# THE CQ PROJECTION METHOD AND EQUILIBRIUM PROBLEMS IN HADAMARD SPACES 

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#### Abstract

We consider equilibrium problems in Hadamard spaces and prove the convergence of an iterative sequence generated by the CQ projection method.


## 1. Introduction

An equilibrium problem was originally proposed by Blum and Oettli [1] in the setting of Banach spaces. This problem generalizes various kinds of nonlinear problems such as convex optimization problems, saddle point problems, fixed point problems, variational inequality problems, and others. In 2005, Combettes and Hirstoaga [2] considered a notion of the resolvent for an equilibrium problem in Hilbert spaces, and Kimura and Kishi [4] generalized their result to Hadamard spaces in 2017. The definition of an equilibrium problem is as follows:

Find $x \in K$ such that $f(x, y) \geq 0$ for all $y \in K$,
where $K$ is a nonempty closed convex subset of a geodesic space and $f: K \times$ $K \rightarrow \mathbb{R}$. Combettes and Hirstoaga proposed the following results for equilibrium problems: Let $H$ be a Hilbert space and $K$ a nonempty closed convex subset of $H$. The resolvent of a bifunction $f: K \times K \rightarrow \mathbb{R}$ is a set-valued operator $J_{f}: H \rightarrow 2^{K}$ defined by

$$
J_{f}(x)=\{z \in K: f(z, y)+\langle z-x, y-z\rangle \geq 0(\forall y \in K)\}
$$

for $x \in H$. We assume the following conditions for $f$.
Condition 1. Let $H$ be a Hilbert space and $K$ a nonempty closed convex subset of $H$. We suppose that a bifunction $f: K \times K \rightarrow \mathbb{R}$ satisfies the following conditions:
(i) $f(x, x)=0$ for any $x \in K$;

[^0](ii) $f(x, y)+f(y, x) \leq 0$ for any $x, y \in K$;
(iii) for every $x \in K, f(x, \cdot): K \rightarrow \mathbb{R}$ is lower semicontinuous and convex;
(iv) for every $y \in K, f(\cdot, y): K \rightarrow \mathbb{R}$ is upper hemicontinuous.

Theorem 1.1 (Combettes and Hirstoaga [2]). subset of $H$. Suppose that $f$ : $K \times K \rightarrow \mathbb{R}$ satisfies Condition 1 and let

$$
S_{f}=\{x \in K: f(x, y) \geq 0(\forall y \in K)\} .
$$

Then:
(i) $D\left(J_{f}\right)=H$;
(ii) $J_{f}$ is single-valued and firmly nonexpansive;
(iii) $F\left(J_{f}\right)=S_{f}$;
(iv) $S_{f}$ is closed and convex.

The result (i) was proved by Blum and Oettli [1]. To prove it, we need to use the Schauder fixed point theorem. However, we cannot apply a similar technique to a Hadamard space; the structure of convex combinations is different from that of a Hilbert space. Hence we need to assume a condition called the convex hull finite property to a Hadamard space.

In 2012, Kimura and Satô proved the CQ projection method in CAT(1) spaces. Since a Hadamard space is also CAT(1) space, it means that the CQ projection method in CAT(1) spaces can be applied to the case of Hadamard spaces. In this paper, we attempt to apply the CQ projection method in Hadamard spaces to equilibrium problems by using resolvent of the bifunction and obtain a convergence theorem to a solution to this problem.

## 2. Preliminaries

Let $X$ be a metric space. For $x, y \in X$, a mapping $c:[0, l] \rightarrow X$ is called a geodesic if $c$ satisfies $c(0)=x, c(l)=y$, and $d(c(u), c(v))=|u-v|$ for every $u, v \in[0, l]$. If for any points $x, y \in X$, there exists a geodesic with endpoints $x$ and $y$, then $X$ is called a geodesic metric space. In what follows, we assume that a geodesic always exists uniquely for each pair of endpoints. Such a space is said to be uniquely geodesic.

For a uniquely geodesic space $X$, the image of geodesic with endpoints $x, y \in X$ is denoted by $[x, y]$. For $x, y, z \in X$, a geodesic triangle $\triangle(x, y, z)$ is defined by $\triangle(x, y, z)=[y, z] \cup[z, x] \cup[x, y]$. For a triangle $\triangle(x, y, z) \subset X$, let $\triangle(\bar{x}, \bar{y}, \bar{z}) \subset \mathbb{E}^{2}$ be such that each corresponding edge has the same length as that of the original triangle. It is called a comparison triangle of $\triangle(x, y, z)$. A point $\bar{p} \in[\bar{x}, \bar{y}]$ is called a comparison point of $p \in[x, y]$ if $d(x, z)=d(\bar{x}, \bar{z})$.
$X$ is called a $\operatorname{CAT}(0)$ space if for every $p, q \in \triangle(x, y, z) \subset X$ and their corresponding points $\bar{p}, \bar{q} \in \triangle(\bar{x}, \bar{y}, \bar{z})$ satisfy that $d(p, q) \leq d_{\mathbb{E}^{2}}(\bar{p}, \bar{q})$, where $d_{\mathbb{E}^{2}}$
is the Euclidean metric. A Hadamard space is defined as a complete CAT(0) space.

Let $X$ be a geodesic metric space and $\left\{x_{n}\right\}$ a bounded sequence of $X$. For $x \in X$, we put $r\left(x,\left\{x_{n}\right\}\right)=\lim \sup _{n \rightarrow \infty} d\left(x, x_{n}\right)$. The asymptotic radius of $\left\{x_{n}\right\}$ is defined by $r\left(\left\{x_{n}\right\}\right)=\inf _{x \in X} r\left(x,\left\{x_{n}\right\}\right)$. Further, the asymptotic center of $\left\{x_{n}\right\}$ is defined by

$$
A C\left(\left\{x_{n}\right\}\right)=\left\{x \in X: r\left(x,\left\{x_{n}\right\}\right)=r\left(\left\{x_{n}\right\}\right)\right\} .
$$

If $A C\left(\left\{x_{n_{k}}\right\}\right)=\left\{x_{0}\right\}$ for any subsequences $\left\{x_{n_{k}}\right\}$ of $\left\{x_{n}\right\}$, that is, their asymptotic center consists of the unique element $x_{0}$, then we say $\left\{x_{n}\right\}$ is $\Delta$-convergent to $x_{0}$ and we denote it by $x_{n} \stackrel{\Delta}{\Delta} x_{0}$.

Lemma 2.1 (Kimura [3]). Let $X$ be a Hadamard space and $\left\{x_{n}\right\}$ a sequence in $X$. Suppose that $\left\{x_{n}\right\}$ is $\Delta$-convergent to $x \in X$ and $\left\{d\left(x_{n}, p\right)\right\}$ converges to $d(x, p)$ for some $p \in X$. Then $\left\{x_{n}\right\}$ converges to $x$.

Let $X$ be a Hadamard space and let $T$ be a mapping from $X$ to $X$. The set of all fixed points of $T$ is denoted by $F(T)$. We say $T$ is nonexpansive if $d(T x, T y) \leq d(x, y)$ for every $x, y \in X . T$ is said to be quasinonexpansive if $F(T) \neq \emptyset$ and $d(T x, z) \leq d(x, z)$ for every $x \in X$ and $z \in F(T)$.

A mapping $T: X \rightarrow X$ is said to be $\Delta$-demiclosed if $x_{0} \in F(T)$ whenever $\left\{x_{n}\right\}$ is $\Delta$-convergent to $x_{0}$ and $\lim _{n \rightarrow \infty} d\left(x_{n}, T x_{n}\right)=0 . T$ is said to be firmly metrically nonspreading if

$$
2 d(T x, T y)^{2} \leq d(x, T y)^{2}+d(T x, y)^{2}-d(x, T x)^{2}-d(y, T y)^{2}
$$

for any $x, y \in X$, see [6]. Moreover, a firmly metrically nonspreading mapping is nonexpansive and $\Delta$-demiclosed. If $F(T)$ is nonempty, then a firmly metrically nonspreading mapping is quasinonexpansive.

Let $X$ be a Hadamard space and $E$ a nonempty finite family of points of $X$. Then a convex hull of $E$ is defined by

$$
\overline{\mathrm{co}} E=\overline{\bigcup_{n=0}^{\infty} X_{n}},
$$

where $X_{0}=E, X_{n}=\left\{t u_{n-1} \oplus(1-t) v_{n-1}: u_{n-1}, v_{n-1} \in X_{n-1}, t \in[0,1]\right\}$ for $n \in \mathbb{N}$.

We say that a Hadamard space $X$ has the Convex Hull Finite Property (CHFP) if every continuous mapping $f: \overline{\mathrm{co}} E \rightarrow \overline{\mathrm{co}} E$ has a fixed point for every finite subset $E$ of $X$.

Following Kimura and Kishi [4], we assume the following conditions for $f$ when we consider an equilibrium problem.

Condition 2 (Kimura-Kishi [4]). Let $X$ be a Hadamard space and $K$ a nonempty closed convex subset of $X$. We suppose that a bifunction $f: K \times K \rightarrow$ $\mathbb{R}$ satisfies the following conditions:
(i) $f(x, x)=0$ for any $x \in K$;
(ii) $f(x, y)+f(y, x) \leq 0$ for any $x, y \in K$;
(iii) for every $x \in K, f(x, \cdot): K \rightarrow \mathbb{R}$ is lower semicontinuous and convex;
(iv) for every $y \in K, f(\cdot, y): K \rightarrow \mathbb{R}$ is upper hemicontinuous.

Following the definition of the resolvent on a Hilbert space, they define a resolvent on this space. Let $X$ be a Hadamard space and $K$ a nonempty closed convex subset of $X$. The resolvent of bifunction $f: K^{2} \rightarrow \mathbb{R}$ is a set-valued operator $J_{f}: X \rightarrow 2^{K}$ defined by

$$
J_{f}(x)=\left\{z \in K: f(z, y)+\frac{1}{2}\left(d(x, y)^{2}-d(x, z)^{2}-d(y, z)^{2}\right) \geq 0(\forall y \in K)\right\}
$$

for $x \in K$.
Theorem 2.2 (Kimura-Kishi [4]). Let $X$ be a Hadamard space with CHFP and let $K$ be a nonempty closed convex subset of $X$. Suppose that $F: K \times K \rightarrow \mathbb{R}$ satisfies Condition 2 and let

$$
J_{f}(x)=\left\{z \in K: f(z, y)+\frac{1}{2}\left(d(x, y)^{2}-d(x, z)^{2}-d(y, z)^{2}\right) \geq 0(\forall y \in K)\right\}
$$

Then,
(i) $D\left(J_{f}\right)=X$;
(ii) $J_{f}$ is single-valued, firmly metrically nonspreading, and $\Delta$-demiclosed;
(iii) $F\left(J_{f}\right)=S_{f}=\{x \in K: f(x, y) \geq 0(\forall y \in K)\}$;
(iv) $S_{f}$ is closed and convex.

## 3. The main result

We obtain the following convergence theorem of an iterative scheme to a solution to an equilibrium problem. The underlying space is a Hadamard space and the approximate sequence is generated by the CQ method.
Theorem 3.1. Let $X$ be a Hadamard space with CHFP and $K$ a closed convex subset of $X$. Suppose that $\{z \in K: d(u, z) \leq d(v, z)\}$ and $\left\{z \in K: d(u, z)^{2} \leq\right.$ $\left.d(v, z)^{2}-d(u, v)^{2}\right\}$ are convex for every $u, v \in X$. Suppose that the set $S=$ $\{x \in K: f(x, y) \geq 0(\forall y \in K)\}$ is nonempty. Let $\left\{\lambda_{n}\right\} \subset \mathbb{R}$ such that $0<\inf _{n \in \mathbb{N}} \lambda_{n} \leq \sup _{n \in \mathbb{N}} \lambda_{n}<\infty$, and
$J_{\lambda_{n} f}(x)=\left\{z \in K: \lambda_{n} f(z, y)+\frac{1}{2}\left(d(x, y)^{2}-d(x, z)^{2}-d(y, z)^{2}\right) \geq 0(\forall y \in K)\right\}$.

For a given initial point $x_{1} \in K$, generate a sequence $\left\{x_{n}\right\}$ as follows:

$$
\begin{aligned}
C_{n+1} & =\left\{z \in K: d\left(J_{\lambda_{n}} f\left(x_{n}\right), z\right) \leq d\left(x_{n}, z\right)\right\} \\
Q_{n+1} & =\left\{z \in K: d\left(x_{n}, z\right)^{2} \leq d\left(x_{1}, z\right)^{2}-d\left(x_{1}, x_{n}\right)^{2}\right\} \\
x_{n+1} & =P_{C_{n+1} \cap Q_{n+1}} x_{1}
\end{aligned}
$$

for each $n \in \mathbb{N}$. Then $\left\{x_{n}\right\}$ is well-defined and converges to $P_{S} x_{1} \in K$, where $P_{C}: K \rightarrow C$ is the metric projection of $C$ onto a nonempty closed convex subset $C$ of $K$.
Proof. We first show that $\left\{x_{n}\right\}$ is well-defined and $S \subset \bigcap_{n \in \mathbb{N}} C_{n+1} \cap Q_{n+1}$ by induction. An initial point $x_{1} \in K$ is given. We have

$$
\begin{aligned}
C_{2} & =\left\{z \in K:\left(J_{\lambda_{1} f}\left(x_{1}\right), z\right) \leq d\left(x_{1}, z\right)\right\}, \\
Q_{2} & =\left\{z \in K: d\left(x_{1}, z\right)^{2} \leq d\left(x_{1}, z\right)^{2}-d\left(x_{1}, x_{1}\right)^{2}\right\} \\
& =\left\{z \in K: d\left(x_{1}, z\right)^{2} \leq d\left(x_{1}, z\right)^{2}\right\} \\
& =K
\end{aligned}
$$

We prove $S \subset C_{2} \cap Q_{2}$. Let $z \in S=F\left(J_{\lambda_{1} f}\right)$. Since $J_{\lambda_{1} f}$ is firmly metrically nonspreading, it is quasinonexpansive. It follows that

$$
d\left(J_{\lambda_{1} f}\left(x_{1}\right), z\right) \leq d\left(x_{1}, z\right)
$$

and thus $z \in C_{2}$. We also have $z \in K=Q_{2}$. Therefore, $z \in C_{2} \cap Q_{2}$. Hence, we get $S \subset C_{2} \cap Q_{2}$.

Suppose that $x_{1}, x_{2}, \ldots, x_{k}$ are defined and both $C_{k+1}$ and $Q_{k+1}$ are closed convex subsets of $K$ such that $S \subset C_{k+1} \cap Q_{k+1}$ for fixed $k \in \mathbb{N}$. Then since $C_{k+1} \cap Q_{k+1}$ is a nonempty closed subset of $K$ by the assumption of the space, we can define $x_{k+1}=P_{C_{k+1} \cap Q_{k+1}} x_{1}$. Further, both $C_{k+2}$ and $Q_{k+2}$ are closed and convex. Let $z \in S$. Since $J_{\lambda_{k+1} f}$ is firmly metrically nonspreading, we get

$$
d\left(J_{\lambda_{k+1} f}\left(x_{k+1}\right), z\right) \leq d\left(x_{k}, z\right),
$$

and thus $z \in C_{k+2}$. This implies that $S \subset C_{k+2}$.
To prove $S \subset Q_{k+2}$, it is sufficient to show that $C_{k+1} \cap Q_{k+1} \subset Q_{k+2}$. For any $z \in C_{k+1} \cap Q_{k+1}$ and $\left.t \in\right] 0,1[$,

$$
t z \oplus(1-t) x_{k+1}=t z \oplus(1-t) P_{C_{k+1} \cap Q_{k+1}} x_{1} \in C_{k+1} \cap Q_{k+1}
$$

It follows that

$$
\begin{aligned}
d\left(x_{1}, x_{k+1}\right)^{2} & \leq d\left(x_{1}, t z \oplus(1-t) x_{k+1}\right)^{2} \\
& \leq t d\left(x_{1}, z\right)^{2}+(1-t) d\left(x_{1}, x_{k+1}\right)^{2}-t(1-t) d\left(z, x_{k+1}\right)^{2}
\end{aligned}
$$

and thus

$$
\begin{aligned}
t(1-t) d\left(x_{k+1}, z\right)^{2} & \leq t d\left(x_{1}, z\right)^{2}+(1-t) d\left(x_{1}, x_{k+1}\right)^{2}-d\left(x_{1}, x_{k+1}\right)^{2} \\
& =t d\left(x_{1}, z\right)^{2}-t d\left(x_{1}, x_{k+1}\right)^{2} .
\end{aligned}
$$

Dividing by $t$ and letting $t \rightarrow 0$, we have

$$
d\left(x_{k+1}, z\right)^{2} \leq d\left(x_{1}, z\right)^{2}-d\left(x_{1}, x_{k+1}\right)^{2}
$$

Thus $z \in Q_{k+2}$. Therefore, we have $C_{k+1} \cap Q_{k+1} \subset Q_{k+2}$. Hence, $C_{k+2} \cap$ $Q_{k+2}$ includes $S$. It follows by induction that $\left\{x_{n}\right\}$ is well-defined and $S \subset$ $\bigcap_{n \in \mathbb{N}} C_{n+1} \cap Q_{n+1}$.

Next, we show $\lim _{n \rightarrow \infty} d\left(J_{\lambda_{n} f}\left(x_{n}\right), x_{n}\right)=0$. For every $n \in \mathbb{N}$, we have

$$
d\left(x_{1}, x_{n}\right)=d\left(x_{1}, P_{C_{n} \cap Q_{n}} x_{1}\right) \leq d\left(x_{1}, P_{S} x_{1}\right)<\infty .
$$

It follows that $\sup _{n \in \mathbb{N}} d\left(x_{1}, x_{n}\right)<\infty$ and $\left\{x_{n}\right\}$ is bounded. Moreover, from the definition of $Q_{n+1}$, we get

$$
d\left(x_{1}, x_{n}\right)=d\left(x_{1}, P_{Q_{n+1}} x_{1}\right) \leq d\left(x_{1}, P_{C_{n+1} \cap Q_{n+1}} x_{1}\right)=d\left(x_{1}, x_{n+1}\right)
$$

for $n \in \mathbb{N}$. Thus $\left\{d\left(x_{1}, x_{n}\right)\right\}$ is an increasing real sequence so that it has a limit $a=\lim _{n \rightarrow \infty} d\left(x_{1}, x_{n}\right)<\infty$. Since $x_{n+1} \in Q_{n+1}$, we have

$$
d\left(x_{n}, x_{n+1}\right)^{2} \leq d\left(x_{1}, x_{n+1}\right)^{2}-d\left(x_{1}, x_{n}\right)^{2} .
$$

for $n \in \mathbb{N}$. Tending $n \rightarrow \infty$, we have

$$
0 \leq \liminf _{n \rightarrow \infty} d\left(x_{n}, x_{n+1}\right)^{2} \leq \limsup _{n \rightarrow \infty} d\left(x_{n}, x_{n+1}\right)^{2} \leq a-a=0
$$

Hence, $\lim _{n \rightarrow \infty} d\left(x_{n}, x_{n+1}\right)=0$. Since $x_{n+1} \in C_{n+1}$, it holds that

$$
d\left(J_{\lambda_{n} f}\left(x_{n}\right), x_{n+1}\right) \leq d\left(x_{n}, x_{n+1}\right) .
$$

These facts imply that

$$
\begin{aligned}
\lim _{n \rightarrow \infty} d\left(J_{\lambda_{n} f}\left(x_{n}\right), x_{n}\right) & \left.\leq \lim _{n \rightarrow \infty}\left(J_{\lambda_{n} f}\left(x_{n}\right), x_{n+1}\right)+d\left(x_{n+1}, x_{n}\right)\right) \\
& \leq \lim _{n \rightarrow \infty}\left(d\left(x_{n}, x_{n+1}\right)+d\left(x_{n+1}, x_{n}\right)\right) \\
& =2 \lim _{n \rightarrow \infty} d\left(x_{n}, x_{n+1}\right) \\
& =0 .
\end{aligned}
$$

Hence $\lim _{n \rightarrow \infty} d\left(J_{\lambda_{n} f}\left(x_{n}\right), x_{n}\right)=0$. In addition, since $\left\{x_{n}\right\}$ is bounded, so is $\left\{J_{\lambda_{n} f}\left(x_{n}\right)\right\}$.

Let $\left\{x_{n_{i}}\right\}$ be an arbitrary subsequence of $\left\{x_{n}\right\}$. Then, since $\sup _{n \in \mathbb{N}} d\left(x_{1}, x_{n}\right)<$ $\infty$, there exists a subsequence $\left\{x_{n_{i_{j}}}\right\}$ of $\left\{x_{n_{i}}\right\}$ such that $x_{n_{i_{j}}} \Delta x_{0}$ and $\left.\lambda_{n_{i_{j}}} \rightarrow \lambda_{0} \in\right] 0, \infty[$.

We first prove that $\left\{f\left(J_{\lambda_{n_{i_{j}}}} f\left(x_{0}\right), J_{\lambda_{0} f}\left(x_{0}\right)\right)\right\}$ is bounded. Since a lower semicontinuous convex function is also lower semicontinuous for $\Delta$-convergent sequences, using the fact that $\left\{J_{\lambda_{i_{i_{j}}}} f\left(x_{n_{i_{j}}}\right)\right\}$ is $\Delta$-convergent to $x_{0}$, we have

$$
\begin{aligned}
f\left(J_{\lambda_{0}}\left(x_{0}\right), x_{0}\right) & \leq \liminf _{j \rightarrow \infty} f\left(J_{\lambda_{0} f}\left(x_{0}\right), J_{\lambda_{n_{i_{j}}}}\left(x_{n_{i_{j}}}\right)\right) \\
& \leq \liminf _{j \rightarrow \infty}\left(-f\left(J_{\lambda_{n_{i_{j}}}} f\left(x_{n_{i_{j}}}\right), J_{\lambda_{0} f} f\left(x_{0}\right)\right)\right) \\
& \leq-\limsup _{j \rightarrow \infty} f\left(J_{\lambda_{n_{i_{j}}}} f\left(x_{n_{i_{j}}}\right), J_{\lambda_{0} f}\left(x_{0}\right)\right)
\end{aligned}
$$

and thus

$$
\limsup _{j \rightarrow \infty} f\left(J_{\lambda_{n_{i_{j}}}}\left(x_{n_{i_{j}}}\right), J_{\lambda_{0}} f\left(x_{0}\right)\right) \leq-f\left(J_{\lambda_{0}}\left(x_{0}\right), x_{0}\right)
$$

It follows that $\left\{f\left(J_{\lambda_{n_{i}}} f\left(x_{n_{i_{j}}}\right), J_{\lambda_{0} f}\left(x_{0}\right)\right)\right\}$ is bounded above. On the other hand, from the definition of the resolvent, we have

$$
\begin{aligned}
\lambda_{n_{i_{j}}} f\left(J_{\lambda_{n_{i_{j}}}} f\left(x_{n_{i_{j}}}\right), y\right)+\frac{1}{2}\left(d(x, y)^{2}-d\left(x, J_{\lambda_{n_{i_{j}}} f}\right.\right. & \left.\left(x_{n_{i_{j}}}\right)\right)^{2} \\
& \left.-d\left(y, J_{\lambda_{n_{i_{j}}}} f\left(x_{n_{i_{j}}}\right)\right)^{2}\right) \geq 0
\end{aligned}
$$

for all $y \in K$. Letting $y=J_{\lambda_{0} f}\left(x_{0}\right)$, we have

$$
\begin{aligned}
& f\left(J_{\lambda_{n_{i_{j}}}} f\left(x_{n_{i_{j}}}\right), J_{\lambda_{0} f}\left(x_{0}\right)\right) \\
\geq & -\frac{1}{2 \lambda_{n_{i_{j}}}}\left(d\left(x, J_{\lambda_{0} f}\left(x_{0}\right)\right)^{2}-d\left(x, J_{\lambda_{n_{i_{j}}} f}\left(x_{n_{i_{j}}}\right)\right)^{2}-d\left(J_{\lambda_{0} f}\left(x_{0}\right), J_{\lambda_{n_{i_{j}}} f}\left(x_{n_{i_{j}}}\right)\right)^{2}\right) .
\end{aligned}
$$

Since $\inf _{n \in \mathbb{N}} \lambda_{n}>0$ and $\left\{J_{\lambda_{n}}\left(x_{n}\right)\right\}$ is bounded, we get $\left\{f\left(J_{\lambda_{n_{i_{j}}} f}\left(x_{n_{i_{j}}}\right), J_{\lambda_{0} f}\left(x_{0}\right)\right)\right\}$ is bounded below. Consequently, we obtain $\left\{f\left(J_{\lambda_{n_{i_{j}}} f}\left(x_{n_{i_{j}}}\right), J_{\lambda_{0} f}\left(x_{0}\right)\right)\right\}$ is a bounded real sequence. We put

$$
M=\sup _{j \in \mathbb{N}} f\left(J_{\lambda_{n_{i_{j}}}}\left(x_{n_{i_{j}}}\right), J_{\lambda_{0} f}\left(x_{0}\right)\right) .
$$

We next show that $\lim \sup _{j \rightarrow \infty} d\left(J_{\lambda_{n_{i_{j}}}}\left(x_{0}\right), J_{\lambda_{0} f}\left(x_{0}\right)\right)=0$. From the definition of the resolvent, we have
$\lambda_{n_{i_{j}}} f\left(J_{\lambda_{n_{i_{j}}}} f\left(x_{0}\right), y\right)+\frac{1}{2}\left(d\left(x_{0}, y\right)^{2}-d\left(x_{0}, J_{\lambda_{n_{i_{j}}} f}\left(x_{0}\right)\right)^{2}-d\left(y, J_{\lambda_{n_{i_{j}}}} f\left(x_{0}\right)\right)^{2}\right) \geq 0$ for every $y \in K$. Letting $y=J_{\lambda_{0} f}\left(x_{0}\right)$, we have

$$
\begin{aligned}
& \lambda_{n_{i_{j}}} f\left(J_{\lambda_{n_{i_{j}}}} f\left(x_{0}\right), J_{\lambda_{0} f}\left(x_{0}\right)\right) \\
& +\frac{1}{2}\left(d\left(x_{0}, J_{\lambda_{0} f}\left(x_{0}\right)\right)^{2}-d\left(x_{0}, J_{\lambda_{n_{i_{j}}} f}\left(x_{0}\right)\right)^{2}-d\left(J_{\lambda_{0} f}\left(x_{0}\right), J_{\lambda_{n_{i_{j}}}} f\left(x_{0}\right)\right)^{2}\right) \geq 0
\end{aligned}
$$

In a similar way, We also have

$$
\begin{aligned}
& \lambda_{0} f\left(J_{\lambda_{0} f}\left(x_{0}\right), J_{\lambda_{n_{i_{j}}}} f\left(x_{0}\right)\right) \\
& +\frac{1}{2}\left(d\left(x_{0}, J_{\lambda_{n_{i_{j}}}} f\left(x_{0}\right)\right)^{2}-d\left(x_{0}, J_{\lambda_{0} f}\left(x_{0}\right)\right)^{2}-d\left(J_{\lambda_{0} f}\left(x_{0}\right), J_{\lambda_{n_{i_{j}}}} f\left(x_{0}\right)\right)^{2}\right) \geq 0 .
\end{aligned}
$$

Summing both sides of these inequalities, we get

$$
\left.\left.\begin{array}{rl}
\lambda_{n_{i_{j}}} f & \left(J_{\lambda_{n_{i_{j}}} f}( \right.
\end{array} x_{0}\right), J_{\lambda_{0} f}\left(x_{0}\right)\right) .
$$

By Condition 2 (ii), we have

$$
\lambda_{0} f\left(J_{\lambda_{0} f}\left(x_{0}\right), J_{\lambda_{n_{i_{j}}} f}\left(x_{0}\right)\right) \leq-\lambda_{0} f\left(J_{\lambda_{n_{i_{j}}} f}\left(x_{0}\right), J_{\lambda_{0} f}\left(x_{0}\right)\right) .
$$

Therefore,

$$
\begin{aligned}
\lambda_{n_{i_{j}}} f\left(J_{\lambda_{n_{i_{j}}} f}\right. & \left.\left(x_{0}\right), J_{\lambda_{0} f}\left(x_{0}\right)\right) \\
& -\lambda_{0} f\left(J_{\lambda_{n_{i_{j}}} f}\left(x_{0}\right), J_{\lambda_{0} f}\left(x_{0}\right)\right)-d\left(J_{\lambda_{0} f}\left(x_{0}\right), J_{\lambda_{n_{i_{j}}}} f\left(x_{0}\right)\right)^{2} \geq 0 .
\end{aligned}
$$

Thus, we have

$$
d\left(J_{\lambda_{0} f}\left(x_{0}\right), J_{\lambda_{n_{i_{j}}} f}\left(x_{0}\right)\right)^{2} \leq\left(\lambda_{n_{i_{j}}}-\lambda_{0}\right) f\left(J_{\lambda_{n_{i_{j}}}}\left(x_{0}\right), J_{\lambda_{0} f}\left(x_{0}\right)\right) \leq\left|\lambda_{n_{i_{j}}}-\lambda_{0}\right| M .
$$

Therefore we obtain

$$
\limsup _{j \rightarrow \infty} d\left(J_{\lambda_{n_{i_{j}}}} f\left(x_{0}\right), J_{\lambda_{0} f}\left(x_{0}\right)\right)=0 .
$$

We show $x_{0}$ is a fixed point of $J_{\lambda_{0} f}$ by contradiction. If $x_{0} \neq J_{\lambda_{0} f}\left(x_{0}\right)$, then

$$
\begin{aligned}
& \limsup _{j \rightarrow \infty} d\left(x_{n_{i_{j}}}, x_{0}\right) \\
& <\limsup _{j \rightarrow \infty} d\left(x_{n_{i_{j}}}, J_{\lambda_{0} f}\left(x_{0}\right)\right) \\
& \leq \limsup _{j \rightarrow \infty} d\left(x_{n_{i_{j}}}, J_{\lambda_{n_{i_{j}}}}\left(x_{n_{i_{j}}}\right)\right)+\underset{j \rightarrow \infty}{\limsup } d\left(J_{\lambda_{n_{i_{j}}} f}\left(x_{n_{i_{j}}}\right), J_{\lambda_{0} f}\left(x_{0}\right)\right) \\
& \leq \limsup _{j \rightarrow \infty} d\left(J_{\lambda_{n_{i_{j}}} f}\left(x_{n_{i_{j}}}\right), J_{\lambda_{n_{i_{j}}}} f\left(x_{0}\right)\right)+\limsup _{j \rightarrow \infty} d\left(J_{\lambda_{n_{i_{j}}} f}\left(x_{0}\right), J_{\lambda_{0} f}\left(x_{0}\right)\right) \\
& \leq \limsup _{j \rightarrow \infty} d\left(x_{n_{i_{j}}}, x_{0}\right)+\limsup _{j \rightarrow \infty} d\left(J_{\lambda_{n_{i_{j}}} f}\left(x_{0}\right), J_{\lambda_{0} f}\left(x_{0}\right)\right) \\
& =\limsup _{j \rightarrow \infty} d\left(x_{n_{i_{j}}}, x_{0}\right),
\end{aligned}
$$

which is a contradiction. Hence $x_{0}=J_{\lambda_{0} f}\left(x_{0}\right)$.

Since $x_{n_{i_{j}}}=P_{C_{n_{i_{j}}} \cap Q_{n_{i_{j}}}} x_{1} \in C_{n_{i_{j}}} \cap Q_{n_{i_{j}}}$ and $S \subset C_{n_{i_{j}}} \cap Q_{n_{i_{j}}}$ for $j \in \mathbb{N}$, we have

$$
\begin{aligned}
& d\left(x_{1}, P_{S} x_{1}\right) \leq d\left(x_{1}, x_{0}\right) \\
& \leq \liminf _{j \rightarrow \infty} d\left(x_{1}, x_{n_{i_{j}}}\right) \leq \limsup _{j \rightarrow \infty} d\left(x_{1}, x_{n_{i_{j}}}\right) \\
&=\limsup _{j \rightarrow \infty} d\left(x_{1}, P_{C_{n_{i_{j}}}} \cap Q_{n_{i_{j}}}\right. \\
&\left.x_{1}\right) \leq d\left(x_{1}, P_{F} x_{1}\right) .
\end{aligned}
$$

Thus, we have

$$
\lim _{j \rightarrow \infty} d\left(x_{1}, x_{n_{i_{j}}}\right)=d\left(x_{1}, x_{0}\right)=d\left(x_{1}, P_{S} x_{1}\right)
$$

Thus we get $x_{0}=P_{S} x_{1}$. By Lemma 2.1, the facts that $x_{n_{i_{j}}} \xrightarrow{\Delta} x_{0}=P_{S} x_{1}$ and $\lim _{j \rightarrow \infty} d\left(x_{1}, x_{n_{i_{j}}}\right)=d\left(x_{1}, P_{S} x_{1}\right)$ imply $x_{n_{i_{j}}} \rightarrow P_{S} x_{1}$. Consequently, we obtain $x_{n} \rightarrow P_{S} x_{1}$, which is the desired result.

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