Proceedings of the International Conference on
Nonlinear Analysis and Convex Analysis &
International Conference on Optimization:
Techniques and Applications -I-
(Hakodate, Japan, 2019), 277–290

PROJECTION METHODS FOR THE VARIATIONAL INEQUALITIES WITH UNRELATED NONEXPANSIVE MAPPINGS IN HILBERT SPACES

JONG KYU KIM

Abstract. In this paper, we introduce new iteration methods for finding a common point of the solution set of a class of pseudomonotone variational inequalities and the fixed point set of a finite system of unrelated nonexpansive mappings in a real Hilbert space. The main iteration step in the proposed methods computes only one projection and does not require any Lipschitz continuity for the cost mapping.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. By $x^k \to \bar{x}$ (resp. $x^k \rightharpoonup \bar{x}$), we denote strong (resp. weak) convergence of the sequence $\{x^k\}$ to $\bar{x}$.

A mapping $F : C \to \mathcal{H}$, usually called *a cost mapping*, is said to be

(1) *monotone* on C, if

$$\langle F(x) - F(y), x - y \rangle \geq 0 \quad \forall x, y \in C;$$

(2) *pseudomonotone* on C, if

$$\langle F(y), x - y \rangle \geq 0 \Rightarrow \langle F(x), x - y \rangle \geq 0 \quad \forall x, y \in C;$$

(3) *Lipschitz continuous* on C with constant $L > 0$, if

$$\|F(x) - F(y)\| \leq L\|x - y\| \quad \forall x, y \in C.$$

A *variational inequality problem*, shortly $VIP(C, F)$, is to find a point in

$$Sol(C, F) = \{x^* \in C : \langle F(x^*), x - x^* \rangle \geq 0 \quad \forall x \in C\}.$$

2010 *Mathematics Subject Classification.* 65K10, 90C25, 47H05, 91B50.

Key words and phrases. Pseudomonotone, variational inequalities, projection method, unrelated nonexpansive mapping.

Let a finite system of mappings $S_i : C_i \to C_i (i \in I := \{1, 2, ..., n\})$, where C_i is nonempty, closed and convex subset of $\mathcal{H}$. For each $i \in I$, S_i is called *nonexpansive* on C_i, if

$$\|S_i x - S_i y\| \leq \|x - y\| \quad \forall x, y \in C_i.$$

Denote the fixed point set of S_i by $Fix(S_i) := \{x \in C_i : S_i x = x\}$.

Problem 1. Let $C \subseteq C_i$ for all $i \in I$. The problem is to find a common element of the solution sets of Problem $VIP(C, F)$ and the set of common fixed points of a finite family of nonexpansive mappings $S_i (i \in I)$, that is,

$$\text{Find } x^* \in \bigcap_{i \in I} Fix(S_i) \cap Sol(C, F).$$

Now we consider the relation of Problem 1 and other problems.

1. *Variational inequality problem:* Let C be a nonempty, closed and convex subset of $\mathcal{H}$ and $F : C \to \mathcal{H}$ be a mapping. We consider the variational inequality problem (see [22, 36]):

$$\text{Find } \bar{x} \in C \text{ such that } \langle F(\bar{x}), y - \bar{x} \rangle \geq 0 \quad \forall y \in C.$$

   Setting $S_i = 0$ for all $i \in I$, it is easy to see that Problem 1 coincides with this variational inequality problem.
2. *Fixed point problem:* Let C be a nonempty, closed and convex subset of $\mathcal{H}$ and the mappings $S_i : C \to \mathcal{C} (i \in I = \{1, 2, ..., n\})$ be nonexpansive. The following problem is called the fixed point problem (see [44]):

$$\text{Find } \hat{x} \in \bigcap_{i \in I} Fix(S_i).$$

   By choosing $F = 0$ and $C_i = C$ for all $i \in I$, we can easily see that Problem 1 is equivalent to this fixed point problem.
3. *Finding a common point of the solution set of variational inequalities and the fixed point set of nonexpansive mappings:* Let C be a nonempty, closed convex subsets of $\mathcal{H}$, the mappings $S_i : C \to \mathcal{C} (i \in I = \{1, 2, ..., n\})$ be nonexpansive, and $F_j : C \to \mathcal{H} (j \in J := \{1, 2, ..., m\})$. The problem is formulated as follows (see [16, 19, 38, 47]):

$$\text{Find } x^* \in \bigcap_{i \in I} Fix(S_i) \bigcap_{j \in J} Sol(C, F_j). \tag{1.1}$$

   Taking $C_i = C$ for all $i \in I, m = 1$ and $F = F_1$, we can see that the problem collapses into Problem 1.
4. *Unrelated variational inequalities:* For each $i \in I = \{1, 2, ..., n + 1\}$, let C_i be a nonempty, closed and convex subset of $\mathcal{H}$ such that $\cap_{i \in I} C_i \neq \emptyset$,

and $F_i : C_i \to \mathcal{H}$. Unrelated variational inequalities in the sense of Censor et al. [17] are first stated as follows:

$$\text{Find } x^* \in Sol(C_i, F_i). \tag{1.2}$$

It is easy to see that Problem 1 is a special case of unrelated variational inequalities in the case $F_i(x) = x - S_i x$ for all $x \in C_i, i \in \{1, 2, ..., n\}$ and $F = F_{n+1}$.

Some notable methods for studying and solving Problem 1 and its special case have been proposed such as hybrid extragradient-viscosity methods of Kim [25] and Maingé [40] for (1.1) in the case $m = n = 1$, extragradient methods of Kim et al. ([30, 27]) and Nakajo et al. [42] for (1.1) where $S_i (i \in I)$ are nonexpansive semigroups, strongly convergent reflection methods of Bauschke et al. [11] for finding the projection onto the intersection of two closed convex sets, projection methods of Censor et al. [17] and Kim et al. [32] for $VIP(C, F)$, projection methods for nonexpansive mappings and inverse-strongly monotone mappings of Iiduka et al. [33], linesearch methods of Anh et al. [5], ergodic iteration methods of Kim et al. [26], parallel extragradient-like projection methods of Anh et al. [8] and others for (1.1); see [4, 6, 9, 15, 22, 31, 29].

The metric projection from $\mathcal{H}$ onto C is denoted by Pr_C and

$$Pr_C(x) := \text{argmin}\{\|x - y\| : \ y \in C\}, \quad \forall x \in \mathcal{H}.$$

It is easy to see that a point $x^* \in Sol(C, F)$ if and only if it is a fixed point of the mapping $T(x) := Pr_C(x - \lambda F(x))$, where $\lambda > 0$. Then, the simplest iterative procedure is the well-known projected gradient method which was given by the following [13]:

$$\begin{cases} x^0 \in \mathcal{R}^n, \\ x^{k+1} = T(x^k). \end{cases}$$

However, in the Euclidean space $\mathcal{H} = \mathcal{R}^n$, its convergence requires either β-strongly monotone property (i.e., $\langle F(x) - F(y), x - y\rangle \geq \beta\|x - y\|^2$, $\forall x, y \in C$) and L-Lipschitz continuity (i.e., $\|F(x) - F(y)\| \leq L\|x - y\|$, $\forall x, y \in C$) on the cost mapping F, or γ-converse strongly monotone property (i.e., $\langle F(x) - F(y), x - y\rangle \geq \gamma\|F(x) - F(y)\|^2$, $\forall x, y \in C$).

In order to overcome the drawbacks, in a real Hilbert space $\mathcal{H}$, Malitsky [39] proposed the projected reflected gradient algorithm with a constant stepsize for solving $VIP(C,F)$, where the iteration scheme is very simple as follows:

$$\begin{cases} x^0 = y^0 \in \mathcal{H}, \\ x^{k+1} = Pr_C(x^k - \lambda F(y^k)), \\ y^{k+1} = 2x^{k+1} - x^k. \end{cases}$$

Only under monotone and L-Lipschitz continuous assumptions of the cost mapping F, by choosing the stepsize $\lambda \in (0, \frac{\sqrt{2}-1}{L})$, the author showed that the sequences $\{x^k\}$ and $\{y^k\}$ weakly converge to a solution of $VIP(C,F)$.

On the one hand, in 1976, Korpelevich [37] first introduced the extragradient method for the saddle point problem and then was extended to $VIP(C,F)$, the sequence $\{x^k\}$ is defined by

$$\begin{cases} x^0 \in \mathcal{R}^n, \\ y^k = T(x^k), \\ x^{k+1} = Pr_C(x^k - \lambda F(y^k)). \end{cases}$$

Under assumptions that F is monotone, L-Lipschitz continuous and $0 < \lambda < \frac{1}{L}$, she showed that the sequences $\{x^k\}$ and $\{y^k\}$ converge to a same point in $Sol(C,F)$. However, in the extragradient method, the sequences only converge to a solution point of $VIP(C,F)$ in $\mathcal{R}^n$.

In 2006, basing on the extragradient method and Mann iteration method [41], another iterative process was proposed by Nadezhkina and Takahashi in [43] for finding a common point of the fixed point set of a nonexpansive mapping S and the set $Sol(C,F)$:

$$\begin{cases} x^0 \in \mathcal{H}, \\ y^k = Pr_C(x^k - \lambda_k F(x^k)), \\ x^{k+1} = \alpha_k x^k + (1-\alpha_k) S Pr_C(x^k - \lambda_k F(y^k)). \end{cases}$$

When $\{\lambda_k\} \subset [a,b] \subset (0, \frac{1}{L})$ and $\{\alpha_k\} \subset [c,d] \subset (0,1)$, where F is monotone and L-Lipschitz continuous on C, they proved that the sequence $\{x^k\}$ converges weakly to some element in $Sol(C,F) \cap Fix(S)$. The method has modified and extended by Anh [2] for finding a common point of the solution set of pseudomonotone equilibrium problems and the fixed point set of nonexpansive mappings in a real Hilbert space. Note that, in the special case that the equilibrium problem is a variational inequality problem, then pseudomonotonicty of the cost bifunction is same as pseudomonotonicity of the cost mapping.

Recently, the extragradient method has been played a crucial role to get a lot of extensions to solve Problem 1: see [7, 36, 40]. Summing up, many of current algorithms for finding a common point of the solution set of a variational inequality problem and the fixed point set of mappings usually requires more than two projections at each iteration step; see, for example [49]. Similarly to the case of the projected reflected gradient algorithm, one could ask the following question:

Could one use the basic projection method for solving Problem 1 such that the method requires only one projection onto the feasible set C and only computing values of the mappings per iteration?

In this paper we propose a solution for the above question by showing that the basic projection method not only works well for solving variational inequalities but also succeeds in solving Problem 1.

This paper, motivated the projected reflected gradient algorithm of Malitsky in [39] and the Mann iteration method in [41], introduces the following one-projection scheme to solve Problem 1 for the variational inequality problem $VIP(C,F)$ and a finite family of nonexpansive mappings $S_i(i \in I)$ in the framework of a real Hilbert space $\mathcal{H}$. This scheme needs only computing one projection of $x^k - \alpha_k F(x^k)$ onto the feasible set C (as in the above) and only n values of nonexpansive mappings $S_i(i \in I)$ per iteration and it has a quite simple and elegant structure. The scheme recalls *one-projection method*; however, stepsize $\alpha_k = \frac{\delta_k}{\gamma_k}$ of a gradient is taken computational complexity in the point $Pr_C(x^k - \alpha_k F(x^k))$, where $\gamma_k = \max\{\lambda_k, \|F(x^k)\|\}$ under some conditions onto parameter sequences $\{\lambda_k\}$ and $\{\delta_k\}$ per each iteration k. We emphasize here that the weak convergence of our scheme follows by continuity, pseudomonotonicity and paramonotonicity of the cost mapping F, and it does not require Lipschitz continuity.

2. Preliminaries

In this paper, we assume that the mapping F, each operator $S_i(i \in I)$, parameter sequences $\{\lambda_k\}, \{\delta_k\}$ and $\{\beta_{k,i}\}$ satisfy the following restrictions:

(C_1) If $\{x^k\} \subset \mathcal{H}$ is bounded, then $\{F(x^k)\}$ is also bounded; for each fixed point $y \in C$, the function $f(x) = \langle F(x), y - x\rangle$ is weakly upper semicontinuous with respect to x [4]; pseudomonotone on C with respect to every solution of Problem 1 and satisfies strict paramonotonicity property that (see [49])

$$\{x \in Sol(C,F), y \in C, \langle F(y), x - y\rangle = 0\} \Rightarrow y \in Sol(C,F);$$

(C_2) The mappings S_i are nonexpansive on C for all $i \in I$;

(C_3) Let $L > \lambda > 0$ and $0 < a < b < 1$. The sequences $\{\delta_k\} \subset (0,1), \{\beta_{k,j}\}$ and $\{\lambda_k\}$ satisfy

$$\sum_{k=0}^{\infty} \delta_k = +\infty, \sum_{k=0}^{\infty} \delta_k^2 < +\infty, a \le \beta_{k,j} \le b \ \ \forall j \in I, \ \text{ and } \{\lambda_k\} \subset [\lambda, L];$$

(example as $\delta_k = \frac{1}{k+1}$ for all $k \geq 0$);

(C_4) The solution set of Problem 1 is nonempty, that is,

$$\Omega := \bigcap_{i\in I} Fix(C_i, S_i) \cap Sol(C, F) \neq \emptyset.$$

Let F be continuous. By Kinderlehrer and Stampacchia [35] (Corollary 4.3), if either there exists $x^0 \in C$ such that

$$\frac{\langle F(y) - F(x^0), y - x^0\rangle}{\|y - x^0\|} \to 0 \quad \text{as } \|y\| \to \infty, y \in C,$$

or C is bounded, then $Sol(C, F) \neq \emptyset$.

Now the one-projection algorithm for solving Problem 1 is formally stated as the following.

Algorithm 2.1. (One-projection algorithm)

Initialization: *Choosing* $x^0 \in C$, *and the parameter sequences* $\{\lambda_k\}$ *and* $\{\delta_k\}$ *satisfy* (C_3).

Iterative step: $k \geq 1$,

$$\begin{cases} \textit{Take } \gamma_k = \max\{\lambda_k, \|F(x^k)\|\}, \alpha_k = \frac{\delta_k}{\gamma_k} \\ \textit{Compute } y^k = Pr_C(x^k - \alpha_k F(x^k)) \\ \textit{For each } j \in I, \textit{compute } u_j^k = (1-\beta_{k,j})x^k + \beta_{k,j} S_j y^k \\ \textit{Set } x^{k+1} = u_{j_k}^k, \ \textit{where } j_k := \textit{argmax}\{\|u_j^k - y^k\| : \ j \in I\}. \end{cases}$$

To investigate the convergence of one-projection method, we recall the following technical lemmas and quasi-Fejér convergence which will be used in the sequel.

Lemma 2.2 ([51]). *Let* $\{a_k\}$ *and* $\{b_k\}$ *be sequences of nonnegative real numbers such that*

$$a_{k+1} \leq a_k + b_k \quad \forall k \geq 0,$$

where $\sum_{k=0}^{\infty} b_k < \infty$. *Then, the sequence* $\{a_k\}$ *is convergent.*

Lemma 2.3 ([44]). *Let* $\mathcal{H}$ *be a real Hilbert space,* $\{\alpha_k\}$ *be a sequence of real numbers such that* $0 < a \leq \alpha_k \leq b < 1$ *for all* $k \geq 0$, *and let* $\{v^k\}, \{w^k\}$ *be sequences in* $\mathcal{H}$ *such that*

$$\limsup_{k\to\infty} \|v^k\| \leq c, \quad \limsup_{k\to\infty} \|w^k\| \leq c$$

and

$$\lim_{k\to\infty} \|\alpha_k v^k + (1-\alpha_k)w^k\| = c.$$

Then, $\lim_{k\to\infty} \|v^k - w^k\| = 0$.

It is well known that the metric projection Pr_C has the following properties.

Lemma 2.4 ([12])**.** *Let C be a nonempty, closed and convex subset of $\mathcal{H}$ and $Pr_C : \mathcal{H} \to C$ be a metric projection on C. Then the following properties hold:*

(i) $\langle x - Pr_C(x), y - Pr_C(x)\rangle \le 0, \quad \forall y \in C, x \in \mathcal{H}$;
(ii) $\langle Pr_C(x) - Pr_C(y), x - y\rangle \ge \|Pr_C(x) - Pr_C(y)\|^2, \quad \forall x, y \in \mathcal{H}$;
(iii) $\|x - Pr_C(x)\|^2 \le \|x - y\|^2 - \|y - Pr_C(x)\|^2, \quad \forall x \in \mathcal{H}, y \in C$.

Lemma 2.5 ([46] **(Opial Condition)**)**.** *Let $\{x^k\}$ be a sequence in $\mathcal{H}$ such that $x^k \rightharpoonup \bar{x}$. Then, for all $y \ne \bar{x}$, we have*

$$\liminf_{k\to\infty} \|x^k - \bar{x}\| < \liminf_{k\to\infty} \|x^k - y\|.$$

We next deal with the so called quasi-Fejér convergence and its properties:

Definition 2.6. *Let S be a nonempty subset of $\mathcal{H}$. A sequence $\{x^k\}$ in $\mathcal{H}$ is said to be quasi-Fejér convergent to S if for all $x^* \in S$, there exist $k_0 \ge 0$ and a sequence $\{\alpha_k\} \subset (0,\infty)$ with $\sum_{k=0}^{\infty} \alpha_k < \infty$ such that*

$$\|x^{k+1} - x^*\|^2 \le \|x^k - x^*\|^2 + \alpha_k, \quad \forall k \ge k_0.$$

This definition originates in [21] and has been used further in [20, 34].

Lemma 2.7 ([34, Theorem 4.1])**.** *Let S be a nonempty subset of $\mathcal{H}$. If $\{x^k\}$ is quasi-Fejér convergent to S. Then we have the followings:*

(i) *The sequence $\{x^k\}$ is bounded;*
(ii) *If all weak cluster points of $\{x^k\}$ belong to S, then the sequence $\{x^k\}$ is weakly convergent to a point of S.*

3. Convergence theorems

Thefollowing lemma is important to prove the convergence theorem.

Lemma 3.1 ([28])**.** *Let $F : C \to \mathcal{H}$ be pseudomonotone and the conditions $(C_2) - (C_4)$ be satisfied. Then we have the following statements.*

(i) *The sequence $\{x^k\}$ generated by Algorithm 2.1 is quasi-Fejér convergent to a point in Ω. More detailed as the following:*

$$\|x^{k+1} - x^*\|^2 \le \|x^k - x^*\|^2 + 2\beta_{k,j_k}\delta_k^2, \quad \forall x^* \in \Omega.$$

(ii) *For each $x^* \in \Omega$,*

$$\limsup_{k\to\infty} \langle F(x^k), x^* - x^k\rangle = 0.$$

Now, we are in a position to indroduce the main convergence theorem.

Theorem 3.2 ([28]). *Let C be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$. Suppose that conditions $(C_1)-(C_4)$ are satisfied. Let $\{x^k\}$ be a sequence generated by Algorithm 2.1. Then, the sequences $\{x^k\}$ and $\{y^k\}$ converge weakly to the same point $x^* \in \Omega$.*

Proof. By Lemma 2.7(i), the sequence $\{x^k\}$ is quasi–Fejér convergent to Ω. Hence, by Lemma 3.1(i), the sequence $\{x^k\}$ is bounded and if all the weak cluster points of $\{x^k\}$ belong to Ω, then the sequence $\{x^k\}$ converges weakly to a point of Ω. Let $\bar{x}$ be any weak cluster point of $\{x^k\}$. Then, since C is weakly closed, $\bar{x} \in C$ and without loss of generality, we can assume that

$$x^{k_j} \rightharpoonup \bar{x} \ \text{ and } \limsup_{k\to\infty} \langle F(x^k), x^* - x^k\rangle = \lim_{j\to\infty} \langle F(x^{k_j}), x^* - x^{k_j}\rangle, \tag{3.1}$$

where $x^* \in \Omega$.

It remains to prove that $\bar{x} \in \Omega$ to get that $\{x^k\}$ converges weakly to a point in Ω, and from the fact that $\|x^k - y^k\| \le \delta_k$, $\{y^k\}$ converges weakly to the same point. The proof of $\bar{x} \in \Omega$ is divided into two steps (see [28]).

Step 1. Claim $\bar{x} \in Sol(C, F)$.

Step 2. Claim $\bar{x} \in Fix(S_j)$ for all $j \in I$.

Now we will show that $S_j(\bar{x}) = \bar{x}$ for all $j \in I$. The proof is similar as in the proof of ([46], Lemma 2) by using the Opial condition in Lemma 2.5. Thus $y^{k_j} \rightharpoonup \bar{x} \in \cap_{j\in I} Fix(S_j)$ as $j \to \infty$. This completes the proof. □

4. Equilibrium problems and common fixed point problems

Let C be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$, $f : C \times C \to \mathcal{R}$ be a bifunction such that $f(x, x) = 0$ for all $x \in C$ (usually called *cost bifunction*), and $f(x, \cdot)$ be convex for each $x \in C$. The equilibrium problem, shortly $EP(C, f)$ [14], is formulated by: Find $x^* \in C$ such that

$$f(x^*, x) \ge 0, \quad \forall x \in C.$$

Let us denote the solution set of $EP(C, f)$ by $Sol(C, f)$. For each $i \in I := \{1, 2, ..., n\}$, let $S_i : C \to C$ be a nonexpansive mapping.

In this section, we consider the following problem of finding a common point of the equilibrium solution set $Sol(C, f)$ and the fixed point set $\cap_{i\in I} Fix(S_i)$:

Problem 2. Find $x^* \in \bigcap_{i\in I} Fix(S_i) \cap Sol(C, f)$.

Methods for solving Problem 2 have been studied extensively by many researchers; see [1, 3, 18, 45, 48, 50]. In the special case $f(x, y) = \langle F(x), y - x\rangle$ for all $x, y \in C$, where the mapping $F : C \to \mathcal{H}$, Problem $EP(C, f)$ becomes the variational inequality problem $VIP(C, F)$. Thus, we can say that Problem 2 is an extension formulation of Problem 1.

Many of the existing methods for solving Problem 2 require a strict assumption on the strong monotonicity or Lipschitz-type continuity of the cost bifunction f. Here, we only assume that f is monotone and satisfies the paramonotonicity property, and not neccesary Lipschitz-type continuous.

We recall that a bifunction $f : C \times C \to \mathcal{R}$ is called (see [5], Definition 1):

(1) *strongly monotone* on C with constant $\beta > 0$, if

$$f(x, y) + f(y, x) \leq -\beta\|x - y\|^2, \quad \forall x, y \in C;$$

(2) *monotone* on C, if

$$f(x, y) + f(y, x) \leq 0, \quad \forall x, y \in C;$$

(3) *pseudomonotone* on C, if

$$f(x, y) \leq 0 \Rightarrow f(y, x) \leq 0, \quad \forall x, y \in C;$$

(4) *Lipschitz-type continuous* on C, if

$$f(x, y) + f(y, z) \leq f(x, z) - c_1\|x - y\|^2 - c_2\|y - z\|^2, \quad \forall x, y, z \in C.$$

For solving Problem 2, we assume that for each $i \in I$, $S_i : C \to C$ is nonexpansive such that

$$\Gamma := \bigcap_{i \in I} Fix(S_i) \cap Sol(C, f) \neq \emptyset$$

and the bifunction f satisfies the following conditions:

(D_1) If $\{x^k\}$ is bounded, then $\{w^k\} \subset \partial_2^{\epsilon_k} f(x^k, x^k)$ is also bounded and f is continuous on $C \times C$ in the sense that if $x, y \in C$, and $\{x^k\}$ and $\{y^k\}$ are two sequences in C converging weakly to x and y, respectively, then $f(x^k, y^k) \to f(x, y)$.

(D_2) f is pseudomonotone on C with respect to every solution of Problem $EP(C, f)$ and satisfies the following condition, called strict paramonotonicity property:

$$\{x \in Sol(C, f), y \in C, f(y, x) = 0\} \Rightarrow y \in Sol(C, f).$$

Next, we propose a modification of Algorithm 2.1 for solving Problem 2.

Algorithm 4.1. (Modified parallel projection algorithm)

Initialization: *Choosing* $x^0 \in C$*, and the parameter sequences* $\{\lambda_k\}$ *and* $\{\delta_k\}$ *satisfy* (C_3)*, and* $\{\epsilon_k\} \subset (0, \infty)$ *such that* $\sum_{k=0}^{\infty} \delta_k \epsilon_k < \infty$.

Iterative step: $k \geq 1$,

$$\begin{cases} \textit{Take } w^k \in \partial_2^{\epsilon_k} f(x^k, x^k), \gamma_k = \max\{\lambda_k, \|w^k\|\}, \alpha_k = \frac{\delta_k}{\gamma_k}; \\ \textit{Compute } y^k = Pr_C(x^k - \alpha_k w^k); \\ \textit{For each } j \in I, \textit{compute } u_j^k = (1 - \beta_{k,j})x^k + \beta_{k,j} S_j y^k; \\ \textit{Set } x^{k+1} = u_{j_k}^k, \ \textit{where } j_k := \textit{argmax}\{\|u_j^k - y^k\| : \ j \in I\}. \end{cases}$$

Note that for each $\epsilon > 0, x \in C$, $\partial_2^\epsilon f(x, y)$ stands for ϵ-subdifferential of the convex function $f(x, \cdot)$ at $y \in C$ i.e.,

$$\partial_2^\epsilon f(x, y) := \{w_y \in \mathcal{H} : f(x, z) - f(x, y) \geq \langle w_y, z - y\rangle - \epsilon \ \ \forall z \in C\}.$$

When $f(x, y) = \langle F(x), y - x\rangle$ for every $x, y \in C$, Problem $EP(C, f)$ becomes the variational inequality problem $VIP(C, F)$. In that case, we can choose $w^k = F(x^k)$ and Algorithm 4.1 can be written as Algorithm 2.1.

Now we need the following lemma for the main theorem.

Lemma 4.2 ([28]). *Let $f : C \times C \to \mathcal{R}$ be pseudomonotone and the conditions (C_3) be satisfied. Then, we have the following statements.*

(i) *The sequence $\{x^k\}$ generated by Algorithm 4.1 is quasi-Fejér convergent to Γ and the following inequality holds:*

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 + \frac{2}{\lambda}\beta_{k,j_k}\delta_k\epsilon_k + 2\beta_{k,j_k}\delta_k^2, \quad \forall x^* \in \Gamma.$$

(ii) *For each $x^* \in \Gamma$,*

$$\limsup_{k\to\infty} f(x^k, x^*) = 0.$$

Theorem 4.3 ([28]). *Let C be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$. Suppose that cnditions (C_3) and $(D_1) - (D_2)$ are satisfied. Then, the sequences $\{x^k\}$ and $\{y^k\}$ generated by Algorithm 4.1 converge weakly to the same point $\bar{x} \in \Gamma$.*

Proof. By the similarly method as Theorem 3.2, we assume that for each $\{x^{k_j}\} \subset \{x^k\}$ and

$$x^{k_j} \rightharpoonup \bar{x} \ \text{ and } \limsup_{k\to\infty} f(x^k, x^*) = \lim_{j\to\infty} f(x^{k_j}, x^*). \tag{4.1}$$

If $\bar{x} \in \Gamma$, then by Lemma 2.7(ii) that the sequence $\{x^k\}$ converges weakly to a solution $\bar{x} \in \Gamma$ and hence $\{y^k\}$ also converges weakly to $\bar{x} \in \Gamma$.

From condition (D_1), (4.1) and Lemma 4.2(ii), we have

$$f(\bar{x}, x^*) = 0$$

Combining this and the pseudomonotonicity of f

$$f(x^*, \bar{x}) \geq 0 \Rightarrow f(\bar{x}, x^*) \leq 0,$$

it implies that

$$f(\bar{x}, x^*) = 0.$$

Since f is pseudomonotone on C with respect to every solution of Problem 2 and satisfies strict paramonotonicity in condition (D_1), we obtain $\bar{x} \in Sol(C, f)$.

By similar arguments as in Step 2 of Theorem 3.2, we also have $\bar{x} \in Fix(S_j)$ for all $j \in I$. Consequently, the sequences $\{x^k\}$ and $\{y^k\}$ generated by Algorithm 4.1 converge weakly to the same point $\bar{x} \in \Gamma$. $\square$

Acknowledgments. This work was supported by the Basic Science Research Program through the National Research Foundation(NRF) Grant funded by Ministry of Education of the republic of Korea (2018R1D1A1B07045427).

References

[1] P. N. Anh, *A hybrid extragradient method extended to fixed point problems and equilibrium problems*, Optim. **62** (2013), 271–283.

[2] P. N. Anh, *A hybrid extragradient method for pseudomonotone equilibrium problems and fixed point problems*, B. Malaysian Math. Sc. Soc. **36** (2013), 107–116.

[3] P. N. Anh, *Strong convergence theorems for nonexpansive mappings and Ky Fan inequalities*, J. Optim. Theory Appl. **154** (2012), 303–320.

[4] P. N. Anh, T. T. H. Anh and T. Kuno, *Strong convergence theorems for variational inequalities on the solution set of Ky Fan inequalities*, Acta Math. Vietnamica **42** (2017), 761–773.

[5] P. N. Anh and H. A. Le Thi, *An Armijo-type method for pseudomonotone equilibrium problems and its applications*, J. Glob. Optim. **57** (2013), 803–820.

[6] P. N. Anh, L. D. Muu, V. H. Nguyen and J. J. Strodiot, *Using the Banach contraction principle to implement the proximal point method for multivalued monotone variational inequalities*, J. Optim. Theory Appl. **124** (2005), 285–306.

[7] P. N. Anh and N. X. Phuong, *Linesearch methods for variational inequalities involving strict pseudocontractions*, Optim. **64** (2015), 1841–1854.

[8] P. N. Anh and N. X. Phuong, *A parallel extragradient-like projection method for unrelated variational inequalities and fixed point problems*, J. Fixed Point Theory Appl. **20** (2018), 1–17.

[9] P. N. Anh, L. Q. Thuy, T. T. H. Anh, *Strong convergence theorem for the lexicographic Ky Fan inequality*, Vietnam J. Math. **46** (2018), 517–530.

[10] H. H.Bauschke and J. Borwein, *On projection algorithms for solving convex feasibility problems*, SIAM Review **38** (1996), 367–426.

[11] H. H. Bauschke, P. L. Combettes and D. R. Luke, *A strongly convergent reflection method for finding the projection onto the intersection of two closed convex sets in a Hilbert space*, J. Approx. Theory **141** (2006), 63–69.

[12] D. P. Bertsekas, A. Nedic and A. E. Ozdaglar, *Convex Analysis and Optimization*, Athena Scientific, Belmont, Massachusetts, 2003.

[13] D. P. Bertsekas and E. M. Gafni, *Projection methods for variational inequalities with applications to the traffic assignment problem*, Math. Prog. Study **17** (1982), 139–159.

[14] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student **63** (1994), 123–145.

[15] A. Cegielski, A. Gibali, S. Reich and R. Zalas, *An algorithm for solving the variational inequality problem over the fixed point set of a quasinonexpansive operator in Euclidean space*, Num. Funct. Anal. Optim. **34** (2013), 1067–1096.

[16] Y. Censor, A. Gibali and S. Reich, *The subgradient extragradient method for solving variational inequalities in Hilbert space*, J. Optim. Theory Appl. **148** (2011), 318–335.

[17] Y. Censor, A. Gibali, S. Reich and S. Sabach, *Common solutions to variational inequalities*, Set Val. Var. Anal. **20** (2012), 229–247.

[18] Y. J. Cho, I. K. Argyros and N. Petrot, *Approximation methods for common solutions of generalized equilibrium, systems of nonlinear variational inequalities and fixed point problems*, Comput. Math. Appl. **60** (2010), 2292–2301.

[19] R. Chugh and R. Rani, *Variational inequalities and fixed point problems a survey*, Int. J. Appl. Math. Res. **3** (2014), 301–326.

[20] P. L. Combettes, *Quasi-Fejérian analysis of some optimization algorithms*, in: Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications, Studies in Computational Mathematics, North-Holland, Amsterdam, 2001, pp. 115–152.

[21] Y. M. Ermoliev, *On the method of generalized stochastic gradients and quasi-Fejér sequences*, Cybern **5** (1969), 208–220.

[22] A. Gibali, S. Reich and R. Zalas, *Outer approximation methods for solving variational inequalities in Hilbert space*, Optim. **66** (2017), 417–437.

[23] P. T. Harker and J. S. Pang, *A damped-Newton method for the linear complementarity problem*, Lectures in Appl. Math. **26** (1990), 265–284.

[24] P. D. Khanh, *A modified extragradient method for infinite-dimensional variational inequalities*, Acta Math. Vietnam **41** (2016), 251–263.

[25] J. K. Kim, *Strong convergence theorems by hybrid projection methods for equilibrium problems and fixed point problems of the asymptotically quasi-ϕ-nonexpansive mappings*, Fixed Point Theory Appl., 2011, doi: 10,1186/1687-1812-2011-10

[26] J. K. Kim, P. N. Anh and T. N. Hai, *The Bruck's ergodic iteration method for the Key Fan inequality over the fixed point set*, Inter. J. Comput. Math. **94** (2017), 1–15.

[27] J. K. Kim, P. N. Anh and Y. M. Nam, *Strong convergence of an extended extragradient method for equibrium problems and fixed point problems*, J. Kor. Math. Soc. **49** (2012), 187–200.

[28] J. K. Kim, P. N. Anh and T. H. Tu, *Projection methods for solving the variational inequalities involving unrelated nonexpansive mappings*, J. Nonlinear Convex Anal. **21** (2020), 2517–2537.

[29] J. K. Kim and N. Buong, *A new iterative method for equilibrium problems and fixed point problems for infinite family of nonself strictly pseudocontractive mappings*, Fixed Point Theory Appl. **2013** (2013):286 .

[30] J. K. Kim and Salahuddin, *Extragradient methods for generalized mixed equilibrium problems and fixed point problems in Hilbert spaces*, Nonlinear Funct. Anal. Appl. **22** (2017), 693–709.

[31] J. K. Kim and T. M. Tuyen, *Alternating resolvent algorithms for finding a common zero of two accretive operator in Banach spaces*, J. Korean Math. Soc. **54** (2017), 1905–1926.

[32] J. K. Kim and T. M. Tuyen, *Regularization proximal point algorithm for finding a common fixed point of a finite family of nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl. 2011, doi: 10,1186/1687-1812-2011-52,

[33] H. Iiduka and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings*, Nonlinear Anal. **61** (2005), 341–350.

[34] A. N. Iusem, B. F. Svaiter and M. Teboulle, *Entropy-like proximal methods in convex programming*, Math. Oper. Res. **19** (1994), 790–814.

[35] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980.

[36] I. V. Konnov, *Combined Relaxation Methods for Variational Inequalities*, Springer-Verlag, Berlin, 2000.

[37] G. M. Korpelevich, *The extragradient method for finding saddle points and other problems*, Ekon. Mat. Metody **12** (1976), 747–756.

[38] Y. Liua, Z. Yao, Y.-C .Liou, L.-J. Zhu, *Algorithms for the variational inequalities and fixed point problems*, J. Nonlinear. Sci. Appl. **9** (2016), 61–74.

[39] Y. Malitsky, *Projected relected gradient methods for monotone variational inequalities*, SIAM J. Optim. **25** (2015), 502–520.

[40] P. E. Maingés, *A hybrid extragradient-viscosity method for monotone operators and fixed point problems*, SIAM J. Control Optim. **47** (2008), 1499–1515.

[41] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1950), 506–510.

[42] N. Nakajo and W. Takahashi, *Strong convergence theorem for nonexpansive mappings and nonexpansive semigroups*, J. Math. Anal. Appl. **279** (2003), 372–379.

[43] N. Nadezhkina and W. Takahashi, *Strong convergence theorem by a hybrid method for nonexpansive mappings and Lipschitz-continuous monotone mappings*, SIAM Optim. **16** (2006), 1230–1241.

[44] J. Schu, *Weak and strong conergence to fixed points of asymptotically nonexpansive mappings*, Bull. Australian Math. Soc. **43** (1991), 153–159.

[45] J. J. Strodiot, P. T. Vuong and N. T. T. Van, *A class of shrinking projection extragradient methods for solving non-monotone equilibrium problems in Hilbert spaces*, J. Global Optim. **64** (2016), 159–178.

[46] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591–597.

[47] S. Plubtieng and R. Punpaeng, *Fixed-point solutions of variational inequalities for nonexpansive semigroups in Hilbert spaces*, Math. Comput. Modell. **48** (2008), 279–286.

[48] S. Takahashi and W. Takahashi, *Viscosity approximation methods for equilibrium problems and fixed point in Hilbert space*, J. Math. Anal. Appl. **331** (2007), 506–515.

[49] P. T. Vuong and Y. Shehu, *Convergence of an extragradient-type method for variational inequality with applications to optimal control problems*, Numer Algor, https://doi.org/10.1007/ s11075-018-0547-6 (2018)

[50] P. T. Vuong, J. J., Strodiot and V. H. Nguyen, *Extragradient methods and linesearch algorithms for solving Ky Fan inequalities and fixed point problems*, J. Optim. Theory Appl. **155** (2012), 605–627.

[51] H. K. Xu, *Viscosity approximation methods for nonexpansive mappings*, Math. Anal. Appl. **298** (2004), 279–291.

Jong Kyu Kim
Department of Mathematics Education, Kyungnam University, Changwon, Gyeongnam 51767, Korea
E-mail address: jongkyuk@kyungnam.ac.kr