Proceedings of the International Conference on
Nonlinear Analysis and Convex Analysis &
International Conference on Optimization:
Techniques and Applications -I-
(Hakodate, Japan, 2019), 291–301

CONVERGENCE OF A SEQUENCE GENERATED BY MULTIPLE ANCHOR POINTS IN HADAMARD SPACES

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Dedicated to Professor Wataru Takahashi on his 75th birthday

Abstract. We consider a generalized approximation method of Browder type in a Hadamard space with multiple anchor points. We obtain its convergence to a minimizer of a certain function defined by anchor points on the set of all fixed points of a mapping.

1. Introduction

Approximation of fixed points of a nonexpansive mapping is a central topic in nonlinear analysis and it has been investigated by a large number of mathematicians. There are many kinds of approximation methods. We will focus on the sequences metrically converging to a fixed point of a given mapping.

One of the most important methods is a convergence theorem proved by Browder [3] as follows: Let T be a nonexpansive mapping defined on a closed convex subset X of a uniformly convex Banach space satisfying certain conditions. Then, for $u \in X$ and $t \in]0, 1[$, there exists a unique point $x_t \in X$ such that $x_t = tu + (1-t)Tx_t$. Moreover $\{x_t\}$ converges to a fixed point of T as $t \to 0$, which is closest to u.

Another important method is the iterative scheme of Halpern type. This sequence is defined as follows: For a mapping $T\colon X \to X$ and given points $u, x_1 \in X$, let

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n$$

for $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a coefficient sequence for convex combination. In 1992, Wittmann [10] proved that if T is a nonexpansive mapping defined on a closed convex subset X of a Hilbert space, then $\{x_n\}$ converges to a fixed point of T which is closest to u under certain conditions of $\{\alpha_n\}$.

2010 *Mathematics Subject Classification.* 47H09.
Key words and phrases. Nonexpansive mapping, fixed point, approximation, Browder type, Hadmard space.

These methods have been studied in various generalized settings and a large number of related results have been proved. We focus on the fact that these two approximating sequences have the same limit point, a fixed point closest to u. A point u is called an anchor point of the scheme. Kimura and Wada [7] considered the iterative scheme of Halpern type with multiple anchor points in the setting of a Hadamard space.

In this work, we consider a generalized approximation method of Browder type with multiple anchor points in Hadamard spaces. The sequence generated by this method converges to a minimizer of a certain function defined by anchor points on the set of all fixed points of a mapping. We use a notion of generalized convex combination among more than two points, which is proposed by [4].

2. Preliminaries

Let X be a metric space. For $x, y \in X$, a geodesic $c\colon [0, l] \to X$ connecting these two points is defined as a mapping satisfying $l = d(x, y)$, $c(0) = x$, $c(l) = y$, and $d(c(s), c(t)) = |s - t|$ for any $s, t \in [0, l]$. If a geodesic exists for every two points, the space X is called a geodesic space. In what follows, a geodesic c is assumed to be unique for each $x, y \in X$ and the image of c is denoted by $[x, y]$. Such a space is referred to as a uniquely geodesic space. In this case, c is a unique isometry from $[0, d(x, y)]$ to $[x, y]$.

In a uniquely geodesic space, we can define a convex combination between two points as follows. For $x, y \in X$ and $t \in [0, 1]$, there exists a unique $z \in [x, y]$ such that $d(x, z) = (1 - t)d(x, y)$ and $d(y, z) = td(x, y)$. We denote it by $z = tx \oplus (1 - t)y$. Using this notion, we define a convex subset of X in a natural way; $C \subset X$ is convex if $tx \oplus (1 - t)y \in C$ for any $x, y \in C$ and $t \in [0, 1]$.

A Hadamard space is defined as a complete geodesic space having a certain geometric property. This property is usually described by using a comparison triangle defined on a model space; see [1, 2] for instance. We define it by the following equivalent form. Namely, a Hadamard space is a complete uniquely geodesic space satisfying that

$$d(tx \oplus (1 - t)y, z)^2 \le td(x, z)^2 + (1 - t)d(y, z)^2 - t(1 - t)d(x, y)^2$$

for every $x, y, z \in X$ and $t \in [0, 1]$. We know that the following inequality holds [2]:

$$d(u, x)^2 + d(v, y)^2 - d(v, x)^2 - d(u, y)^2 \le 2d(u, v)d(x, y)$$

for any four points $u, v, x, y \in X$.

Let $\{x_n\}$ be a bounded sequence in a metric space X. We say that z is an asymptotic center of $\{x_n\}$ if it satisfies

$$\limsup_{n\to\infty} d(x_n, z) = \inf_{y\in X} \limsup_{n\to\infty} d(x_n, y).$$

In a Hadamard space, an asymptotic center of every bounded sequence is known to be unique. A bounded sequence $\{x_n\}$ is said to be Δ-convergent [9] to z if an asymptotic center of every subsequence of $\{x_n\}$ is equal to z. It is known that every bounded sequence in a Hadamard space has a Δ-convergent subsequence. We also know [5] that if $\{x_n\}$ is a Δ-convergent sequence in a Hadamard space with a Δ-limit z, then

$$d(z, y) \leq \liminf_{n\to\infty} d(x_n, y)$$

for every $y \in X$. For more details of Δ-convergence, see [8, 9].

The following lemma describes the relation between Δ-convergence and convergence in metric, which is known as the Kadec-Klee property in Banach spaces.

Lemma 2.1 (Kimura [6]). *Let $\{x_n\}$ be a Δ-convergent sequence in a Hadamard space X with its Δ-limit $x \in X$. If $\{d(x_n, u)\}$ converges to $d(x, u)$ for some $u \in X$, then $\{x_n\}$ converges to x.*

Let X be a Hadamard space. A mapping $T\colon X \to X$ is called a nonexpansive mapping if

$$d(Tx, Ty) \leq d(x, y)$$

for all $x, y \in X$. We denote the set of fixed points of T by $\operatorname{Fix} T$, that is

$$\operatorname{Fix} T = \{z \in X \mid Tz = z\}.$$

We know that $\operatorname{Fix} T$ is closed and convex, which is the same as the case where the underlying space is a Hilbert space.

A contraction $U\colon X \to X$ is defined as a mapping such that for some $r \in [0, 1[$, it satisfies that

$$d(Ux, Uy) \leq r d(x, y)$$

for every $x, y \in X$. The famous Banach contraction principle guarantees the existence and uniqueness of a fixed point of U.

3. A convergent sequence to a fixed point

In the main result of this work, the following lemma, which is essentially proved in [4], plays an important role.

Lemma 3.1 (Hasegawa-Kimura [4])**.** *Let X be a Hadamard space and C a nonempty closed convex subset of X. Let $u_1, u_2, \dots, u_r \in X$ and $\beta_1, \beta_2, \dots, \beta_r \in [0,1]$ such that $\sum_{k=1}^r \beta_k = 1$. Define a function $f\colon C \to \mathbb{R}$ by*

$$f(y) = \sum_{k=1}^{r} \beta_k d(y, u_k)^2$$

for every $y \in C$. Then, f has a unique minimizer on C.

The next theorem guarantees the well-definedness of the sequence considered in our main result.

Theorem 3.2. *Let X be a Hadamard space and $T\colon X \to X$ a nonexpansive mapping. Let $u_1, u_2, \dots, u_r$ be points in X. Let $\alpha \in]0,1[$ and $\beta^1, \beta^2, \dots, \beta^r \in [0,1]$ such that $\sum_{k=1}^r \beta^k = 1$. Then there exists a unique point $x_0 \in X$ satisfying*

$$x_0 \in \operatorname*{argmin}_{y \in X} \left(\alpha \sum_{k=1}^{r} \beta^k d(y, u_k)^2 + (1-\alpha) d(y, Tx_0)^2 \right).$$

Proof. Define a mapping $U\colon X \to X$ by

$$Ux = \operatorname*{argmin}_{y \in X} \left(\alpha \sum_{k=1}^{r} \beta^k d(y, u_k)^2 + (1-\alpha) d(y, Tx)^2 \right)$$

for every $x \in X$. Then, by Lemma 3.1, U is well defined as a single-valued mapping on X. Moreover, we may show that U is a contraction. Indeed, for $x, x' \in X$ and $t \in]0,1[$, we have

$$\begin{aligned}
&\alpha \sum_{k=1}^{r} \beta^k d(Ux, u_k)^2 + (1-\alpha) d(Ux, Tx)^2 \\
&\le \alpha \sum_{k=1}^{r} \beta^k d(tUx \oplus (1-t)Ux', u_k)^2 + (1-\alpha) d(tUx \oplus (1-t)Ux', Tx)^2 \\
&\le \alpha \sum_{k=1}^{r} \beta^k \left(td(Ux, u_k)^2 + (1-t) d(Ux', u_k)^2 - t(1-t) d(Ux, Ux')^2 \right) \\
&\quad + (1-\alpha) \left(td(Ux, Tx)^2 + (1-t) d(Ux', Tx)^2 - t(1-t) d(Ux, Ux')^2 \right) \\
&\le t \left(\alpha \sum_{k=1}^{r} \beta^k d(Ux, u_k)^2 + (1-\alpha) d(Ux, Tx)^2 \right) \\
&\quad + (1-t) \left(\alpha \sum_{k=1}^{r} \beta^k d(Ux', u_k)^2 + (1-\alpha) d(Ux', Tx)^2 \right) - t(1-t) d(Ux, Ux')^2,
\end{aligned}$$

and thus

$$\alpha \sum_{k=1}^{r} \beta^k d(Ux, u_k)^2 + (1-\alpha) d(Ux, Tx)^2 \\ \leq \alpha \sum_{k=1}^{r} \beta^k d(Ux', u_k)^2 + (1-\alpha) d(Ux', Tx)^2 - t d(Ux, Ux')^2.$$

Letting $t \to 1$, we have

$$d(Ux, Ux')^2 \leq \alpha \sum_{k=1}^{r} \beta^k \left(d(Ux, u_k)^2 - d(Ux', u_k)^2 \right) \\ + (1-\alpha) \left(d(Ux, Tx)^2 - d(Ux', Tx)^2 \right).$$

In the same way, we also have

$$d(Ux', Ux)^2 \leq \alpha \sum_{k=1}^{r} \beta^k \left(d(Ux', u_k)^2 - d(Ux, u_k)^2 \right) \\ + (1-\alpha) \left(d(Ux', Tx')^2 - d(Ux, Tx')^2 \right).$$

Summing up these inequalities, we obtain

$$\begin{aligned} &2d(Ux', Ux)^2 \\ &\leq (1-\alpha) \left(d(Ux, Tx)^2 - d(Ux', Tx)^2 + d(Ux', Tx')^2 - d(Ux, Tx')^2 \right) \\ &\leq 2(1-\alpha) d(Tx, Tx') d(Ux, Ux') \\ &\leq 2(1-\alpha) d(x, x') d(Ux, Ux'), \end{aligned}$$

and hence

$$d(Ux', Ux) \leq (1-\alpha) d(x, x').$$

Since $0 < \alpha < 1$, it implies that U is a contraction. Thus there exists a unique fixed point $x_0 \in X$. That is,

$$x_0 = Ux_0 = \operatorname*{argmin}_{y \in X} \left(\alpha \sum_{k=1}^{r} \beta^k d(y, u_k)^2 + (1-\alpha) d(y, Tx_0)^2 \right).$$

This is the desired result. □

Now we show our main result. The following is a convergence theorem of the Browder type with multiple anchor points in Hadamard spaces.

Theorem 3.3. *Let X be a Hadamard space and $T\colon X \to X$ a nonexpansive mapping such that $\operatorname{Fix} T \neq \emptyset$. Let $u_1, u_2, \ldots, u_r$ be a finite family of anchor points in X. Let $\{\alpha_n\} \subset]0,1[$ be a real sequence such that $\alpha_n \to 0$ as $n \to \infty$,* *and for $k = 1, 2, \dots, r$, let $\{\beta_n^k\} \subset [0,1]$ be real sequences such that $\sum_{k=1}^r \beta_n^k = 1$ for every $n \in \mathbb{N}$ and $\beta_n^k \to \beta^k \in [0,1]$ as $n \to \infty$. Define $\{x_n\} \subset X$ by*

$$x_n = \operatorname*{argmin}_{y \in X} \left(\alpha_n \sum_{k=1}^r \beta_n^k d(y, u_k)^2 + (1 - \alpha_n) d(y, Tx_n)^2 \right)$$

for each $n \in \mathbb{N}$. Then $\{x_n\}$ converges to a unique minimizer of a function g on $\operatorname{Fix} T$ as $n \to \infty$, where $g \colon X \to \mathbb{R}$ is defined by

$$g(y) = \sum_{k=1}^r \beta^k d(y, u_k)^2$$

for $y \in X$.

Proof. We know that Theorem 3.2 imples the well-definedness of x_n for every $n \in \mathbb{N}$. It is also known by Lemma 3.1 that there exists a unique minimizer p of g on $\operatorname{Fix} T$, that is,

$$p = \operatorname*{argmin}_{y \in \operatorname{Fix} T} \sum_{k=1}^r \beta^k d(y, u_k)^2.$$

Then, from the definition of x_n, for $t \in]0,1[$ we have

$$\begin{aligned}
&\alpha_n \sum_{k=1}^r \beta_n^k d(x_n, u_k)^2 + (1 - \alpha_n) d(x_n, Tx_n)^2 \\
&\le \alpha_n \sum_{k=1}^r \beta_n^k d(tx_n \oplus (1-t)p, u_k)^2 + (1 - \alpha_n) d(tx_n \oplus (1-t)p, Tx_n)^2 \\
&\le t \left(\alpha_n \sum_{k=1}^r \beta_n^k d(x_n, u_k)^2 + (1 - \alpha_n) d(x_n, Tx_n)^2 \right) \\
&\quad + (1-t) \left(\alpha_n \sum_{k=1}^r \beta_n^k d(p, u_k)^2 + (1 - \alpha_n) d(p, Tx_n)^2 \right) - t(1-t) d(x_n, p)^2
\end{aligned}$$

and thus it follows that

$$\begin{aligned}
&\alpha_n \sum_{k=1}^{r} \beta_n^k d(x_n, u_k)^2 + (1-\alpha_n) d(x_n, Tx_n)^2 \\
&\le \alpha_n \sum_{k=1}^{r} \beta_n^k d(p, u_k)^2 + (1-\alpha_n) d(p, Tx_n)^2 - t d(x_n, p)^2 \\
&\le \alpha_n \sum_{k=1}^{r} \beta_n^k d(p, u_k)^2 + (1-\alpha_n) d(p, x_n)^2 - t d(x_n, p)^2.
\end{aligned}$$

Letting $t \to 1$, we have

$$\alpha_n \sum_{k=1}^{r} \beta_n^k d(x_n, u_k)^2 + (1-\alpha_n) d(x_n, Tx_n)^2 \le \alpha_n \left(\sum_{k=1}^{r} \beta_n^k d(p, u_k)^2 - d(p, x_n)^2 \right). \tag{3.1}$$

From this inequality, we obtain

$$0 \le \sum_{k=1}^{r} \beta_n^k d(p, u_k)^2 - d(p, x_n)^2$$

and thus

$$\begin{aligned}
d(p, x_n)^2 &\le \sum_{k=1}^{r} \beta_n^k d(p, u_k)^2 \\
&\le \max_{k \in \{1,2,\dots,r\}} d(p, u_k)^2
\end{aligned}$$

for every $n \in \mathbb{N}$. It implies that $\{x_n\}$ is bounded. By (3.1), we also have

$$\begin{aligned}
\alpha_n \sum_{k=1}^{r} \beta_n^k d(x_n, u_k)^2 &\le \alpha_n \left(\sum_{k=1}^{r} \beta_n^k d(p, u_k)^2 - d(p, x_n)^2 \right) \\
&\le \alpha_n \sum_{k=1}^{r} \beta_n^k d(p, u_k)^2.
\end{aligned}$$

Since $\alpha_n > 0$, it follows that

$$\sum_{k=1}^{r} \beta_n^k d(x_n, u_k)^2 \le \sum_{k=1}^{r} \beta_n^k d(p, u_k)^2 \tag{3.2}$$

for every $n \in \mathbb{N}$.

To show that $\{x_n\}$ is Δ-convergent to p, we take a subsequence $\{x_{n_i}\} \subset \{x_n\}$ arbitrarily with its asymptotic center $v \in X$, and will prove $v = p$. Taking a subsequence repeatedly, we can find a subsequence $\{x'_j\}$ of $\{x_{n_i}\}$ such that

$$\lim_{j\to\infty} d(x'_j, p) = \limsup_{i\to\infty} d(x_{n_i}, p)$$

and $\{x'_j\}$ is Δ-convergent to some $q \in X$. We first show that $q = p$. Let $\{\alpha'_j\}$ and $\{\beta'^k_j\}$ be subsequences of $\{\alpha_{n_i}\}$ and $\{\beta^k_{n_i}\}$ corresponding to $\{x'_j\}$, respectively. That is, since we may write $x'_j = x_{n_{i_j}}$, using this subscript, we define $\beta'^k_j = \beta^k_{n_{i_j}}$ and $\alpha'_j = \alpha_{n_{i_j}}$ for every $j \in \mathbb{N}$. Then, by (3.1) we have

$$0 \le (1-\alpha'_j) d(x'_j, Tx'_j)^2 \le \alpha'_j \left(\sum_{k=1}^{r} \beta'^k_j d(p, u_k)^2 - d(p, x'_j)^2 \right).$$

Since $\alpha'_j \to 0$ as $j \to \infty$ and $\{x'_j\}$ is bounded, we have $\lim_{j\to\infty} d(x'_j, Tx'_j) = 0$. It follows that

$$\begin{aligned}\limsup_{j\to\infty} d(x'_j, Tq) &\le \limsup_{j\to\infty} (d(x'_j, Tx'_j) + d(Tx'_j, Tq)) \\ &\le \limsup_{j\to\infty} d(x'_j, Tx'_j) + \limsup_{j\to\infty} d(Tx'_j, Tq) \\ &\le \limsup_{j\to\infty} d(x'_j, q).\end{aligned}$$

Since q is an asymptotic center of $\{x'_j\}$, so is Tq. From the uniqueness of the asymptotic center of a bounded sequence, we have $q = Tq$, or equivalently, $q \in \operatorname{Fix} T$. By (3.2), we have

$$\sum_{k=1}^{r} \beta'^k_j d(x'_j, u_k)^2 \le \sum_{k=1}^{r} \beta'^k_j d(p, u_k)^2$$

for every $j \in \mathbb{N}$, and thus

$$\begin{aligned}\liminf_{j\to\infty} \sum_{k=1}^{r} \beta'^k_j d(x'_j, u_k)^2 &\le \liminf_{j\to\infty} \sum_{k=1}^{r} \beta'^k_j d(p, u_k)^2 \\ &= \sum_{k=1}^{r} \beta^k d(p, u_k)^2 = g(p).\end{aligned}$$

Since q is a Δ-limit of $\{x'_j\}$, we have

$$\begin{aligned} g(q) = \sum_{k=1}^{r} \beta^k d(q, u_k)^2 &\le \sum_{k=1}^{r} \beta^k \liminf_{j\to\infty} d(x'_j, u_k)^2 \\ &\le \liminf_{j\to\infty} \sum_{k=1}^{r} \beta^k d(x'_j, u_k)^2 \\ &= \liminf_{j\to\infty} \sum_{k=1}^{r} \beta'^k_j d(x'_j, u_k)^2 \le g(p). \end{aligned}$$

This shows that q is a minimizer of g on $\operatorname{Fix} T$. From its uniqueness, we obtain $q = p$, that is, p is an asymptotic center of $\{x'_j\}$. From the assumptions of $\{x'_j\}$, we have

$$\begin{aligned} \limsup_{i\to\infty} d(x_{n_i}, p) &= \lim_{j\to\infty} d(x'_j, p) \\ &\le \limsup_{j\to\infty} d(x'_j, v) \\ &\le \limsup_{i\to\infty} d(x_{n_i}, v). \end{aligned}$$

Hence p is also an asymptotic center of $\{x_{n_i}\}$ and it implies that $v = p$. Since v is an asymptotic center of a subsequence $\{x_{n_i}\}$ of $\{x_n\}$, which is arbitrarily chosen, and it coincides with p, we conclude that $\{x_n\}$ is Δ-convergent to $p = \operatorname{argmin}_{y\in \operatorname{Fix} T} \sum_{k=1}^{r} \beta^k d(y, u_k)^2$.

We finally show the convergence of $\{x_n\}$ to p. Since $\{x_n\}$ is Δ-convergent to p, by (3.2) we have

$$\begin{aligned} \sum_{k=1}^{r} \beta^k d(p, u_k)^2 &\le \sum_{k=1}^{r} \beta^k \liminf_{n\to\infty} d(x_n, u_k)^2 \\ &\le \liminf_{n\to\infty} \sum_{k=1}^{r} \beta^k_n d(x_n, u_k)^2 \le \limsup_{n\to\infty} \sum_{k=1}^{r} \beta^k_n d(x_n, u_k)^2 \\ &\le \limsup_{n\to\infty} \sum_{k=1}^{r} \beta^k_n d(p, u_k)^2 \\ &= \sum_{k=1}^{r} \beta^k d(p, u_k)^2, \end{aligned}$$

and hence

$$\lim_{n\to\infty} \sum_{k=1}^{r} \beta^k_n d(x_n, u_k)^2 = \sum_{k=1}^{r} \beta^k d(p, u_k)^2.$$

Since $\beta^1, \beta^2, \dots, \beta^r \in [0,1]$ and $\sum_{k=1}^r \beta^k = 1$, there exists $k_0 \in \{1, 2, \dots, r\}$ such that $\beta^{k_0} > 0$. Then, it follows that $d(p, u_{k_0}) = \lim_{n\to\infty} d(x_n, u_{k_0})$. Indeed, since we know

$$d(p, u_{k_0}) \le \liminf_{n\to\infty} d(x_n, u_{k_0}) \le \limsup_{n\to\infty} d(x_n, u_{k_0}),$$

it is sufficient to show $\limsup_{n\to\infty} d(x_n, u_{k_0}) \le d(p, u_{k_0})$. We prove it by contradiction. If it does not hold, there exists a subsequence $\{x_{n_l}\} \subset \{x_n\}$ such that $\limsup_{n\to\infty} d(x_n, u_{k_0})^2 = \lim_{l\to\infty} d(x_{n_l}, u_{k_0})^2 > d(p, u_{k_0})^2$. Then we have

$$\begin{aligned}
\sum_{k\neq k_0} \beta^k d(p, u_k)^2 &= \sum_{k=1}^r \beta^k d(p, u_k)^2 - \beta^{k_0} d(p, u_{k_0})^2 \\
&= \lim_{n\to\infty} \sum_{k=1}^r \beta_n^k d(x_n, u_k)^2 - \beta^{k_0} d(p, u_{k_0})^2 \\
&> \lim_{l\to\infty} \sum_{k=1}^r \beta_{n_l}^k d(x_{n_l}, u_k)^2 - \beta^{k_0} \lim_{l\to\infty} d(x_{n_l}, u_{k_0})^2 \\
&= \lim_{l\to\infty} \sum_{k=1}^r \beta^k d(x_{n_l}, u_k)^2 - \lim_{l\to\infty} \beta^{k_0} d(x_{n_l}, u_{k_0})^2 \\
&= \lim_{l\to\infty} \sum_{k\neq k_0} \beta^k d(x_{n_l}, u_k)^2 \\
&\ge \sum_{k\neq k_0} \liminf_{l\to\infty} \beta^k d(x_{n_l}, u_k)^2 \\
&\ge \sum_{k\neq k_0} \beta^k d(p, u_k)^2.
\end{aligned}$$

It is a contradiction and thus we have $\limsup_{n\to\infty} d(x_n, u_{k_0}) \le d(p, u_{k_0})$. It implies that $\lim_{n\to\infty} d(x_n, u_{k_0}) = d(p, u_{k_0})$ and by Lemma 2.1, we obtain $p = \lim_{n\to\infty} x_n$, which is the desired result. □

In the end of this paper, we remark the case where the domain X of T is a closed convex subset of a Hilbert space. Under this assumption, it follows from the parallelogram law that

$$g(y) = \sum_{k=1}^r \beta^k \|y - u_k\|^2 = \left\| y - \sum_{k=1}^r \beta^k u_k \right\|^2 + \sum_{j=i+1}^n \sum_{i=1}^{n-1} \beta^i \beta^j \|u_i - u_j\|^2 .$$

Then, putting $u = \sum_{k=1}^r \beta^k u_k$, we have a minimizer of g on $\operatorname{Fix} T$ is a unique closest point to u. That is, we obtain the following result.

Theorem 3.4. *Let X be a closed convex subset of a Hilbert space and $T\colon X \to X$ a nonexpansive mapping such that* $\operatorname{Fix} T \neq \emptyset$. *Let $u_1, u_2, \ldots, u_r$ be a finite family of anchor points in X. Let $\{\alpha_n\} \subset]0,1[$ be a real sequence such that $\alpha_n \to 0$ as $n \to \infty$, and for $k = 1, 2, \ldots, r$, let $\{\beta_n^k\} \subset [0,1]$ be real sequences such that $\sum_{k=1}^{r} \beta_n^k = 1$ for every $n \in \mathbb{N}$ and $\beta_n^k \to \beta^k \in [0,1]$ as $n \to \infty$. Define $\{x_n\} \subset X$ by*

$$x_n = \alpha_n \sum_{k=1}^{r} \beta_n^k u_k + (1 - \alpha_n) T x_n$$

for each $n \in \mathbb{N}$. Then $\{x_n\}$ converges to $P_{\operatorname{Fix} T} \sum_{k=1}^{r} \beta^k u_k$ as $n \to \infty$, where $P_{\operatorname{Fix} T}$ is a metric projection of X onto $\operatorname{Fix} T$.

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