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ON THE OPTIMAL ALLOCATION OF THE COSTS OF MAINTAINING THE COMPONENTS OF A SYSTEM

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Abstract. In our former works, we proved some properties of the Barlow and Proschan's structural importance of components' performances with different costs of maintenance in binary coherent systems. The structural importance was derived from the cost-related Barlow and Proschan's conditional prior distribution of components' performances. In this article, we make further investigations on that conditional prior distribution. We derive a general form of the cost-related Barlow and Proschan's joint distribution of components' performances from that conditional prior distribution. With the general form of the cost-related Barlow and Proschan's joint distribution, we obtain an allocation of the costs to the components to optimize the performance of the system.

1. Introduction

In [3](2016), we investigated Barlow and Proschan's(BP) interpretation for their structural importance of components in binary-state coherent systems, then rederived the BP conditional prior distribution of the components' performances in coherent systems. Inspired by the cost-based defined by Wu and Coolen [12](2013), Hsiao and Chiou [7](2018) extended it to a binary coherent system where components require some costs of maintenance in the system, then generalized it to a cost-related BP conditional prior distribution of the components' performance in coherent systems.

In this article, with the cost-related BP conditional prior distribution, we find its joint distribution of components' performances in coherent systems.

In reliability theory, optimal allocation is an important concept used for improving performance of a coherent system. Please see the literatures [6, 8]. In this article, with the cost-related BP joint distribution, we obtain an

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allocation of the costs to the components to optimize the performance of the system.

2. Definition, notations nd basic theorem

Following [1, 2, 9], we have definitions and notations as follows. Consider a binary system (C, ϕ) composed of n components, where $C = \{1, 2, ..., n\}$ denotes the set of the n components, and $\phi : \{0, 1\}^n \to \{0, 1\}$ denotes the structure function of the system. For brevity, we denote $\mathbb{S} = \{0, 1\}$ and $\mathbb{S}^n = \{0, 1\}^n$.

The state x_i of component i is defined by

$$x_i = \begin{cases} 1 & \text{if component } i \text{ is functioning} \\ 0 & \text{if component } i \text{ is failed.} \end{cases}$$

Similarly, the state ϕ of the system is a deterministic binary function of the state vector $\boldsymbol{x} = (x_1, x_2, ..., x_n)$ of components, defined by

$$\phi(\boldsymbol{x}) = \begin{cases} 1 & \text{if the system is functioning} \\ 0 & \text{if the system is failed.} \end{cases}$$

Given $\boldsymbol{x} \in \mathbb{S}^n$, following [1], we denote $(\cdot_i, \boldsymbol{x}) = (x_1, ..., x_{i-1}, \cdot_i, x_{i+1}, ..., x_n)$, i.e., $(0_i, \boldsymbol{x}) = (x_1,\ ...,\ x_{i-1},\ 0,\ x_{i+1},\ ...,x_n)$ and $(1_i, \boldsymbol{x}) = (x_1,\ ...,\ x_{i-1},\ 1,\ x_{i+1},\ ...,x_n)$.

Definition 2.1. Given a binary system (C, ϕ), a component i is irrelevant to ϕ if

$$\phi(1_i, \boldsymbol{x}) = \phi(0_i, \boldsymbol{x})$$

for all $(\cdot_i, \boldsymbol{x})$, i.e., component i is relevant to ϕ if there exists a vector $(\cdot_i, \boldsymbol{x})$ such that $\phi(1_i, \boldsymbol{x}) = 1$ and $\phi(0_i, \boldsymbol{x}) = 0$.

Definition 2.2. A binary coherent system is a binary system (C, ϕ) such that (i) $\phi(\boldsymbol{x})$ is nondecreasing in each component, (ii) each component $i \in C$ is relevant to ϕ.

Let the components of a system ϕ be stochastically independent. The reliability function $h(\boldsymbol{p})$ of ϕ is the probability that ϕ is functioning, as a function of component reliabilities $\boldsymbol{p} = (p_1, p_2, ..., p_n)$.

Given a binary coherent system ϕ, it is well known that Birnbaum's *structural importance* can written as

$$B_\phi(i) = \frac{1}{2^{n-1}} \sum_{(1_i, \boldsymbol{x}) \in S^n} [\phi(1_i, \boldsymbol{x}) - \phi(0_i, \boldsymbol{x})].$$

Wu and Coolen [12](2013) extended Birnbaum's *structural importance* to some cost-based structural important. When $\boldsymbol{p}$ is not available, Barlow and Proschan [1](1975) define their *structural importance* $I_\phi^{BP}(i)$ of component i by

$$I_\phi^{BP}(i) = \int_0^1 [h(1_i, \boldsymbol{p}) - h(0_i, \boldsymbol{p})]dp, \text{ where } p_j = p \ \forall j \neq i.$$

Given $B \subseteq C$, let $\mathbf{e}(B)$ be the binary vector with components $e_j(B)$ such that

$$e_j(B) = \begin{cases} 1 & \text{if } j \in B \\ 0 & \text{otherwise.} \end{cases}$$

For brevity, we let the standard unit vectors $\mathbf{e}(\{j\}) = \mathbf{e}_j$ for all $j \in C$. Throughout this article, we denote $|B|$ number of elements in B, $\mathbf{e}(C) = \mathbf{1} = (1, 1, ..., 1)$ and $\mathbf{e}(\emptyset) = \mathbf{0} = (0, 0, ..., 0)$.

Definition 2.3. A set $P \subseteq C$ is called *path set* if $\phi(\mathbf{e}(P)) = 1$. A path set P is said to be a min path set if $\phi(\mathbf{e}(Z)) = 0$ for any $Z \subset P$.

A set $K \subseteq C$ is called *cut set* if $\phi(\mathbf{e}(K^c)) = 0$. A cut set K is said to be a min cut set if $\phi(\mathbf{e}(Z^c)) = 1$ for any $Z \subset K$.

A *critical path vector for component* i is a vector $(1_i, \boldsymbol{x})$ such that $\phi(1_i, \boldsymbol{x}) = 1$ while $\phi(0_i, \boldsymbol{x}) = 0$. The corresponding critical path set for i is $\{i\} \cup \{j \mid j \neq i, \text{the } j\text{th component of } (1_i, \boldsymbol{x}) = 1\}$.

Barlow and Proschan's *structural importance* in [1](1975) is as follows.

$$\begin{aligned} I_\phi^{BP}(i) &= \sum_{\substack{B \subseteq C \\ i \in B}} \frac{(|B|-1)!(n-|B|)!}{n!} [\phi(\mathbf{e}(B)) - \phi(\mathbf{e}(B \setminus \{i\}))] \\ &= \frac{1}{n} \sum_{r=1}^{n} \binom{n-1}{r-1}^{-1} \Big[\sum_{\substack{B \subseteq C \\ i \in B \\ |B| = r}} [\phi(\mathbf{e}(B)) - \phi(\mathbf{e}(B \setminus \{i\}))] \Big]. \end{aligned} \tag{2.1}$$

Barlow and Proschan have a probability interpretation for their *structural importance* in [1] as follows. In the absence of information concerning component reliabilities, $I_\phi^{BP}(i)$ is the expectation of component i being in a critical path set according to a prior probability that the order of components' failures is uniformly distributed. Therefore given any $j \in C$, Barlow and Proschan's *structural importance* is derived from the following prior distribution.

$$Pr\{\text{component } j \text{ is the } (n - |B| + 1)\text{th failure}\} = \frac{(|B|-1)!(n-|B|)!}{n!} \tag{2.2}$$

In [3], we rederive (2.2) as follows. Given $\boldsymbol{x} \in \mathbb{S}^n$ define $S(\boldsymbol{x}) = \{j \mid x_j \neq 0\}$, then $\mathbf{e}(S(\boldsymbol{x})) = \boldsymbol{x}$. Let $\boldsymbol{X} = (X_1, X_2, ..., X_n)$ be a random vector such that

$$Pr\{\boldsymbol{X} \geq \boldsymbol{x} \mid X_j = 1\} = \frac{1}{|\boldsymbol{x}|}, \text{ where } |\boldsymbol{x}| = \sum_{j \in C} x_j. \tag{2.3}$$

By inclusive and exclusive principle, we see that Barlow and Proschan's *structural importance* can also be derived from the prior distribution as follows.

$$Pr\{\boldsymbol{X} = \boldsymbol{x} \mid X_j = 1\} = \frac{(|S(\boldsymbol{x})| - 1)!(n - |S(\boldsymbol{x})|)!}{n!}$$

and

$$I_\phi^{BP}(j) = E[\phi(\mathbf{X}) - \phi(\boldsymbol{X} - \mathbf{e}_j) \mid X_j = 1].$$

Then, Barlow and Proschan's *structural importance* can also be regard as derived from prior (2.3) which is inversely proportional to the size of $S(\boldsymbol{x})$.

Inspired by El-Neweihi, Proschan and Sethuraman[6](1986), with the cost-related Barlow and Proschan's joint distribution of the components' performances in coherent systems, we apply Majorization Theory to find an allocation of the costs to the components to optimize the performance of the system.

Following [6](El-Neweihi et al., 1986), [8](Kim and Zuo, 2018) and [9](Marshall, Olkin and Arnold, 2011), we have the definitions, notations and theorem from Majorization Theory.

For a vector $\boldsymbol{a} \in \mathbb{R}^m$, we denote by $\boldsymbol{a}^\uparrow \in \mathbb{R}^m$ the vector with the same components, but sorted in descending order. Given $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^m$ $\boldsymbol{a}$ is said to majorize $\boldsymbol{b}$ written as $\boldsymbol{a} \succ \boldsymbol{b}$ if

$$\sum_{i=1}^{k} a_i^\uparrow \geq \sum_{i=1}^{k} b_i^\uparrow, \text{ for } k = 1, ..., m-1,$$

and

$$\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} b_i.$$

A symmetric function $g : \mathbb{R}^m \to \mathbb{R}$ (that is, a function g such that $g(\boldsymbol{x}) = g(\boldsymbol{x}\Pi)$ for every permutation Π) is said to be Schur-concave (or convex) if

$$g(\boldsymbol{x}) \leq (\text{ or } \geq) g(\boldsymbol{y})$$

for all $\boldsymbol{x}$ majorizing $\boldsymbol{y}$.

Please notice that if ϕ is symmetric on a symmetric set $\mathcal{A}$ (that is, a set $\mathcal{A}$ such that $\boldsymbol{x} \in \mathcal{A}$ implies $\boldsymbol{x}\Pi \in \mathcal{A}$ for every permutation Π) and Schur-convex on $\mathcal{D} \cap \mathcal{A}$, where $\mathcal{D} = \{\boldsymbol{x} : x_1 \geq x_2 \ldots \geq x_n\}$, then ϕ is Schur-convex on $\mathcal{A}$.

In page 84 of [9](2011), Schur and Ostrowski proved the following theorem, respectively.

Theorem 2.4 (Schur, 1923; Ostrowski, 1952)**.** *Let $I \subseteq \mathbb{R}$ be an open interval and let $\phi : I^n \to \mathbb{R}$ be continuously differentiable. Necessary and sufficient conditions for ϕ to be Schur-convex on I^n are*

$$\phi \quad \textit{is symmetric on } I^n,$$

and

$$\frac{\partial}{\partial z_i}\phi(\boldsymbol{z}) \quad \textit{is non-increasing in } i = 1, \ldots, n \quad \textit{all } \boldsymbol{z} \in \mathcal{D} \cap I^n.$$

3. Main results

Considering costs incurred by maintaining a system and its components, Wu and Coolen [12](2013) proposed a cost-based importance. Inspired by [12], Hsiao and Chiou [7] extend priors (2.3) to a binary coherent system(BCS) where components require some costs of maintenance in the system: given a BCS (C, ϕ), let $\kappa : C \to R_+$ be such that $\kappa(j)$ is the cost of maintaining the function of component j in ϕ. ("delete $\kappa(j)$ is the cost to maintain component j work in ϕ.") The cost $\kappa(i)$ does not have to be equal to the cost $\kappa(j)$ for $i \neq j$. If $S = \{s_1, \ldots, s_\ell\} \subseteq C$, we denote $\kappa(S)$ to be

$$\kappa(S) = \kappa(s_1) + \kappa(s_2) + \cdots + \kappa(s_\ell).$$

Note that we regard the costs as the precision of components, the moisture resistance of components, the corrosion resistance of components, etc. which can be accurately measured by engineers. In [7] we generalize prior probability (2.3) for BCS to the following cost-related prior probability for BCS.

$$Pr\{\mathbf{X} \geq \boldsymbol{x} \mid X_j = 1\} = \frac{\kappa(j)}{\kappa(S(\boldsymbol{x}))} \ \text{(all } \ \kappa(i) > 0),$$

where we regard $\kappa(S(\boldsymbol{x}))$ the cost to be paid for keeping the components in state vector $\boldsymbol{x}$ and regard $\kappa(j)$ as the cost already paid for keeping component j working. Note that $Pr\{\mathbf{X} \geq \boldsymbol{x} \mid X_j = 1\}$ is defined on all the state vectors with $x_j = 1$ and it is decreasing in $\boldsymbol{x}$. Especially, $Pr\{\mathbf{X} \geq (1_j, \mathbf{0}) \mid X_j = 1\} = \frac{k(j)}{k(j)} = 1$.

By inclusive and exclusive principle, one get

$$Pr\{\mathbf{X} = \boldsymbol{x} \mid X_j = 1\} = \sum_{T \subseteq C \setminus S(\boldsymbol{x})} (-1)^{|T|} \cdot \frac{\kappa(j)}{\kappa(S(\boldsymbol{x})) + \kappa(T)}. \tag{3.1}$$

Now fixed $i \in C$, for each $\boldsymbol{x}$ with $x_i = 1$, we denote (3.1) by

$$p^i_{\boldsymbol{x}} = Pr\{\mathbf{X} = \boldsymbol{x} \mid X_i = 1\} = \sum_{T \subseteq C \setminus S(\boldsymbol{x})} (-1)^{|T|} \frac{\kappa_i}{[\sum_{j \in S(\boldsymbol{x})} \kappa_j] + [\sum_{j \in T} \kappa_j]}.$$

We have

$$p^i_{\boldsymbol{x}} \geq 0 \ \forall i \text{ and } \sum_{\substack{x_i \neq 0 \\ \boldsymbol{x} \in \mathbb{S}^n}} p^i_{\boldsymbol{x}} = 1.$$

Observe that for each fixed $i \in C$, $\{\boldsymbol{x} \mid x_i = 1, \boldsymbol{x} \in \mathbb{S}^n\}$ is a proper subset of $\mathbb{S}^n$, we have that the probability mass function $\{p^i_{\boldsymbol{x}} \mid x_i = 1, \boldsymbol{x} \in \mathbb{S}^n\}$ is a conditional probability mass function of some probability mass function over $\mathbb{S}^n$, say $\{p_{\boldsymbol{x}} : \boldsymbol{x} \in \mathbb{S}^n\}$.

Observing (3.1), let $\kappa(S(\boldsymbol{x})) = z$, $C \setminus S(\boldsymbol{x}) = M = \{1, 2, \ldots, m\}$, we get the following lemma.

Lemma 3.1. *Given a set $M = \{1, 2, \ldots, m\}$, let*

$$f(z, \kappa_1, \ldots, \kappa_m) = \sum_{T \subseteq M} (-1)^{|T|} \frac{1}{z + \sum_{i \in T} \kappa_i},$$

$z > 0$, and $\kappa_i \geq 0 \ \forall i \in M$. Then we have $f(z, \kappa_1, \ldots, \kappa_m) \geq 0$.
Especially, if there is some $j \in M$ with $\kappa_j = 0$ then all $f(z, \kappa_1, \ldots, \kappa_m) = 0$.

Proof. First, $f(z, \kappa_1, \ldots, \kappa_m) \geq 0$ for $z > 0$ and $\kappa_i > 0 \ \forall i \in M$ since $p^i_{\boldsymbol{x}} \geq 0$. Next, suppose $j \in M$ with $\kappa_j = 0$. Then we have

$$\begin{aligned} f(z, \kappa_1, \ldots, \kappa_m) &= \sum_{\substack{T \subseteq M \\ j \in T}} (-1)^{|T|} \frac{1}{z + \left[0 + \sum_{i \in T \setminus \{j\}} \kappa_i\right]} + \sum_{\substack{T \subseteq M \\ j \notin T}} (-1)^{|T|} \frac{1}{z + \sum_{i \in T} \kappa_i} \\ &= \sum_{T \subseteq \{1, \ldots, j-1, j+1, \ldots, M\}} (-1)^{|T|+1} \frac{1}{z + \sum_{i \in T} \kappa_i} \\ &+ \sum_{T \subseteq \{1, \ldots, j-1, j+1, \ldots, M\}} (-1)^{|T|} \frac{1}{z + \sum_{i \in T} \kappa_i} \\ &= 0. \end{aligned}$$

□

The following theorem exhibit a closed form of the probability distribution $\{p_{\boldsymbol{x}} \mid \boldsymbol{x} \in \mathbb{S}^n\}$.

Theorem 3.2. *For each* $\boldsymbol{x} \in \mathbb{S}^n$ *with* $\boldsymbol{x} \neq \mathbf{0} = (0, 0, \ldots, 0)$*, the probability distribution* $\{p_{\boldsymbol{x}} \mid \boldsymbol{x} \in \mathbb{S}^n\}$ *has the following closed form.*

(3.2)

$$p_{\boldsymbol{x}} = \sum_{T \subseteq C \setminus S(\boldsymbol{x})} (-1)^{|T|} \frac{1}{[\sum_{j \in S(\boldsymbol{x})} \kappa_j] + [\sum_{j \in T} \kappa_j]} \times \left[\frac{1 - p_{\mathbf{0}}}{\sum_{\substack{K \subseteq C \\ K \neq \emptyset}} (-1)^{|K|+1} \frac{1}{\sum_{j \in K} \kappa_j}} \right]$$

if $p_{\mathbf{0}} \neq 1$.

Proof. First, please notice that if $p_{\mathbf{0}} = 1$ and hence $p_{\boldsymbol{x}} = 0 \ \forall \boldsymbol{x} \neq \mathbf{0}$, then you won't use such an equipment in the real world. Next, by the *law of total probability* and (3.1) we have

$$\begin{aligned} p_{(1,1,\ldots,1)} &= Pr(X = \mathbf{1} \mid X_i = 1) Pr(X_i = 1) + Pr(X = \mathbf{1} \mid X_i = 0) Pr(X_i = 0) \\ &= \frac{\kappa_i}{\kappa(C)} Pr(X_i = 1) + 0 \cdot Pr(X_i = 0). \end{aligned}$$

This implies that for all $i \in C$ the values $\kappa_i \cdot Pr(X_i = 1)$ are the same. Let $\kappa = \kappa_i \cdot Pr(X_i = 1), i \in C$. Now for $\boldsymbol{x} \neq \mathbf{0}$, say $x_i = 1$, then

$$\begin{aligned} p_{\boldsymbol{x}} &= Pr(X = \boldsymbol{x} \mid X_i = 1) Pr(X_i = 1) + Pr(X = \boldsymbol{x} \mid X_i = 0) Pr(X_i = 0) \\ &= \Big[\sum_{T \subseteq C \setminus S(\boldsymbol{x})} (-1)^{|T|} \frac{\kappa_i}{[\sum_{j \in S(\boldsymbol{x})} \kappa_j] + [\sum_{j \in T} \kappa_j]} \Big] Pr(X_i = 1) + 0 \cdot Pr(X_i = 0) \\ &= \sum_{T \subseteq C \setminus S(\boldsymbol{x})} (-1)^{|T|} \frac{\kappa}{[\sum_{j \in S(\boldsymbol{x})} \kappa_j] + [\sum_{j \in T} \kappa_j]}. \end{aligned}$$

Since (i)$p_{\mathbf{0}} + \sum_{\substack{\boldsymbol{x} \neq \mathbf{0} \\ \boldsymbol{x} \in \mathbb{S}^n}} p_{\boldsymbol{x}} = 1$, (ii)given $\boldsymbol{x} \in \mathbb{S}^n$, we have $S(\boldsymbol{x}) \subseteq C$, and (iii) conversely, given $S \subseteq C$, we can choose $\boldsymbol{x} \in \mathbb{S}^n$ such that $S(\boldsymbol{x}) = S$, one get

$$\begin{aligned} 1 - p_{\mathbf{0}} &= \sum_{\substack{\boldsymbol{x} \neq \mathbf{0} \\ \boldsymbol{x} \in \mathbb{S}^n}} \Big[\sum_{T \subseteq C \setminus S(\boldsymbol{x})} (-1)^{|T|} \frac{\kappa}{[\sum_{j \in S(\boldsymbol{x})} \kappa_j] + [\sum_{j \in T} \kappa_j]} \Big] \\ &= \sum_{\substack{S \neq \emptyset \\ S \subseteq C}} \sum_{T \subseteq C \setminus S} (-1)^{|T|} \frac{\kappa}{[\sum_{j \in S} \kappa_j] + [\sum_{j \in T} \kappa_j]} \ (> 0 \text{ if } p_{\mathbf{0}} \neq 1). \end{aligned}$$

And hence

$$\kappa = \frac{1 - p_{\mathbf{0}}}{\sum_{\substack{S \neq \emptyset \\ S \subseteq C}} \sum_{T \subseteq C \setminus S} (-1)^{|T|} \frac{1}{[\sum_{j \in S} \kappa_j] + [\sum_{j \in T} \kappa_j]}}.$$

Simplify the expression $\sum_{\substack{S\neq\emptyset\\ S\subseteq C}} \sum_{T\subseteq C\setminus S}(-1)^{|T|}\frac{1}{[\sum_{j\in S}\kappa_j]+[\sum_{j\in T}\kappa_j]}$, we see that the coefficient of the term $\frac{1}{\kappa_{j_1}+\kappa_{j_2}+\cdots+\kappa_{j_\ell}}$ in the simplified expression is

$$\begin{aligned}
&C_1^\ell \cdot 1^1 \cdot (-1)^{\ell-1} + C_2^\ell \cdot 1^2 \cdot (-1)^{\ell-2} + \cdots + C_\ell^\ell \cdot 1^\ell \cdot (-1)^0 \\
&= (1+(-1))^\ell - C_0^\ell \cdot 1^0 \cdot (-1)^\ell \\
&= (-1)^{\ell+1} \\
&= (-1)^{|K|+1} \text{ (if } K = \{j_1, \ldots, j_\ell\}).
\end{aligned}$$

We obtain an equality as follows.

$$\sum_{\substack{S\neq\emptyset\\ S\subseteq C}} \sum_{T\subseteq C\setminus S}(-1)^{|T|}\frac{1}{[\sum_{j\in S}\kappa_j]+[\sum_{j\in T}\kappa_j]} = \sum_{\substack{K\neq\emptyset\\ K\subseteq C}}(-1)^{|K|+1}\frac{1}{\sum_{j\in K}\kappa_j}.$$

The result follows.

□

Remark 3.3. In (3.2) we do not assume that $p_{\mathbf{0}}$ is known. However, in the real world, if none of the components of an equipment is working, then we won't buy it. Therefore, we may assume machine $p_{\mathbf{0}} = 0$.

Theorem 3.4. *Given a set $M = \{1, 2, \ldots, m\}$, let*

$$f(z, \kappa_1, \ldots, \kappa_m) = \sum_{T\subseteq M}(-1)^{|T|}\frac{1}{z+\sum_{i\in T}\kappa_i},$$

where $z > 0$, and $\kappa_i \geq 0$ $\forall i \in M$. Then

(a) *$f(z, \kappa_1, \ldots, \kappa_m)$ is non-decreasing in each κ_i.*
(b) *$f(z, \kappa_1, \ldots, \kappa_m)$ is non-increasing in z.*

Proof. (a) Suppose $\kappa_j > \kappa_j^1 \geq 0$. Then $\kappa_j^2 = \kappa_j - \kappa_j^1 > 0$, and

$$\begin{aligned}
f(z, \kappa_1, \ldots, \kappa_m) &= \sum_{\substack{T \subseteq M \\ j \in T}} (-1)^{|T|} \frac{1}{z + [(\kappa_j^1 + \kappa_j^2) + \sum_{i \in T \setminus \{j\}} \kappa_i]} \\
&\quad + \sum_{\substack{T \subseteq M \\ j \notin T}} (-1)^{|T|} \frac{1}{z + \sum_{i \in T} \kappa_i} \\
&= \left[\sum_{\substack{T \subseteq M \\ j \in T}} (-1)^{|T|} \frac{1}{(z + \kappa_j^1) + (\kappa_j^2 + \sum_{i \in T \setminus \{j\}} \kappa_i)} \right. \\
&\quad \left. + \sum_{\substack{T \subseteq M \\ j \notin T}} (-1)^{|T|} \frac{1}{(z + \kappa_j^1) + \sum_{i \in T \setminus \{j\}} \kappa_i} \right] \\
&\quad + \left[- \sum_{\substack{T \subseteq M \\ j \notin T}} (-1)^{|T|} \frac{1}{(z + \kappa_j^1) + \sum_{i \in T \setminus \{j\}} \kappa_i} \right. \\
&\quad \left. + \sum_{\substack{T \subseteq M \\ j \notin T}} (-1)^{|T|} \frac{1}{z + \sum_{i \in T} \kappa_i} \right] \\
&= f(z + \kappa_j^1, \kappa_1, \ldots, \kappa_j^2, \ldots, \kappa_m) \\
&\quad + \left[\sum_{\substack{T \subseteq M \\ j \in T}} (-1)^{|T|} \frac{1}{z + (\kappa_j^1 + \sum_{i \in T \setminus \{j\}} \kappa_i)} \right. \\
&\quad \left. + \sum_{\substack{T \subseteq M \\ j \notin T}} (-1)^{|T|} \frac{1}{z + \sum_{i \in T} \kappa_i} \right] \\
&= f(z + \kappa_j^1, \kappa_1, \ldots, \kappa_j^2, \ldots, \kappa_m) + f(z, \kappa_1, \ldots, \kappa_j^1, \ldots, \kappa_m) \\
&\geq f(z, \kappa_1, \ldots, \kappa_j^1, \ldots, \kappa_m).
\end{aligned}$$

(b) Suppose $z_1 > z > 0$. Then $\kappa_{m+1} = z_1 - z > 0$, and

$$\begin{aligned}
f(z_1, \kappa_1, \ldots, \kappa_m) &= \sum_{T \subseteq M} (-1)^{|T|} \frac{1}{(z + \kappa_{m+1}) + \sum_{i \in T} \kappa_i} \\
&= (-1) \cdot \sum_{\substack{T \subseteq M \cup \{m+1\} \\ m+1 \in T}} (-1)^{|T|} \frac{1}{z + \sum_{i \in T} \kappa_i} \\
&\quad - \sum_{\substack{T \subseteq M \cup \{m+1\} \\ m+1 \notin T}} (-1)^{|T|} \frac{1}{z + \sum_{i \in T} \kappa_i} \\
&\quad + \sum_{\substack{T \subseteq M \cup \{m+1\} \\ m+1 \notin T}} (-1)^{|T|} \frac{1}{z + \sum_{i \in T} \kappa_i} \\
&= - \sum_{T \subseteq M \cup \{m+1\}} (-1)^{|T|} \frac{1}{z + \sum_{i \in T} \kappa_i} \\
&\quad + \sum_{T \subseteq M} (-1)^{|T|} \frac{1}{z + \sum_{i \in T} \kappa_i} \\
&= -f(z, \kappa_1, \ldots, \kappa_m, \kappa_{m+1}) + f(z, \kappa_1, \ldots, \kappa_m).
\end{aligned}$$

We have that

$$f(z_1, \kappa_1, \ldots, \kappa_m) - f(z, \kappa_1, \ldots, \kappa_m) = -f(z, \kappa_1, \ldots, \kappa_m, \kappa_{m+1}) \leq 0$$

by Lemma 3.1. □

Remark 3.5. Theorem 3.4(a) shows that the cost-related prior probability density function $p_{\boldsymbol{x}}^i$ (please see (3.1)) is increasing whenever κ_i is increasing, which is a reasonable model in the real world. Here we regard the cost κ_i as the precision of component, the moisture resistance of component, the corrosion resistance of component, etc. rather than the "cost" defined by economists.

Corollary 3.6. *Given a set* $M = \{1, 2, \ldots, m\}$, *let*

$$f^*(z, \kappa_1, \ldots, \kappa_m) = \sum_{T \subseteq M} (-1)^{|T|} \frac{1}{[z + \sum_{i \in T} \kappa_i]^2},$$

where $z > 0$, *and* $\kappa_i \geq 0$ $\forall i \in M$. *Then*

(a) $f^*(z, \kappa_1, \ldots, \kappa_m) \geq 0$.
(b) $f^*(z, \kappa_1, \ldots, \kappa_m)$ *is non-increasing in* z.

Proof. First, $f^*(z,\kappa_1,\ldots,\kappa_m) = -\frac{\partial f(z,\kappa_1,\ldots,\kappa_m)}{\partial z} \geq 0$ by Theorem 3.4 (b).

Next, with the same method as in the process of the proof in Theorem 3.4 (b), we have

$$\sum_{T\subseteq M}(-1)^{|T|}\frac{1}{[(z+\kappa_{m+1})+\sum_{i\in T}\kappa_i]^2} - \sum_{T\subseteq M}(-1)^{|T|}\frac{1}{[z+\sum_{i\in T}\kappa_i]^2}$$
$$= - \sum_{T\subseteq M\cup\{m+1\}}(-1)^{|T|}\frac{1}{[z+\sum_{i\in T}\kappa_i]^2}.$$

Namely,

$$f^*(z_1,\kappa_1,\ldots,\kappa_m) - f^*(z,\kappa_1,\ldots,\kappa_m) = -f^*(z,\kappa_1,\ldots,\kappa_m,\kappa_{m+1}),$$

here we let $z_1 = z+\kappa_{m+1} > z$. Then (b) follows by (a). □

Theorem 3.7. *Given a set* $M = \{1,2,\ldots,m\}$*, let*

$$g(\kappa_1,\ldots,\kappa_m) = \sum_{T\subseteq M}(-1)^{|T|}\frac{1}{z+\sum_{i\in T}\kappa_i},$$

where $z > 0$ *is a fixed constant, and* $\kappa_i > 0\ \forall i \in M$*. Then* $g(\kappa_1,\ldots,\kappa_m)$ *is a Schur-concave function in* κ_i*'s.*

Proof. First, it is trivial to see that $g(\kappa_1,\ldots,\kappa_m)$ is a symmetric function on $\mathbb{R}^n_{++} = \{\boldsymbol{\kappa} = (\kappa_1,\ldots,\kappa_m) : \kappa_i > 0\ \forall i\}$. Next, observe that

$$\frac{\partial g(\boldsymbol{\kappa})}{\partial\kappa_1} - \frac{\partial g(\boldsymbol{\kappa})}{\partial\kappa_2}$$
$$= \sum_{\substack{T\subseteq M\\1\in T}}(-1)^{|T|+1}\frac{1}{[z+\sum_{i\in T}\kappa_i]^2} - \sum_{\substack{T\subseteq M\\2\in T}}(-1)^{|T|+1}\frac{1}{[(z+\sum_{i\in T}\kappa_i]^2}$$
$$= \left(\sum_{\substack{T\subseteq M\\1\in T,2\notin T}}(-1)^{|T|+1}\frac{1}{[z+\sum_{i\in T}\kappa_i]^2} - \sum_{\substack{T\subseteq M\\1\in T,2\in T}}(-1)^{|T|+1}\frac{1}{[(z+\sum_{i\in T}\kappa_i]^2}\right)$$
$$- \left(\sum_{\substack{T\subseteq M\\2\in T,1\in T}}(-1)^{|T|+1}\frac{1}{[z+\sum_{i\in T}\kappa_i]^2} - \sum_{\substack{T\subseteq M\\2\in T,1\notin T}}(-1)^{|T|+1}\frac{1}{[(z+\sum_{i\in T}\kappa_i]^2}\right)$$
$$= \sum_{T\subseteq M\setminus\{1,2\}}(-1)^{|T|}\frac{1}{[(z+\kappa_1)+\sum_{i\in T}\kappa_i]^2}$$

$$- \sum_{T \subseteq M \setminus \{1,2\}} (-1)^{|T|} \frac{1}{[(z+\kappa_2) + \sum_{i \in T} \kappa_i]^2}.$$

Let $\kappa_1 > \kappa_2$ and $\kappa_{m+1} = \kappa_1 - \kappa_2 = (z+\kappa_1) - (z+\kappa_2)$. Then by Corrollary 3.6, we conclude that

$$\frac{\partial g(\boldsymbol{\kappa})}{\partial \kappa_1} - \frac{\partial g(\boldsymbol{\kappa})}{\partial \kappa_2} = - \sum_{T \subseteq \{3,4,\ldots,m,m+1\}} (-1)^{|T|} \frac{1}{[(z+\kappa_{m+1}) + \sum_{i \in T} \kappa_i]^2} \leq 0.$$

Exactly the same method, we have that $\frac{\partial(-g)(\boldsymbol{\kappa})}{\partial \kappa_j} - \frac{\partial(-g)(\boldsymbol{\kappa})}{\partial \kappa_{j+1}} \geq 0$, for $\kappa_j > \kappa_{j+1}, j = 1, 2, \ldots, m-1$. Therefore, $\frac{\partial}{\partial \kappa_i}(-g)(\boldsymbol{\kappa})$ is nonincreasing in $i = 1, \ldots, n$ all $\boldsymbol{\kappa} \in \mathcal{D} \cap \mathbb{R}^n_{++}$. Then by Theorem 2.4, $-g(\kappa_1, \ldots, \kappa_m)$ is a Schur-convex function in κ_i's, and hence the result follows.

□

In Theorem 3.7, since z is a fixed constant, we have that

$$g(\kappa_1, \ldots, \kappa_m) - \frac{1}{z} = \sum_{\substack{T \subseteq M \\ T \neq \emptyset}} (-1)^{|T|} \frac{1}{z + \sum_{i \in T} \kappa_i},$$

is also Schur-concave in κ_i's. Then it is trivial to see that the function

$$\sum_{\substack{T \subseteq M \\ T \neq \emptyset}} (-1)^{|T|+1} \frac{1}{\sum_{i \in T} \kappa_i} = - \lim_{z \to 0^+} \sum_{\substack{T \subseteq M \\ T \neq \emptyset}} (-1)^{|T|} \frac{1}{z + \sum_{i \in T} \kappa_i}$$

is Schur-convex in κ_i's. Hence, we have the following theorem.

Theorem 3.8. *Given a set $M = \{1, 2, \ldots, n\}$, let*

$$g^*(\kappa_1, \ldots, \kappa_n) = \frac{1}{\sum_{\substack{K \subseteq M \\ K \neq \emptyset}} (-1)^{|K|+1} \frac{1}{\sum_{i \in K} \kappa_i}},$$

where $\kappa_i > 0 \ \forall i \in M$. Then $g^*(\kappa_1, \ldots, \kappa_n)$ is a Schur-concave function in κ_i's.

Conclusions. We have the following theorem as conclusions.

Theorem 3.9. *Suppose $p_{\mathbf{0}} = 0$, then*

$$p_{\mathbf{e}(C)} = \frac{1}{\sum_{\ell \in C} \kappa_\ell} \times \left[\frac{1}{\sum_{\substack{K \subseteq C \\ K \neq \emptyset}} (-1)^{|K|+1} \frac{1}{\sum_{j \in K} \kappa_j}} \right]$$

is Schur-concave in κ_i's where $i \in C$. Furthermore, $p_{\mathbf{e}(C)} = Pr\{x_j = 1,$ for all $j \in C\}$ is the maximum whenever $\kappa_1 = \kappa_2 = \ldots = \kappa_n$.

Proof. By formula (3.2) $p_{\mathbf{e}(C)} = h(\kappa_1, \ldots, \kappa_n) \times g^*(\kappa_1, \ldots, \kappa_n)$, where

$$h(\kappa_1, \ldots, \kappa_n) = \frac{1}{\sum_{\ell \in C} \kappa_\ell} \text{ and } g^*(\kappa_1, \ldots, \kappa_n) = \left[\frac{1}{\sum_{\substack{K \subseteq C \\ K \neq \emptyset}} (-1)^{|K|+1} \frac{1}{\sum_{j \in K} \kappa_j}} \right].$$

First, notice that by Theorem 3.8 we have that $g^*(\kappa_1, \ldots, \kappa_n)$ is Schur-concave on $\mathbb{R}^n_{++}$ and hence $(-{g^*}_i) - (-{g^*}_{i+1}) \geq 0$ on $\mathcal{D} \cap \mathbb{R}^n_{++}$ by Theorem 2.4. Next, observe $h_i(\kappa_1, \ldots, \kappa_n) = \frac{-1}{\left[\sum_{\ell \in C} \kappa_\ell\right]^2}$ for all i, then we have

$$\begin{aligned}
(h \cdot g^*)_i - (h \cdot g^*)_{i+1} &= [h_i \cdot g^* + h \cdot {g^*}_i] - [h_{i+1} \cdot g^* + h \cdot {g^*}_{i+1}] \\
&= [h_i - h_{i+1}] \cdot g^* + h \cdot [{g^*}_i - {g^*}_{i+1}] \\
&= 0 \cdot g^* + h \cdot [{g^*}_i - {g^*}_{i+1}] \leq 0
\end{aligned}$$

on $\mathcal{D} \cap \mathbb{R}^n_{++}$. It is trivial to see $-h \cdot g^*$ is symmetric on $\mathbb{R}^n_{++}$, we have that $-h \cdot g^*$ is Schur-convex in κ_i's where $\kappa_i > 0, i \in C$ by Theoem 2.4. In views of $p_{\mathbf{e}(C)} = h \cdot g^*$ being Schur-concave and symmetric on $\mathbb{R}^n_{++}$, we see that $p_{\mathbf{e}(C)}$ attains maximum at $\kappa_1 = \kappa_2 = ... = \kappa_n$. □

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