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AN APPROXIMATION THEOREM OF LAX TYPE FOR EVOLUTION OPERATORS OF LIPSCHITZ OPERATORS IN A METRIC SPACE

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Abstract. In this paper, we consider a class of evolution operators of Lipschitz operators in a metric space, which includes the class proposed by Iwamiya, Oharu and Takahashi [1]. An analogue of Lax's theorem for semigroups of Lipschitz operators in Banach spaces due to Oharu and the authors [2] is extended to the case of the class of evolution operators. To this end, a stability condition and a generalized consistency condition are defined for a family of Lipschitz operators in a metric space.

Let X be a metric space with a metric $d(\cdot,\cdot)$ and $T \in (0,\infty)$. Let $\mathcal{F}$ denote the class of $f \in L^1([0,T)\times[0,T);[0,\infty))$ satisfying the following two conditions:

(f1) $f(r,r) = 0$ for $r \in [0,T)$.

(f2) For any $h \in (0,T)$ and $(r,s) \in [0,T-h) \times [0,T-h)$,

$$\limsup_{(\hat{r},\hat{s})\to(r,s)} \int_0^h f(\sigma+\hat{r},\sigma+\hat{s})\,d\sigma \le \int_0^h f(\sigma+r,\sigma+s)\,d\sigma.$$

Example 1 (Iwamiya-Oharu-Takahashi[1])**.** Let Y be a Banach space with norm $\|\cdot\|_Y$ and let

$$f(r,s) = \gamma(|r-s|) + \|g(r)-g(s)\|_Y \quad \text{for } r,s \in [0,T).$$

Then $f \in \mathcal{F}$ if γ is a continuous, nonnegative and non-decreasing function on $[0,T)$ such that $\gamma(0)=0$ and g is a Y-valued Bochner integrable function on $[0,T)$.

In this paper, let $\{D(t); t \in [0,T)\}$ denote a family of nonempty subsets of X and $D = \{(t,x); x \in D(t), t \in [0,T)\}$.

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Definition 2. A family $U = \{U(t,s); 0 \leq s \leq t < T\}$ of operators in X is called an evolution operator of Lipschitz operators on D if it satisfies the following conditions (E1) and (E2):

(E1) $U(s,r) : D(r) \to D(s)$, $U(r,r)x = x$ and $U(t,s)(U(s,r)x) = U(t,r)x$ for $x \in D(r)$ and $0 \leq r \leq s \leq t < T$.

(E2) If $0 \leq s \leq t < T$, $(s,x) \in D$, $0 \leq s_n \leq t_n < T$ for $n \geq 1$, $(s_n, x_n) \in D$ for $n \geq 1$, $\lim_{n\to\infty} s_n = s$, $\lim_{n\to\infty} t_n = t$ and $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} U(t_n, s_n)x_n = U(t,s)x$.

(E3) There exists $L \in (0,\infty)$ such that

$$d(U(t,s)x, U(t,s)y) \leq Ld(x,y)$$

for $(s,x), (s,y) \in D$ and $0 \leq s \leq t < T$.

By $\mathcal{E}(D)$ we denote the set of all evolution operators of Lipschitz operators on D. An evolution operator of Lipschitz operators on D was characterized in [3] by a dissipativity condition with respect to a family of metric-like functionals, a subtangential condition and a connectedness condition of D when its infinitesimal generator is continuous from D into X, where X is a real Banach space and d is the metric induced by its norm.

A family $U = \{U(t,s); 0 \leq s \leq t < T\}$ of operators in X satisfying condition (E1) is called an evolution operator of class $\mathcal{E}(D,f)$ if it satisfies the following conditions (E4) and (E5):

(E4) For $x \in D(s)$ and $s \in [0,T)$, the mapping $t \mapsto U(t,s)x$ is continuous on $[s,T)$ in X.

(E5) There exist $L \in (0,\infty)$ and $f \in \mathcal{F}$ such that

$$d(U(r+h,r)x, U(s+h,s)y) \leq L\left(d(x,y) + \int_0^h f(r+\sigma, s+\sigma)\,d\sigma\right)$$

for (r,x), $(s,y) \in D$ and $0 \leq r \leq s \leq s+h < T$.

Note that $\mathcal{E}(D,f) \subset \mathcal{E}(D)$. Indeed, assume that $0 \leq s \leq t < T$, $(s,x) \in D$, $0 \leq s_n \leq t_n < T$ for $n \geq 1$, $(s_n,x_n) \in D$ for $n \geq 1$, $\lim_{n\to\infty} s_n = s$, $\lim_{n\to\infty} t_n = t$ and $\lim_{n\to\infty} x_n = x$. Then there exists an integer $N \geq 1$ such that $t_n - s_n + \max(s_n, s) < T$ for $n \geq N$. Condition (E2) can be verified by using condition (E4) and the inequality

$$\begin{aligned} &d(U(s_n + (t_n - s_n), s_n)x_n, U(s + (t_n - s_n), s)x) \\ &\quad \leq L\left(d(x_n, x) + \int_0^{t_n - s_n} (f(s_n+\sigma, s+\sigma) + f(s+\sigma, s_n+\sigma))\,d\sigma\right) \end{aligned}$$

for $n \geq N$, which follows from condition (E5). The following condition (E6) implies condition (E5):

(E6) There exist $\omega \in [0,\infty)$ and $f \in \mathcal{F}$ such that

$$d(U(r+h,r)x, U(s+h,s)y) \le e^{h\omega} d(x,y) + \int_0^h e^{(h-\sigma)\omega} f(r+\sigma, s+\sigma)\, d\sigma$$

for (r,x), $(s,y) \in D$ and $0 \le r \le s \le s+h < T$.

A family $U = \{U(t,s); 0 \le s \le t < T\}$ of operators in X satisfying conditions (E1), (E4) and (E6) is a metric-setting version of the evolution operator proposed in [1].

For j, $k = 0, 1, 2, \ldots$, we define

$$\prod_{i=j}^{j} T_i = T_j, \quad \prod_{i=j}^{k+1} T_i = T_{k+1} \prod_{i=j}^{k} T_i \text{ if } j \le k \text{ and } \prod_{i=j}^{k} T_i = I \text{ otherwise.}$$

Let $C = \{C_h(t)\,;\, 0 \le t < T,\ 0 < h \le h_0\}$ be a family of operators in X. We consider the following conditions (C1) and (C2).

(C1) $C_h(r) : D(r) \to D(r+h)$ for $0 \le r \le r+h < T$ and $h \in (0, h_0]$.

(C2) There exist $L \in (0,\infty)$ and a family $\{f_h; 0 < h \le h_0\}$ in $\mathcal{F}$ such that

$$d\Big(\prod_{i=1}^{n} C_h(r+(i-1)h)x, \prod_{i=1}^{n} C_h(s+(i-1)h)y\Big) \le L\Big(d(x,y) + \int_0^{nh} f_h(r+\sigma, s+\sigma)\, d\sigma\Big)$$

for (r,x), $(s,y) \in D$, $h \in (0,h_0]$, $0 \le r \le s \le s+nh < T$ and $n = 1, 2, \ldots$.

The following provides a sufficient condition for stability condition (C2).

Proposition 3. *Assume that there exist $\omega \in [0,\infty)$ and a family $\{f_h; 0 < h \le h_0\}$ in $\mathcal{F}$ such that*

$$d(C_h(r)x, C_h(s)y) \le e^{h\omega} d(x,y) + \int_0^h e^{(h-\sigma)\omega} f_h(r+\sigma, s+\sigma)\, d\sigma$$

for (r,x), $(s,y) \in D$, $0 \le r \le s \le s+h < T$ and $h \in (0,h_0]$. Then condition (C2) *holds with $L = e^{T\omega}$.*

Proof. A straightforward induction argument on n gives

$$d\Big(\prod_{i=1}^{n} C_h(r+(i-1)h)x, \prod_{i=1}^{n} C_h(s+(i-1)h)y\Big) \le e^{nh\omega} d(x,y) + \int_0^{nh} e^{(nh-\sigma)\omega} f_h(r+\sigma, s+\sigma)\, d\sigma,$$

for (r,x), $(s,y) \in D$, $h \in (0,h_0]$, $0 \le r \le s \le s+nh < T$ and $n = 1, 2, \ldots$. □

Lemma 4. *Let u be a continuous function on $[0,T)$ to X such that $u(t) \in D(t)$ for $t \in [0,T)$. Let $C = \{C_h(t)\,;\, 0 \le t < T,\ 0 < h \le h_0\}$ be a family of operators in X. Assume conditions* (C1) *and* (C2) *to be satisfied. Let $h \in (0,h_0]$. Then for each integer $k \ge 1$ such that $kh < T$, the function $s \mapsto \prod_{i=1}^{k} C_h(s+(i-1)h)u(s)$ is continuous on $[0, T-kh)$.*

Proof. Let $h \in (0,h_0]$. We apply condition (C2) with $n = 1$ to obtain

$$\lim_{s\to r, y\to x, (s,y)\in D} C_h(s)y = C_h(r)x$$

for $(r,x) \in D$ with $r + h < T$. Assertion with $k = 1$ can be verified by this and the continuity of u. The conclusion follows inductively. $\square$

Lemma 5. *Let u be a continuous function on $[0,T)$ to X such that $u(t) \in D(t)$ for $t \in [0,T)$. Let $C = \{C_h(t)\,;\, 0 \le t < T,\ 0 < h \le h_0\}$ be a family of operators in X. Assume that conditions* (C1) *and* (C2) *are satisfied. Then*

$$\begin{aligned} (1) \qquad & d\Big(u(t), \prod_{i=1}^{[t/h]} C_h((i-1)h)u(0)\Big) \\ & \le \sup\{d(u(s),u(t))\,;\, s \in [0,T) \text{ such that } |s-t| \le h\} \\ & \quad + \frac{L}{h}\int_0^h \Big(d(u(s),u(0)) + \int_0^{[t/h]h} f_h(s+\sigma,\sigma)\,d\sigma \Big)\,ds \\ & \quad + \frac{L}{h}\int_0^{[t/h]h} d\big(u(s+h), C_h(s)u(s)\big)\,ds \end{aligned}$$

for $0 \le t < t+h < T$ and $h \in (0,h_0]$.

Proof. Let $0 \le t < t+h < T$ and $h \in (0,h_0]$. Define

$$\phi_h(s) = d\Big(\prod_{i=1}^{[t/h]-[s/h]} C_h(s+(i-1)h)u(s), \prod_{i=1}^{[t/h]} C_h((i-1)h)u(0) \Big)$$

for $s \ge 0$ with $[s/h] \le [t/h]$. From Lemma 4 we infer that the function ϕ_h is measurable. If $0 \le s < [t/h]h$, then $(s+h)/h < [t/h]+1$, and so we see that $[(s+h)/h] \le [t/h]$ and $\phi_h(s+h)$ makes sense. If $[t/h]h \le s < ([t/h]+1)h$, then $\phi_h(s) = d\left(u(s), \prod_{i=1}^{[t/h]} C_h((i-1)h)u(0)\right)$. If $0 \le s < h$, then $\phi_h(s) = d\left(\prod_{i=1}^{[t/h]} C_h(s+(i-1)h)u(s), \prod_{i=1}^{[t/h]} C_h((i-1)h)u(0)\right)$, and we apply condition (C2) to get $\phi_h(s) \le L\left(d(u(s),u(0)) + \int_0^{[t/h]h} f_h(s+\sigma,\sigma)\,d\sigma\right)$

for $0 \le s < h$. Therefore, we have

$$\begin{aligned}
&\int_0^{[t/h]h} \big(\phi_h(s+h) - \phi_h(s)\big)\, ds = \int_h^{([t/h]+1)h} \phi_h(s)\, ds - \int_0^{[t/h]h} \phi_h(s)\, ds \\
&= \int_{[t/h]h}^{([t/h]+1)h} \phi_h(s)\, ds - \int_0^h \phi_h(s)\, ds \\
&\ge \int_{[t/h]h}^{([t/h]+1)h} d\Big(u(s), \prod_{i=1}^{[t/h]} C_h((i-1)h)u(0)\Big)\, ds \\
&\quad - \int_0^h L\left(d(u(s),u(0)) + \int_0^{[t/h]h} f_h(s+\sigma,\sigma)\, d\sigma\right)\, ds.
\end{aligned}$$

The first term on the right-hand side is estimated as follows:

$$\begin{aligned}
&\int_{[t/h]h}^{([t/h]+1)h} d\Big(u(s), \prod_{i=1}^{[t/h]} C_h((i-1)h)u(0)\Big)\, ds \\
&\ge \int_{[t/h]h}^{([t/h]+1)h} \Big(d\Big(u(t), \prod_{i=1}^{[t/h]} C_h((i-1)h)u(0)\Big) - d(u(s),u(t))\Big)\, ds \\
&= hd\Big(u(t), \prod_{i=1}^{[t/h]} C_h((i-1)h)u(0)\Big) - \int_{[t/h]h}^{([t/h]+1)h} d(u(s),u(t))\, ds.
\end{aligned}$$

If $0 \le s < [t/h]h$, then

$$\begin{aligned}
&\phi_h(s+h) - \phi_h(s) \\
&\le d\Big(\prod_{i=1}^{[t/h]-[s/h]-1} C_h(s+ih)u(s+h), \prod_{i=0}^{[t/h]-[s/h]-1} C_h(s+ih)u(s)\Big) \\
&\le Ld\big(u(s+h), C_h(s)u(s)\big).
\end{aligned}$$

The inequality (1) can be obtained by combining these inequalities. □

Corollary 6. *Let u be a continuous function on $[0,T)$ to X such that $u(t) \in D(t)$ for $t \in [0,T)$. Let $U = \{U(t,s)\,;\, 0 \le s \le t < T\}$ be an evolution operator* *of class* $\mathcal{E}(D,f)$. *Then*

$$\begin{aligned}(2) \qquad & d\,(u(t), U([t/h]h,0)u(0)) \\ & \le \sup\{d(u(s),u(t))\,;\, s\in[0,T) \text{ such that } |s-t|\le h\} \\ & \quad + \frac{L}{h}\int_0^h \left(d(u(s),u(0)) + \int_0^{[t/h]h} f(s+\sigma,\sigma)\,d\sigma \right) ds \\ & \quad + \frac{L}{h}\int_0^{[t/h]h} d\big(u(s+h), U(s+h,s)u(s)\big)\,ds\end{aligned}$$

for $0\le t<t+2h<T$.

Proof. Assume that $U=\{U(t,s)\,;\,0\le s\le t<T\}$ is of class $\mathcal{E}(D,f)$. Let $0\le \hat{t}<\hat{t}+2\hat{h}<T$. Choose $T_0>0$ so that $\hat{t}+\hat{h}<T_0<T_0+\hat{h}<T$ and set $h_0=T-T_0$. Define

$$C_h(t)x = U(t+h,t)x$$

for $t\in[0,T_0)$, $h\in(0,h_0]$ and $x\in D(t)$. Then conditions (C1) and (C2) are satisfied with $f_h(r,s)=f(r,s)$ for $(r,s)\in[0,T_0)\times[0,T_0)$ and $T=T_0$. Therefore, (1) implies the desired inequality (2) with $t=\hat{t}$ and $h=\hat{h}$. □

The following is a direct consequence of Lemma 5.

Theorem 7 (Lax-Richtmyer's theorem [5]). *Let* $U=\{U(t,s)\,;\,0\le s\le t<T\}$ *be an evolution operator of class* $\mathcal{E}(D,f)$ *and* $C=\{C_h(t)\,;\,0\le t<T,\ 0<h\le h_0\}$ *be a family of operators in* X. *Assume that conditions* (C1) *and* (C2) *are satisfied and*

$$\lim_{h\downarrow 0}\frac{1}{h}\int_0^h\left(\int_0^{T-h} f_h(s+\sigma,\sigma)\,d\sigma\right)ds = 0.$$

Let $x\in D(0)$. *If the consistency condition*

$$\lim_{h\downarrow 0}\frac{1}{h}\int_0^{T-h} d\big(U(s+h,0)x, C_h(s)U(s,0)x\big)\,ds = 0$$

is satisfied, then

$$\lim_{h\downarrow 0}\prod_{i=1}^{[t/h]} C_h((i-1)h)x = U(t,0)x$$

uniformly for t *in any compact subinterval of* $[0,T)$.

The following is a well-known Van Kampen's theorem [4, Theorem 1.20.2].

Theorem 8. *Let u be a continuous function on $[0,T)$ to X such that $u(t) \in D(t)$ for $t \in [0,T)$. Let $U = \{U(t,s)\,;\, 0 \le s \le t < T\}$ be an evolution operator of class $\mathcal{E}(D,f)$. If*

$$\lim_{h\downarrow 0} \frac{1}{h}\int_0^{T-h} d\big(u(s+h), U(s+h,s)u(s)\big)\,ds = 0,$$

then $u(t) = U(t,0)u(0)$ for $0 \le t < T$.

Proof. Let $0 < T_0 < T$ and set $h_0 = T - T_0$. By condition (f2) we see that the function $s \to \int_0^{T_0} f(s+\sigma,\sigma)\,d\sigma$ is upper semicontinuous on $[0,h_0)$. Since

$$\frac{1}{h}\int_0^h \left(\int_0^{T_0-h} f(s+\sigma,\sigma)\,d\sigma\right) ds \le \frac{1}{h}\int_0^h \left(\int_0^{T_0} f(s+\sigma,\sigma)\,d\sigma\right) ds$$

for $h \in (0,h_0)$, and the right-hand side tends to zero as $h \to 0$, by condition (f1). From Corollary 6 we infer that $u(t) = U(t,0)u(0)$ for $0 \le t < T_0$. Since $T_0 \in (0,T)$ is arbitrary, the conclusion follows. □

Example 9 (Agarwal-Lakshmikantham [4])**.** Conditions (E1) and (E2) are not sufficient for the conclusion of Theorem 8. In fact, let $X = [0,\infty)$ and $d(x,y) = |x-y|$ for $x, y \in X$, and define

$$U(t,s)x = \left\{x^{1/3} + (t-s)/3\right\}^3 \text{ for } x \in D(s) = X \text{ and } 0 \le s \le t < T < \infty.$$

Then (E1) and (E2) are satisfied, but (E3) is not satisfied, since

$$\lim_{(x,y)\to(0,0)} \frac{|U(r+h,r)x - U(r+h,r)y|}{|x-y|} = \infty$$

for $0 \le r < r+h < T$. Let $u(t) = 0$ for $0 \le t < T$. Then

$$\frac{1}{h}\int_0^{T-h} |u(s+h) - U(s+h,s)u(s)|\,ds = \frac{T-h}{3}\left(\frac{h}{3}\right)^2 \to 0,$$

as $h \downarrow 0$, but $u(t) \ne U(t,0)u(0) = (t/3)^3$ for $0 \le t < T$.

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