \in \mathbb{N}$. Thus, we obtain $\sum_{n=1}^{\infty}(\alpha_n \sigma_n + (1-\alpha_n)\tau_n) = \infty$. □

**Lemma 3.6** (Kimura, Nakagawa and Wada [6]). *Let $\{s_n\}$ and $\{t_n\} \subset [0,\infty[$. then*

$$\limsup_{n\to\infty}(s_n \cdot t_n) \geq \overline{\lim_{n\to\infty}}\, s_n \cdot \underline{\lim_{n\to\infty}}\, t_n.$$

**Lemma 3.7** (Kimura, Nakagawa and Wada [6]). *Let $\{d_n\} \subset [0,\pi/2[$ and $t \in ]0,1[$. Suppose*

$$\lim_{n\to\infty} \frac{\sin(td_n) + \sin((1-t)d_n)}{\sin d_n} = 1.$$

**Corollary 3.8** (Kimura and Satô [8]). *Let $\triangle(x,y,z)$ be a geodesic triangle in a* CAT(1) *space such that $d(x,y)+d(y,z)+d(z,x) < 2\pi$. Let $u = tx \oplus (1-t)y$ for some $t \in [0,1]$. Then*

$$\cos d(u,z) \geq t \cos d(x,z) + (1-t)\cos d(y,z).$$

**Theorem 3.9** (Kimura and Satô [7]). *Let $\triangle(x,y,z)$ be a geodesic triangle in a CAT(1) space such that $d(x,y)+d(y,z)+d(z,x) < 2\pi$. Let $u = tx \oplus (1-t)y$ for some $t \in [0,1]$. Then*

$$\cos d(u,z)\sin d(x,y) \geq \cos d(x,z)\sin td(x,y) + \cos d(y,z)\sin(1-t)d(x,y).$$

**Theorem 3.10** (Espínola and Fernández-León [3]). *Let $X$ be a complete CAT(1) space and $\{x_n\}$ a sequence in $X$. If $r(\{x_n\}) < \pi/2$, then the following hold.*

(i) AC($\{x_n\}$) *consists of exactly one point;*
(ii) *$\{x_n\}$ has a $\Delta$-convergent subsequence.*

## 4. Main result

**Theorem 4.1.** *Let $X$ be a complete* CAT(1) *space with $M = \operatorname{diam} X < \pi/2$. Let $R\colon X \to X$ be a nonexpansive mapping and $S,T\colon X \to X$ strongly quasinonexpansive and $\Delta$-demiclosed mappings. For $r \in ]0,1/2[$, let $\{\alpha_n\} \subset [r,1-r] \subset ]0,1[$, $\{\beta_n\},\{\gamma_n\} \subset ]0,1[$ be real sequences satisfying $\beta_n \to 0$ or 1, $\gamma \to 0$ and $\sum_{n=1}^{\infty}\gamma_n = \infty$. Define $\{x_n\} \subset X$ by $x_1 = u \in X$ and*

$$\begin{cases} s_n = \gamma_n u \oplus (1-\gamma_n)Sx_n \\ t_n = \gamma_n u \oplus (1-\gamma_n)Tx_n \\ y_n = \beta_n s_n \oplus (1-\beta_n)t_n \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)Ry_n \end{cases}$$

*for all $n \in \mathbb{N}$. Then*

- *if $F(T) \cap F(R) \neq \emptyset, F(S) \neq \emptyset$, and $\beta_n \to 0$ then $x_n \to P_{F(T)\cap F(R)}u$;*
- *if $F(S) \cap F(R) \neq \emptyset, F(T) \neq \emptyset$, and $\beta_n \to 1$ then $x_n \to P_{F(S)\cap F(R)}u$.*

*Proof.* We consider the case that $F(T) \cap F(R) \neq \emptyset, F(S) \neq \emptyset$, and $\beta_n \to 0$. Put $p = P_{F(T)\cap F(R)}u$,

$$\begin{aligned}
a_n &= 1 - \cos d(x_n, p),\\
b_n &= 1 - \frac{\cos d(u,p)}{\sin d(u, Sx_n)\tan(\frac{\gamma_n}{2} d(u, Sx_n)) + \cos d(u, Sx_n)},\\
c_n &= 1 - \frac{\cos d(u,p)}{\sin d(u, Tx_n)\tan(\frac{\gamma_n}{2} d(u, Tx_n)) + \cos d(u, Tx_n)},\\
\sigma_n &= \begin{cases} 1 - \dfrac{\sin(1-\gamma_n)d(u, Sx_n)}{\sin d(u, Sx_n)} & (u \neq Sx_n),\\ \gamma_n & (u = Sx_n),\end{cases}\\
\tau_n &= \begin{cases} 1 - \dfrac{\sin(1-\gamma_n)d(u, Tx_n)}{\sin d(u, Tx_n)} & (u \neq Tx_n),\\ \gamma_n & (u = Tx_n).\end{cases}
\end{aligned}$$

for $n \in \mathbb{N}$. Since $\beta_n\sigma_n + (1-\beta_n)\tau_n > 0$ for any $n \in \mathbb{N}$, by Lemma 3.2 and Corollary 3.8, we have the following inequality:

$$\begin{aligned}
a_{n+1} &= 1 - \cos d(\alpha_n x_n \oplus (1-\alpha_n)Ry_n, p)\\
&\leq 1 - (\alpha_n \cos d(x_n, p) + (1-\alpha_n)\cos d(Ry_n, p))\\
&\leq \alpha_n a_n + (1-\alpha_n)(1 - \cos d(y_n, p))\\
&\leq \alpha_n a_n + (1-\alpha_n)(1 - (\beta_n \cos d(s_n, p) + (1-\beta_n)\cos d(t_n, p)))\\
&\leq \alpha_n a_n + (1-\alpha_n)(\beta_n((1-\sigma_n)a_n + \sigma_n b_n) + (1-\beta_n)\cos d(t_n, p))\\
&= (\alpha_n + (1-\alpha_n)(\beta_n(1-\sigma_n) + (1-\beta_n)(1-\tau_n)))a_n\\
&\quad + (1-\alpha_n)(\beta_n\sigma_n b_n + (1-\beta_n)\tau_n c_n)\\
&= (1 - (1-\alpha_n)(\beta_n\sigma_n + (1-\beta_n)\tau_n))a_n\\
&\quad + (1-\alpha_n)(\beta_n\sigma_n + (1-\beta_n)\tau_n)\frac{\beta_n\sigma_n b_n + (1-\beta_n)\tau_n c_n}{\beta_n\sigma_n + (1-\beta_n)\tau_n},
\end{aligned}$$

Note that $\alpha_n, 1-\alpha_n \leq 1-r$. To apply Lemma 3.1, we will show the following:

(i) $\displaystyle\sum_{n=1}^{\infty}(\beta_n\sigma_n + (1-\beta_n)\tau_n) = \infty$,

(ii) $\overline{\lim}_{i\to\infty}\left(\dfrac{\beta_{\varphi(i)}\sigma_{\varphi(i)}b_{\varphi(i)}+(1-\beta_{\varphi(i)})\tau_{\varphi(i)}c_{\varphi(i)}}{\beta_{\varphi(i)}\sigma_{\varphi(i)}+(1-\beta_{\varphi(i)})\tau_{\varphi(i)}}\right)\le 0$
for any nondecreasing functions $\varphi:\mathbb{N}\to\mathbb{N}$ satisfying $\lim_{i\to\infty}\varphi(i)=\infty$ and $\underline{\lim}_{i\to\infty}(a_{\varphi(i)+1}-a_{\varphi(i)})\ge 0$.

For (i), we obtain $\sum_{n=1}^{\infty}(\beta_n\sigma_n+(1-\beta_n)\tau_n)=\infty$ by Lemma 3.5.

Next we consider (ii). Let $n_i=\varphi(i)$ for any $i\in\mathbb{N}$. Then

$$
\begin{aligned}
0 &\le \varliminf_{i\to\infty}(a_{n_i+1}-a_{n_i})\\
&= \varliminf_{i\to\infty}(\cos d(x_{n_i},p)-\cos d(x_{n_{i+1}},p))\\
&= \varliminf_{i\to\infty}(\cos d(x_{n_i},p)-\cos d(\alpha_{n_i}x_{n_i}\oplus(1-\alpha_{n_i})Ry_{n_i},p))\\
&\le \varliminf_{i\to\infty}(\cos d(x_{n_i},p)-(\alpha_{n_i}\cos d(x_{n_i},p)+(1-\alpha_{n_i})\cos d(Ry_{n_i},p)))\\
&= \varliminf_{i\to\infty}(1-\alpha_{n_i})(\cos d(x_{n_i},p)-\cos d(Ry_{n_i},p))\\
&\le \varliminf_{i\to\infty}(1-\alpha_{n_i})(\cos d(x_{n_i},p)-\cos d(y_{n_i},p))\\
&= \varliminf_{i\to\infty}(1-\alpha_{n_i})(\cos d(x_{n_i},p)-\cos d(\beta_{n_i}\cos d(s_{n_i},p)+(1-\beta_{n_i})\cos d(t_{n_i},p)))\\
&\le \varliminf_{i\to\infty}(1-\alpha_{n_i})(\cos d(x_{n_i},p)-(\beta_{n_i}\cos d(s_{n_i},p)+(1-\beta_{n_i})\cos d(t_{n_i},p)))\\
&= \varliminf_{i\to\infty}(1-\alpha_{n_i})(\beta_{n_i}(\cos d(x_{n_i},p)-\cos d(s_{n_i},p))\\
&\quad +(1-\beta_{n_i})(\cos d(x_{n_i},p)-\cos d(t_{n_i},p)))\\
&\le \varliminf_{i\to\infty}(1-\alpha_{n_i})(\beta_{n_i}(\cos d(x_{n_i},p)-(\gamma_{n_i}\cos d(u,p)+(1-\gamma_{n_i})\cos d(Sx_{n_i},p)))\\
&\quad +(1-\beta_{n_i})(\cos d(x_{n_i},p)-(\gamma_{n_i}\cos d(u,p)+(1-\gamma_{n_i})\cos d(Tx_{n_i},p))))\\
&= \varliminf_{i\to\infty}(1-\alpha_{n_i})(\cos d(x_{n_i},p)-\cos d(Tx_{n_i},p))\\
&\le \varliminf_{i\to\infty}(1-r)(\cos d(x_{n_i},p)-\cos d(Tx_{n_i},p))\\
&\le \varlimsup_{i\to\infty}(1-r)(\cos d(x_{n_i},p)-\cos d(Tx_{n_i},p))\\
&\le 0.
\end{aligned}
$$

Therefore, we get

$$\lim_{i\to\infty}(\cos d(x_{n_i},p)-\cos d(Tx_{n_i},p))=0.$$

Further, we have

$$\begin{aligned}\left|1-\frac{\cos d(x_{n_i},p)}{\cos d(Tx_{n_i},p)}\right| &= \left|\frac{\cos d(Tx_{n_i},p)-\cos d(x_{n_i},p)}{\cos d(Tx_{n_i},p)}\right| \\ &= \frac{1}{\cos d(Tx_{n_i},p)}\left|\cos d(Tx_{n_i},p)-\cos d(x_{n_i},p)\right| \\ &\leq \frac{1}{\cos M}\left|\cos d(Tx_{n_i},p)-\cos d(x_{n_i},p)\right| \\ &\to 0\ (i\to\infty).\end{aligned}$$

Therefore, we get

$$\lim_{i\to\infty}\frac{\cos d(x_{n_i},p)}{\cos d(Tx_{n_i},p)}=1.$$

Since $T$ is strongly quasinonexpansive, we get

$$\lim_{i\to\infty} d(x_{n_i},Tx_{n_i})=0. \tag{4.1}$$

Further, by Corollary 3.8, and Theorem 3.9, we obtain

$$\begin{aligned}\cos d(y_{n_i},p) &= \cos d(\beta_{n_i}s_{n_i}\oplus(1-\beta_{n_i})t_{n_i},p) \\ &\geq \beta_{n_i}\cos d(s_{n_i},p)+(1-\beta_{n_i})\cos d(t_{n_i},p) \\ &= \beta_{n_i}(\cos d(\gamma_{n_i}u\oplus(1-\gamma_{n_i})Sx_{n_i},p) \\ &\quad+(1-\beta_{n_i})\cos d(\gamma_{n_i}u\oplus(1-\gamma_{n_i})Tx_{n_i},p) \\ &\geq \beta_{n_i}(\gamma_{n_i}\cos d(u,p)+(1-\gamma_{n_i})\cos d(x_{n_i},p)) \\ &\quad+(1-\beta_{n_i})(\gamma_{n_i}\cos d(u,p)+(1-\gamma_{n_i})\cos d(x_{n_i},p)) \\ &\geq \beta_{n_i}(\gamma_{n_i}\cos d(u,p)+(1-\gamma_{n_i})\cos d(Sx_{n_i},p)) \\ &\quad+(1-\beta_{n_i})(\gamma_{n_i}\cos d(u,p)+(1-\gamma_{n_i})\cos d(x_{n_i},p)) \\ &= \gamma_{n_i}\cos d(u,p) \\ &\quad+(1-\gamma_{n_i})(\beta_{n_i}\cos d(Sx_{n_i},p)+(1-\beta_{n_i})\cos d(x_{n_i},p)).\end{aligned}$$

Further, by Theorem 3.9 and Corollary 3.8, we obtain

$$
\begin{aligned}
&\cos d(x_{n_i+1},p)\sin d(x_{n_i},Ry_{n_i})\\
&\geq \cos d(x_{n_i},p)\sin\alpha_{n_i}d(x_{n_i},Ry_{n_i})+\cos d(Ry_{n_i},p)\sin(1-\alpha_{n_i})d(x_{n_i},Ry_{n_i})\\
&\geq \cos d(x_{n_i},p)\sin\alpha_{n_i}d(x_{n_i},Ry_{n_i})+\cos d(y_{n_i},p)\sin(1-\alpha_{n_i})d(x_{n_i},Ry_{n_i})\\
&\geq \cos d(x_{n_i},p)\sin\alpha_{n_i}d(x_{n_i},Ry_{n_i})+(\gamma_{n_i}\cos d(u,p)\\
&\quad+(1-\gamma_{n_i})(\beta_{n_i}\cos d(Sx_{n_i},p)\\
&\quad+(1-\beta_{n_i})\cos d(x_{n_i},p))\sin(1-\alpha_{n_i})d(x_{n_i},Ry_{n_i})\\
&=\gamma_{n_i}\sin(\alpha_{n_i})d(x_{n_i},Ry_{n_i})(\cos d(u,p)\\
&\quad-(\beta_{n_i}\cos d(Sx_{n_i},p)+(1-\beta_{n_i})\cos d(x_{n_i},p)))\\
&\qquad+(\cos d(x_{n_i},p)(\sin\alpha_{n_i}d(x_{n_i},Ry_{n_i})+(1-\beta_{n_i})s_{n_i}(1-\alpha_{n_i})d(x_{n_i},Ry_{n_i}))\\
&\qquad\quad+\beta_{n_i}\cos d(Sx_{n_i},p)\sin(1-\alpha_{n_i})d(x_{n_i},Ry_{n_i}).
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
1&\geq \gamma_{n_i}\frac{\sin(1-\alpha_{n_i})d(x_{n_i},Ry_{n_i})}{\sin d(x_{n_i},Ry_{n_i})}\cdot\frac{\cos d(u,p)-\beta_{n_i}\cos d(Sx_{n_i},p)}{\cos d(x_{n_{i+1}},p)}\\
&\quad+(1-\beta_{n_i})\cos d(x_{n_i},p)))\\
&\quad+\frac{\cos d(x_{n_i},p)}{\cos d(x_{n_i+1},p)}\cdot\frac{\sin\alpha_{n_i}d(x_{n_i},Ry_{n_i})+(1-\beta_{n_i})\sin(1-\alpha_{n_i})d(x_{n_i},Ry_{n_i})}{\sin d(x_{n_i},Ry_{n_i})}\\
&\quad+\frac{\beta_{n_i}\cos d(Sx_{n_i},p)}{\cos d(x_{n_{i+1}},p)}\cdot\frac{\sin(1-\alpha_{n_i})d(x_{n_i},Ry_{n_i})}{\sin d(x_{n_i},Ry_{n_i})}\\
&\geq -\gamma_{n_i}\frac{\sin(1-r)d(x_{n_i},Ry_{n_i})}{\sin d(x_{n_i},Ry_{n_i})}\\
&\quad\times\frac{|\cos d(u,p)-(\beta_{n_i}\cos d(Sx_{n_i},p)+(1-\beta_{n_i})\cos d(x_{n_i},p))|}{\cos M}\\
&\quad+\frac{\cos d(x_{n_i},p)}{\cos d(x_{n_{i+1}},p)}\cdot\frac{\sin\alpha_{n_i}d(x_{n_i},Ry_{n_i})+(1-\beta_{n_i})\sin(1-\alpha_{n_i})d(x_{n_i},Ry_{n_i})}{\sin d(x_{n_i},Ry_{n_i})}\\
&\quad+\frac{\beta_{n_i}\cos d(Sx_{n_i},p)}{\cos d(x_{n_i},p)}\cdot\frac{\sin(1-\alpha_{n_i})d(x_{n_i},Ry_{n_i})}{\sin d(x_{n_i},Ry_{n_i})}\\
&\geq -\gamma_{n_i}(1-r)\frac{d(x_{n_i},Ry_{n_i})}{\sin d(x_{n_i},Ry_{n_i})}\cdot\frac{\sin(1-r)d(x_{n_i},Ry_{n_i})}{(1-r)d(x_{n_i},Ry_{n_i})}\cdot\frac{2}{\cos M}\\
&\quad+\frac{\cos d(x_{n_i},p)}{\cos d(x_{n_{i+1}},p)}\cdot\frac{\sin\alpha_{n_i}d(x_{n_i},Ry_{n_i})+(1-\beta_{n_i})\sin(1-\alpha_{n_i})d(x_{n_i},Ry_{n_i})}{\sin d(x_{n_i},Ry_{n_i})}\\
&\quad+\frac{\beta_{n_i}\cos d(Sx_{n_i},p)}{\cos d(x_{n_i},p)}\cdot\frac{\sin(1-\alpha_{n_i})d(x_{n_i},Ry_{n_i})}{\sin d(x_{n_i},Ry_{n_i})}.
\end{aligned}
$$

Since $\beta_n \to 0$, $\gamma_n \to 0$ and $\underline{\lim}_{i\to\infty}(\cos d(x_{n_i}, p) - \cos d(x_{n_{i+1}}, p)) \geq 0$, we get

$$\begin{aligned}
1 &\geq \overline{\lim_{i\to\infty}} \left( \frac{\cos d(x_{n_i}, p)}{\cos d(x_{n_{i+1}}, p)} \right. \\
&\qquad \left. \cdot \frac{\sin \alpha_{n_i} d(x_{n_i}, Ry_{n_i}) + (1-\beta_{n_i}) \sin(1-\alpha_{n_i}) d(x_{n_i}, Ry_{n_i})}{\sin d(x_{n_i}, Ry_{n_i})} \right) \\
&\geq \underline{\lim}_{i\to\infty} \frac{\cos d(x_{n_i}, p)}{\cos d(x_{n_{i+1}}, p)} \\
&\qquad \overline{\lim_{i\to\infty}} \frac{\sin \alpha_{n_i} d(x_{n_i}, Ry_{n_i}) + (1-\beta_{n_i}) \sin(1-\alpha_{n_i}) d(x_{n_i}, Ry_{n_i})}{\sin d(x_{n_i}, Ry_{n_i})} \\
&\geq \overline{\lim_{i\to\infty}} \frac{\sin \alpha_{n_i} d(x_{n_i}, Ry_{n_i}) + \sin(1-\alpha_{n_i}) d(x_{n_i}, Ry_{n_i})}{\sin d(x_{n_i}, Ry_{n_i})} \\
&\geq \overline{\lim_{i\to\infty}} \frac{\alpha_{n_i} \sin d(x_{n_i}, Ry_{n_i}) + (1-\alpha_{n_i}) \sin d(x_{n_i}, Ry_{n_i})}{\sin d(x_{n_i}, Ry_{n_i})} \\
&= 1.
\end{aligned}$$

Since $\{\alpha_{n_i}\} \subset [r, 1-r]$ and by Lemma 3.7 we have

$$\lim_{i\to\infty} d(x_{n_i}, Ry_{n_i}) = 0.$$

It follows that

$$\begin{aligned}
d(y_{n_i}, Tx_{n_i}) &= d(\beta_{n_i} s_{n_i} \oplus (1-\beta_{n_i}) t_{n_i}, Tx_{n_i}) \\
&\leq d(\beta_{n_i} s_{n_i} \oplus (1-\beta_{n_i}) t_{n_i}, t_{n_i}) + d(t_{n_i}, Tx_{n_i}) \\
&= \beta_{n_i} d(s_{n_i}, t_{n_i}) + \gamma_{n_i} d(u, Tx_{n_i}) \\
&< \beta_{n_i} \frac{\pi}{2} + \gamma_{n_i} \frac{\pi}{2} \to 0.
\end{aligned}$$

Therefore

$$\lim_{i\to\infty} d(y_{n_i}, Tx_{n_i}) = 0.$$

It also follows that

$$\begin{aligned}
d(x_{n_i}, RTx_{n_i}) &\leq d(x_{n_i}, Ry_{n_i}) + d(Ry_{n_i}, RTx_{n_i}) \\
&\leq d(x_{n_i}, Ry_{n_i}) + d(y_{n_i}, Tx_{n_i}) \to 0.
\end{aligned}$$

We get

$$\lim_{i\to\infty} d(x_{n_i}, RTx_{n_i}) = 0.$$

Further,

$$d(x_{n_i}, p) \leq d(x_{n_i}, RTx_{n_i}) + d(RTx_{n_i}, p) \leq d(x_{n_i}, RTx_{n_i}) + d(Tx_{n_i}, p).$$

Moreover,

$$\begin{aligned}0 &\le |\cos d(x_{n_i}, p) - \cos d(Tx_{n_i}, p)| \\ &= 2\sin\frac{d(x_{n_i}, p) + d(Tx_{n_i}, p)}{2}\left|\sin\frac{d(x_{n_i}, p) - d(Tx_{n_i}, p)}{2}\right| \\ &\le 2\sin M \sin\frac{d(x_{n_i}, RTx_{n_i})}{2} \\ &\to 0 \quad (i \to \infty).\end{aligned}$$

Therefore, we get

$$\begin{aligned}d(x_{n_i}, Rx_{n_i}) &\le d(x_{n_i}, RTx_{n_i}) + d(RTx_{n_i}, Rx_{n_i}) \\ &\le d(x_{n_i}, RTx_{n_i}) + d(Tx_{n_i}, x_{n_i}) \\ &\to 0.\end{aligned} \tag{4.2}$$

Further, by taking a subsequence again, we may asuume that subsequence $\{z_k\}$ of $\{x_{n_{i_j}}\}$ satisfies $z_k \overset{\Delta}{\rightharpoonup} x_0 \in X$. Then by (4.1) and $\Delta$-demiclosedness of $T$, we get $x_0 \in F(T)$. Since $R$ is nonexpansive, it follows from (4.2) that $x_0 \in F(R)$. Moreover, by Lemma 3.4, there exist $\delta \ge 0$ and subsequennce $\{z_{k_l}\}$ of $\{z_k\}$ such that

$$\delta = \lim_{l\to\infty} d(u, z_{k_l}) = \varliminf_{k\to\infty} d(u, z_k) \ge d(u, x_0) \ge d(u, p).$$

Also, we obtain

$$\begin{aligned}\lim_{l\to\infty} d(u, z_{k_l}) &\le \lim_{l\to\infty}(d(u, Tz_{k_l}) + d(Tz_{k_l}, z_{k_l})) = \lim_{l\to\infty} d(u, Tz_{k_l}) \\ &\le \lim_{l\to\infty}(d(u, z_{k_l}) + d(z_{k_l}, Tz_{k_l})) = \lim_{l\to\infty} d(u, z_{k_l}).\end{aligned}$$

Therefore, we obtain $\lim_{l\to\infty} d(u, z_{k_l}) = \lim_{l\to\infty} d(u, Tz_{k_l})$. We write $\{\alpha_k\}$, $\{\sigma_k\}$, $\{\tau_k\}$, $\{b_k\}$, $\{c_k\}$ as subsequences of $\{\alpha_n\}, \{\sigma_n\}, \{\tau_n\}, \{b_n\}, \{c_n\}$ corresponding to a subsequence $\{z_k\}$ of $\{x_n\}$. Then, we obtain that

$$\begin{aligned}\varlimsup_{i\to\infty}\left(\frac{\beta_{n_i}\sigma_{n_i}b_{n_i} + (1-\beta_{n_i})\tau_{n_i}c_{n_i}}{\beta_{n_i}\sigma_{n_i} + (1-\beta_{n_i})\tau_{n_i}}\right) &= \lim_{j\to\infty}\left(\frac{\beta_{n_{i_j}}\sigma_{n_{i_j}}b_{n_{i_j}} + (1-\beta_{n_{i_j}})\tau_{n_{i_j}}c_{n_{i_j}}}{\beta_{n_{i_j}}\sigma_{n_{i_j}} + (1-\beta_{n_{i_j}})\tau_{n_{i_j}}}\right) \\ &= \varlimsup_{j\to\infty}\left(\frac{\beta_{n_{i_j}}\sigma_{n_i}b_{n_{i_j}} + (1-\beta_{n_{i_j}})\tau_{n_{i_j}}c_{n_{i_j}}}{\beta_{n_{i_j}}\sigma_{n_{i_j}} + (1-\beta_{n_{i_j}})\tau_{n_{i_j}}}\right) \\ &= \lim_{k\to\infty}\left(\frac{\beta_k\sigma_k b_k + (1-\beta_k)\tau_k c_k}{\beta_k\sigma_k + (1-\beta_k)\tau_k}\right) \\ &= \varliminf_{k\to\infty}\left(\frac{\beta_k\sigma_k b_k + (1-\beta_k)\tau_k c_k}{\beta_k\sigma_k + (1-\beta_k)\tau_k}\right)\end{aligned}$$

$$
\begin{aligned}
&= \lim_{l\to\infty} \left( \frac{\beta_{k_l}\sigma_{k_l}b_{k_l} + (1-\beta_{k_l})\tau_{k_l}c_{k_l}}{\beta_{k_l}\sigma_{k_l} + (1-\beta_{k_l})\tau_{k_l}} \right) \\
&= \lim_{l\to\infty} c_{k_l} \\
&= 1 - \frac{\cos d(u,p)}{\cos\delta} \\
&\le 0.
\end{aligned}
$$

Thus, we get

$$
\overline{\lim_{i\to\infty}} \left( \frac{\beta_{\varphi(i)}\sigma_{\varphi(i)}b_{\varphi(i)} + (1-\beta_{\varphi(i)})\tau_{\varphi(i)}c_{\varphi(i)}}{\beta_{\varphi(i)}\sigma_{\varphi(i)} + (1-\beta_{\varphi(i)})\tau_{\varphi(i)}} \right) \le 0.
$$

By Lemma 3.1, we get $\lim_{n\to\infty} a_n = 0$. It implies that $\{x_n\}$ converges to $p$. In a similar fashion, we have $\{x_n\}$ converges to $P_{F(S)\cap F(R)}u$ if $\beta_n \to 1$. Hence we obtain the desired result. □

The following theorem generalizes Theorem 1.2 and Theorem 4.3. The limit of an iterative sequence can be represented by a single mapping.

**Theorem 4.2.** *Let $X$ be a complete* CAT(1) *space with $M = \operatorname{diam} X < \pi/2$. Let $R\colon X \to X$ be a nonexpansive mapping and $S, T\colon X \to X$ strongly quasinonexpansive and $\Delta$-demiclosed mappings from $X$ into itself. For $r \in ]0, 1/2[$, let $\{\alpha_n\} \subset [r, 1-r] \subset ]0,1[, \{\beta_n\}, \{\gamma_n\} \subset ]0,1[$ be real sequences satisfying $\beta_n \to [0,1]$, $\gamma_n \to 0$ and $\sum_{n=1}^{\infty}\gamma_n = \infty$. Define $\{x_n\} \subset X$ by $x_1 = u \in X$ and*

$$
\begin{cases}
s_n = \gamma_n u \oplus (1-\gamma_n)Sx_n \\
t_n = \gamma_n u \oplus (1-\gamma_n)Tx_n \\
y_n = \beta_n s_n \oplus (1-\beta_n)t_n \\
x_{n+1} = \beta_n x_n \oplus (1-\beta_n)Ry_n
\end{cases}
$$

*for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges to $P_{F(\beta S\oplus(1-\beta)T)\cap F(R)}u$.*

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