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ITERATIVE SEQUENCES FOR A BALANCED MAPPING OF RESOLVENTS

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Abstract. In this paper, we find a common minimizer of a finite convex functions by using Halpern's and Mann's iterative scheme. Furthermore, we use iterative sequences which are generated by a finite resolvent operators without regard of order.

1. Introduction

We know that there are various kinds of iterative scheme which is effective to find fixed points of nonexpansive mappings. In this paper, the authors pay attention to Halpern's [3] and Mann's [8] iterative scheme. A number of authors have proved theorems by using these scheme. Wittmann [13] proved the convergence of Halpern iterative scheme in a Hilbert space. Reich [10] proved that of Mann iterative scheme in a Banach space. Takahashi and Tamura [12] proved that of Halpern iterative scheme by using two nonexpansive mappings in a Banach space. Dhompongsa and Panyanak [2] proved that of Mann type iteration in a CAT(0) space. Saejung [11] proved that of Halpern type iteration in a CAT(0) space. We especially note that Kimura and Hasegawa proved the convergence of Mann [4] and Halpern [5] type iteration by using a balanced mapping in a CAT(0) space.

Theorem 1.1 (Hasegawa-Kimura [4]). *Let X be a complete* CAT(0) *space. Let T^k be a nonexpansive mapping from X to X for every $k = 1, 2, \ldots, N$ such that $F = \bigcap_{k=1}^{N} F(T^k) \neq \emptyset$. For a given real number $a \in \left]0, \frac{1}{2}\right]$, let $\{\alpha_n^k\}, \{\beta_n\} \subset [a, 1-a]$ for every $k = 1, 2, \ldots, N$ and $n \in \mathbb{N}$ such that $\sum_{k=1}^{N} \alpha_n^k = 1$. Define U_n be a mapping from X to X by*

$$U_n x = \operatorname*{argmin}_{y \in X} \sum_{k=1}^{N} \alpha_n^k d(T^k x, y)^2$$

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for every $x \in X$ and $n \in \mathbb{N}$. For a given point $x_1 \in X$, let $\{x_n\}$ be a sequence in X generated by

$$x_{n+1} = \beta_n x_n \oplus (1-\beta_n) U_n x_n$$

for every $n \in \mathbb{N}$. Then $\{x_n\}$ Δ-converges to a point in F.

Theorem 1.2 (Hasegawa-Kimura [5]). *Let X be a complete* CAT(0) *space. Let T^k be a nonexpansive mapping from X to X for every $k = 1, 2, \dots, N$ such that $F = \bigcap_{k=1}^N F(T^k) \neq \emptyset$. For a given real number $a \in \left]0, \frac{1}{2}\right]$, let $\{\beta_n\} \subset]0,1[$, $\left\{\alpha_n^k\right\} \subset [a, 1-a]$ for every $k = 1, 2, \dots, N$ and $n \in \mathbb{N}$ such that $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^\infty \beta_n = \infty$, $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$, $\sum_{k=1}^N \alpha_n^k = 1$ and $\sum_{n=1}^\infty \sum_{k=1}^N \left|\alpha_{n+1}^k - \alpha_n^k\right| < \infty$. Define a mapping U_n from X to X by*

$$U_n x = \operatorname*{argmin}_{y \in X} \sum_{k=1}^N \alpha_n^k d(T^k x, y)^2$$

for every $x \in X$ and $n \in \mathbb{N}$. For given points $u, x_1 \in X$, let $\{x_n\}$ be a sequence in X generated by

$$x_{n+1} = \beta_n u \oplus (1-\beta_n) U_n x_n$$

for every $n \in \mathbb{N}$. Then $\{x_n\}$ converges to $P_F u$.

In this paper, the authors prove two theorems based on Theorems 1.1 and 1.2 with the resolvent of a convex function in a complete CAT(0) space.

2. Preliminaries

Let X be a metric space and let $\{x_n\}$ be a sequence in X. An element $z \in X$ is said to be an asymptotic center of $\{x_n\} \subset X$ if

$$\limsup_{n\to\infty} d(x_n, z) = \inf_{x \in X} \limsup_{n\to\infty} d(x_n, x)$$

Moreover, we say $\{x_n\}$ Δ-converges to a Δ-limit z if z is the unique asymptotic center of any subsequences of $\{x_n\}$. For $x, y \in X$, a mapping $c : [0, l] \to X$ is called a geodesic if c satisfies

$$c(0) = x, c(l) = y, \text{ and } d(c(u), c(v)) = |u - v|$$

for every $u, v \in [0, l]$. If a geodesic exists for every $x, y \in X$, then we call X a geodesic space. Moreover, if a geodesic exists uniquely for every $x, y \in X$, then we call X a uniquely geodesic space.

Let X be a uniquely geodesic space. An image $[x, y]$ of c is called a geodesic segment joining x and y. For a triangle $\triangle(x, y, z) \subset X$, a comparison triangle $\triangle(\bar{x}, \bar{y}, \bar{z})$ in the Euclidean plane $\mathbb{R}^2$ is defined as a triangle such that each corresponding edge has the same length as that of the original triangle. If for every $x, y, z \in X$, every $p, q \in \triangle(x, y, z)$ and their corresponding points $\bar{p}, \bar{q} \in \triangle(\bar{x}, \bar{y}, \bar{z})$ satisfy that

$$d(p, q) \leq \|\bar{p} - \bar{q}\|,$$

X is called a CAT(0) space.

Let X be a CAT(0) space. For every $x, y \in X$ with $\alpha \in [0, 1]$, if $z \in [x, y]$ satisfies that $d(y, z) = \alpha d(x, y)$ and $d(x, z) = (1 - \alpha)d(x, y)$, then we denote z by $z = \alpha x \oplus (1 - \alpha)y$.

Let X be a CAT(0) space and let T be a mapping from X to X such that the set $F(T) = \{z \in X : z = Tz\}$ of fixed points of T is not empty. If $d(Tx, Ty) \leq d(x, y)$ for every $x, y \in X$, then we call T a nonexpansive mapping. Let X be a complete CAT(0) space and let C be a nonempty closed convex subset of X. Then for every $x \in X$, there exists a unique point $x_0 \in C$ satisfying

$$d(x, x_0) = \inf_{y \in C} d(x, y).$$

We define the metric projection P_C from X onto C by $P_C x = x_0$. We know that the metric projection P_C is a nonexpansive mapping such that $F(P_C) = C$.

Let X be a complete CAT(0) space. Let f be a proper lower semicontinuous convex function from X into $]-\infty, \infty]$. For $\lambda > 0$, the resolvent $R_{\lambda f}$ of λf is defined by

$$R_{\lambda f} x = \operatorname*{argmin}_{y \in X} \{\lambda f(y) + d(y, x)^2\}$$

for all $x \in X$ [6, 9]. We know that $R_{\lambda f}$ is a single-valued mapping from X to X. We also know that the resolvent $R_{\lambda f}$ is nonexpansive such that $F(R_{\lambda f}) = \operatorname{argmin}_{x \in X} f$.

Let X be a complete CAT(0) space. Let T^k be a nonexpansive mapping from X to X for every $k = 1, 2, \ldots, N$. Let $\{\alpha^k\} \subset]0, 1[$ for every $k = 1, 2, \ldots, N$ such that $\sum_{k=1}^{N} \alpha^k = 1$. Hasegawa and Kimura [4] define a balanced mapping U from X to X by

$$Ux = \operatorname*{argmin}_{y \in X} \sum_{k=1}^{N} d(T^k x, y)^2$$

for every $x \in X$. They find that this mapping U is defined as a single-valued mapping, has nonexpansiveness and $F(U) = \bigcap_{k=1}^{N} F(T^k)$. We introduce some lemmas used for our results.

Lemma 2.1. (Hasegawa-Kimura [4]) *Let X be a complete* CAT(0) *space. Let T^k be a nonexpansive mapping from X to X for every $k = 1, 2, \ldots, N$. Let $\{\alpha^k\} \subset]0, 1[$ for every $k = 1, 2, \ldots, N$ such that $\sum_{k=1}^{N} \alpha^k = 1$. Define a* *balanced mapping* $U : X \to X$ *by* $Ux = \operatorname{argmin}_{y\in X} \sum_{k=1}^{N} \alpha^k d(T^k x, y)^2$ *for every* $x \in X$*. Then we have*

$$\sum_{k=1}^{N} \alpha^k d(T^k x, Ux)^2 \leq \sum_{k=1}^{N} \alpha^k d(T^k x, Uy)^2 - \sum_{k=1}^{N} \alpha^k d(Uy, Ux)^2$$

for every $x, y \in X$*.*

Lemma 2.2 (Hasegawa-Kimura [5])**.** *Let* X *be a complete* CAT(0) *space. Let* U *be a nonexpansive mapping from* X *to* X*. Suppose* $\{x_n\} \subset X$ *is* Δ*-convergent to* $x_0 \in X$ *and* $\{d(x_n, Ux_n)\}$ *is convergent to* 0*. Then* $x_0 \in F(U)$*.*

Lemma 2.3 (Kimura-Kohsaka [7])**.** *Let* X *be a complete* CAT(0) *space. Let* f *be a proper lower semicontinuous convex function from* X *into* $]-\infty, \infty]$*. Let* $\lambda, \mu > 0$*, and* $R_{\lambda f}, R_{\mu f}$ *be the resolvent of* $\lambda f, \mu f$*. Then we have*

$$\begin{aligned}(\lambda + \mu) d(R_{\lambda f} x, R_{\mu f} x)^2 + \mu d(R_{\lambda f} x, x)^2 + \lambda d(R_{\mu f} x, x)^2 \\ \leq \lambda d(R_{\lambda f} x, x)^2 + \mu d(R_{\mu f} x, x)^2\end{aligned}$$

for every $x \in X$*.*

Lemma 2.4 (Aoyama-Kimura-Takahashi-Toyoda [1])**.** *Let* $\{s_n\}, \{u_n\} \subset]0, \infty[$*,* $\{t_n\} \subset \mathbb{R}$ *and* $\{\alpha_n\} \subset [0, 1]$ *such that* $\sum_{n=1}^{\infty} \alpha_n = \infty$*,* $\sum_{n=1}^{\infty} u_n < \infty$ *and* $\limsup_{n\to\infty} t_n \leq 0$*. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n) s_n + \alpha_n t_n + u_n$$

for all $n \in \mathbb{N}$*. Then* $\lim_{n\to\infty} s_n = 0$*.*

3. Main results

Theorem 3.1. *Let* X *be a complete* CAT(0) *space. Let* f^k *be a proper lower semicontinuous convex function from* X *into* $]-\infty, \infty]$ *for every* $k = 1, 2, \dots, N$ *such that* $F = \bigcap_{k=1}^{N} \operatorname{argmin}_X f^k \neq \emptyset$*. For a given real number* $a \in \left]0, \frac{1}{2}\right]$*, let* $\left\{\alpha_n^k\right\}, \{\beta_n\} \subset [a, 1 - a]$ *and* $\left\{\lambda_n^k\right\} \subset [a, \infty[$ *for every* $k = 1, 2, \dots, N$ *and* $n \in \mathbb{N}$ *such that* $\sum_{k=1}^{N} \alpha_n^k = 1$*. Let* $R_{\lambda_n^k f^k}$ *be the resolvent of* $\lambda_n^k f^k$ *for every* $k = 1, 2, \dots, N$ *and* $n \in \mathbb{N}$*. Define* U_n *be a mapping from* X *to* X *by*

$$U_n x = \operatorname*{argmin}_{y\in X} \sum_{k=1}^{N} \alpha_n^k d\left(R_{\lambda_n^k f^k} x, y\right)^2$$

for every $x \in X$ *and* $n \in \mathbb{N}$*. For a given point* $x_1 \in X$*, let* $\{x_n\}$ *be a sequence in* X *generated by*

$$x_{n+1} = \beta_n x_n \oplus (1 - \beta_n) U_n x_n$$

for every $n \in \mathbb{N}$*. Then* $\{x_n\}$ Δ*-converges to a point in* F*.*

Proof. Let $z \in F$. Then we have

$$\begin{aligned} d(x_{n+1}, z)^2 &= d(\beta_n x_n \oplus (1-\beta_n) U_n x_n, z)^2 \\ &\le \beta_n d(x_n, z)^2 + (1-\beta_n) d(U_n x_n, z)^2 - \beta_n(1-\beta_n) d(U_n x_n, x_n)^2 \\ &\le d(x_n, z)^2 - \beta_n(1-\beta_n) d(U_n x_n, x_n)^2 \\ &\le d(x_n, z)^2. \end{aligned}$$

Thus, we obtain $d(x_{n+1}, z) \le d(x_n, z)$ for all $n \in \mathbb{N}$ and there exists

$$D = \lim_{n\to\infty} d(x_n, z) \le d(x_1, z).$$

Since $0 < a^2 \le \beta_n(1-\beta_n)$, we have $\lim_{n\to\infty} d(U_n x_n, x_n) = 0$. From boundedness of $\{x_n\}$, it follows that

$$\begin{aligned} \lim_{n\to\infty} d(x_n, z) &\le \lim_{n\to\infty} (d(x_n, U_n x_n) + d(U_n x_n, z)) \\ &= \lim_{n\to\infty} d(U_n x_n, z) \\ &= \lim_{n\to\infty} d(U_n x_n, U_n z) \\ &\le \lim_{n\to\infty} d(x_n, z). \end{aligned}$$

Thus we get $\lim_{n\to\infty} d(x_n, z) = \lim_{n\to\infty} d(U_n x_n, z) = D$. By Lemma 2.1,

$$\begin{aligned} \sum_{k=1}^{N} \alpha_n^k d\left(R_{\lambda_n^k f^k} x_n, U_n x_n\right)^2 &\le \sum_{k=1}^{N} \alpha_n^k d\left(R_{\lambda_n^k f^k} x_n, z\right)^2 - d(z, U_n x_n)^2 \\ &\le \sum_{k=1}^{N} \alpha_n^k d(x_n, z)^2 - d(z, U_n x_n)^2 \\ &= d(x_n, z)^2 - d(z, U_n x_n)^2. \end{aligned}$$

Since $0 < a \le \alpha_n^k$, we obtain $\lim_{n\to\infty} d(R_{\lambda_n^k f^k} x_n, U_n x_n) = 0$ for every $k = 1, 2, \dots, N$. Since $\lim_{n\to\infty} d(U_n x_n, x_n) = 0$, we also get $\lim_{n\to\infty} d(R_{\lambda_n^k f^k} x_n, x_n) = 0$ for every $k = 1, 2, \dots, N$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_r}\}$ of $\{x_n\}$ which Δ-converges to a point $x_0 \in X$. Assume $x_0 \notin \operatorname{argmin}_X f^1$. Then we get

$$\begin{aligned} \limsup_{r\to\infty} d\left(x_{n_r}, x_0\right) &< \limsup_{r\to\infty} d\left(x_{n_r}, R_{\lambda_n^1 f^1} x_0\right) \\ &\le \limsup_{r\to\infty} \left(d\left(x_{n_r}, R_{\lambda_n^1 f^1} x_{n_r}\right) + d\left(R_{\lambda_n^1 f^1} x_{n_r}, R_{\lambda_n^1 f^1} x_0\right)\right) \\ &\le \limsup_{r\to\infty} d\left(x_{n_r}, x_0\right). \end{aligned}$$

We obtain a contradiction and $x_0 \in \operatorname{argmin}_X f^1$. Similarly, we can show $x_0 \in \operatorname{argmin}_X f^k$ for all $k = 1, 2, \dots, N$. Suppose that there are two subsequences

$\{u_i\}$ and $\{v_i\}$ of $\{x_n\}$ which Δ-converges to u_0 and v_0, respectively. Then we obtain that $u_0, v_0 \in \bigcap_{k=1}^N \operatorname{argmin}_X f^k$ and both $\{d(x_n, u_0)\}$ and $\{d(x_n, v_0)\}$ have limits. Assume that $u_0 \neq v_0$, then we obtain

$$\begin{aligned}\lim_{n\to\infty} d(x_n, u_0) &= \lim_{i\to\infty} d(u_i, u_0)\\ &< \lim_{i\to\infty} d(u_i, v_0)\\ &= \lim_{n\to\infty} d(x_n, v_0)\\ &= \lim_{i\to\infty} d(v_i, v_0)\\ &< \lim_{i\to\infty} d(v_i, u_0)\\ &= \lim_{n\to\infty} d(x_n, u_0)\end{aligned}$$

It is a contradiction and thus $u_0 = v_0$. Hence we obtain $\{x_n\}$ Δ-converges to $x_0 \in F$. □

Theorem 3.2. *Let X be a complete* CAT(0) *space. Let f^k be a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ for every $k = 1, 2, \dots, N$ such that $F = \bigcap_{k=1}^N \operatorname{argmin}_X f^k \neq \emptyset$. For a given real number $a \in \left]0, \frac{1}{2}\right]$, let $\{\beta_n\} \subset]0,1[$, $\left\{\alpha_n^k\right\} \subset [a, 1-a]$ and $\left\{\lambda_n^k\right\} \subset [a, \infty[$ for every $k = 1, 2, \dots, N$ and $n \in \mathbb{N}$ such that $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^\infty \beta_n = \infty$, $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$, $\sum_{k=1}^N \alpha_n^k = 1$, $\sum_{n=1}^\infty \sum_{k=1}^N \left|\alpha_{n+1}^k - \alpha_n^k\right| < \infty$ and $\sum_{n=1}^\infty \sum_{k=1}^N \left|\lambda_{n+1}^k - \lambda_n^k\right| < \infty$. Let $R_{\lambda_n^k f^k}$ be the resolvent of $\lambda_n^k f^k$ for every $k = 1, 2, \dots, N$ and $n \in \mathbb{N}$. Define U_n be a mapping from X to X by*

$$U_n x = \operatorname*{argmin}_{y\in X} \sum_{k=1}^N \alpha_n^k d\left(R_{\lambda_n^k f^k} x, y\right)^2$$

for every $x \in X$ and $n \in \mathbb{N}$. For given points $u, x_1 \in X$, let $\{x_n\}$ be a sequence in X generated by

$$x_{n+1} = \beta_n u \oplus (1-\beta_n) U_n x_n$$

for every $n \in \mathbb{N}$. Then $\{x_n\}$ converges to $P_F u$.

Proof. We show boundedness of $\{x_n\}$ and $\{U_n x_n\}$. Let $z \in F$. Then we have

$$\begin{aligned}d(x_{n+1}, z) &= d(\beta_n u \oplus (1-\beta_n)U_n x_n, z)\\ &\le \beta_n d(u, z) + (1-\beta_n) d(U_n x_n, z)\\ &\le \beta_n d(u, z) + (1-\beta_n) d(x_n, z)\\ &\le \max\{d(u, z), d(x_n, z)\}\\ &\le \max\{d(u, z), d(x_1, z)\}.\end{aligned}$$

Thus we obtain $\{x_n\}$ and $\{U_n x_n\}$ are bounded. We also have

$$\begin{aligned}
d(x_{n+2}, x_{n+1}) &\leq d(\beta_{n+1} u \oplus (1-\beta_{n+1}) U_{n+1} x_{n+1}, \beta_n u \oplus (1-\beta_n) U_n x_n) \\
&\leq d(\beta_{n+1} u \oplus (1-\beta_{n+1}) U_{n+1} x_{n+1}, \beta_n u \oplus (1-\beta_n) U_{n+1} x_{n+1}) \\
&\quad + d(\beta_n u \oplus (1-\beta_n) U_{n+1} x_{n+1}, \beta_n u \oplus (1-\beta_n) U_n x_n) \\
&\leq |\beta_{n+1} - \beta_n| \, d(U_{n+1} x_{n+1}, u) + (1-\beta_n) d(U_{n+1} x_{n+1}, U_n x_n) \\
&\leq |\beta_{n+1} - \beta_n| \, d(U_{n+1} x_{n+1}, u) + (1-\beta_n)(d(U_{n+1} x_{n+1}, U_n x_{n+1}) \\
&\quad + d(U_n x_{n+1}, U_n x_n)) \\
&\leq (1-\beta_n) d(x_{n+1}, x_n) + |\beta_{n+1} - \beta_n| \, d(U_{n+1} x_{n+1}, u) \\
&\quad + d(U_{n+1} x_{n+1}, U_n x_{n+1}).
\end{aligned}$$

We show $\sum_{n=1}^{\infty} d(U_{n+1} x_{n+1}, U_n x_{n+1}) < \infty$. Let $t \in \,]0,1[$. For all $x \in X$, we have

$$\begin{aligned}
&\sum_{k=1}^{N} \alpha_n^k d\left(R_{\lambda_n^k f^k} x, U_n x\right)^2 \\
&\quad \leq \sum_{k=1}^{N} \alpha_n^k d\left(R_{\lambda_n^k f^k} x, t U_n x \oplus (1-t) U_{n+1} x\right)^2 \\
&\quad \leq t \sum_{k=1}^{N} \alpha_n^k d\left(R_{\lambda_n^k f^k} x, U_n x\right)^2 + (1-t) \sum_{k=1}^{N} \alpha_n^k d\left(R_{\lambda_n^k f^k} x, U_{n+1} x\right)^2 \\
&\qquad - t(1-t) \sum_{k=1}^{N} \alpha_n^k d(U_n x, U_{n+1} x)^2 \\
&\quad = t \sum_{k=1}^{N} \alpha_n^k d\left(R_{\lambda_n^k f^k} x, U_n x\right)^2 + (1-t) \sum_{k=1}^{N} \alpha_n^k d\left(R_{\lambda_n^k f^k} x, U_{n+1} x\right)^2 \\
&\qquad - t(1-t) d(U_n x, U_{n+1} x)^2.
\end{aligned}$$

Since $1 - t > 0$, we obtain

$$t d(U_{n+1} x, U_n x)^2 \leq \sum_{k=1}^{N} \alpha_n^k d\left(R_{\lambda_n^k f^k} x, U_{n+1} x\right)^2 - \sum_{k=1}^{N} \alpha_n^k d\left(R_{\lambda_n^k f^k} x, U_n x\right)^2.$$

Tending $t \to 1$, we have

$$d(U_{n+1} x, U_n x)^2 \leq \sum_{k=1}^{N} \alpha_n^k d\left(R_{\lambda_n^k f^k} x, U_{n+1} x\right)^2 - \sum_{k=1}^{N} \alpha_n^k d\left(R_{\lambda_n^k f^k} x, U_n x\right)^2.$$

Similarly, we have

$$d(U_{n+1}x, U_nx)^2 \le \sum_{k=1}^{N} \alpha_{n+1}^k d\left(R_{\lambda_{n+1}^k f^k} x, U_nx\right)^2 - \sum_{k=1}^{N} \alpha_{n+1}^k d\left(R_{\lambda_{n+1}^k f^k} x, U_{n+1}x\right)^2.$$

From the above two inequalities, we get

$$d(U_{n+1}x, U_nx)^2 \le \frac{1}{2}\sum_{k=1}^{N}\Bigg(\alpha_n^k d\left(R_{\lambda_n^k f^k}x, U_{n+1}x\right)^2 - \alpha_n^k d\left(R_{\lambda_n^k f^k}x, U_nx\right)^2 + \alpha_{n+1}^k d\left(R_{\lambda_{n+1}^k f^k}x, U_nx\right)^2 - \alpha_{n+1}^k d\left(R_{\lambda_{n+1}^k f^k}x, U_{n+1}x\right)^2\Bigg).$$

Put $D = d(R_{\lambda_n^k f^k}x, R_{\lambda_{n+1}^k f^k}x)$. We obtain

$$\begin{aligned}
&\alpha_n^k d\left(R_{\lambda_n^k f^k}x, U_{n+1}x\right)^2 - \alpha_{n+1}^k d\left(R_{\lambda_{n+1}^k f^k}x, U_{n+1}x\right)^2 \\
&\quad \le \alpha_n^k \left(D + d\left(R_{\lambda_{n+1}^k f^k}x, U_{n+1}x\right)\right)^2 - \alpha_{n+1}^k d\left(R_{\lambda_{n+1}^k f^k}x, U_{n+1}x\right)^2 \\
&\quad = \alpha_n^k \left(D^2 + 2Dd\left(R_{\lambda_{n+1}^k f^k}x, U_{n+1}x\right) + d\left(R_{\lambda_{n+1}^k f^k}x, U_{n+1}x\right)^2\right) \\
&\qquad - \alpha_{n+1}^k d\left(R_{\lambda_{n+1}^k f^k}x, U_{n+1}x\right)^2 \\
&\quad = \alpha_n^k \left(D^2 + 2Dd\left(R_{\lambda_{n+1}^k f^k}x, U_{n+1}x\right)\right) \\
&\qquad + \left|\alpha_{n+1}^k - \alpha_n^k\right| d\left(R_{\lambda_{n+1}^k f^k}x, U_{n+1}x\right)^2.
\end{aligned}$$

Summarizing above inequalities, we get

$$\begin{aligned}
&d(U_{n+1}x, U_nx)^2 \\
&\le \frac{1}{2}\sum_{k=1}^{N}\Bigg(\left|\alpha_{n+1}^k - \alpha_n^k\right| \left(d\left(R_{\lambda_n^k f^k}x, U_nx\right)^2 + d\left(R_{\lambda_{n+1}^k f^k}x, U_{n+1}x\right)^2\right) \\
&\quad + \alpha_n^k \left(D^2 + 2Dd\left(R_{\lambda_{n+1}^k f^k}x, U_{n+1}x\right)\right) + \alpha_{n+1}^k \left(D^2 + 2Dd\left(R_{\lambda_n^k f^k}x, U_nx\right)\right)\Bigg) \\
&\le \sum_{k=1}^{N}\left(4\left|\alpha_{n+1}^k - \alpha_n^k\right| d(x,z)^2 + D^2 + 4Dd(x,z)\right).
\end{aligned}$$

On the other hand, by Lemma 2.3, we have

$$
\begin{aligned}
&d\left(R_{\lambda_{n+1}^k f^k}x, R_{\lambda_n^k f^k}x\right)^2 \\
&\quad\le \frac{\lambda_{n+1}^k-\lambda_n^k}{\lambda_{n+1}^k+\lambda_n^k}\left(d\left(R_{\lambda_{n+1}^k f^k}x, x\right)^2 - d\left(R_{\lambda_n^k f^k}x, x\right)^2\right) \\
&\quad\le \frac{\left|\lambda_{n+1}^k-\lambda_n^k\right|}{2a}\left(d\left(R_{\lambda_{n+1}^k f^k}x, x\right) + d\left(R_{\lambda_n^k f^k}x, x\right)\right) \\
&\qquad\qquad\qquad\qquad \left|d\left(R_{\lambda_{n+1}^k f^k}x, x\right) - d\left(R_{\lambda_n^k f^k}x, x\right)\right| \\
&\quad\le \frac{\left|\lambda_{n+1}^k-\lambda_n^k\right|}{2a}\left(d\left(R_{\lambda_{n+1}^k f^k}x, x\right) + d\left(R_{\lambda_n^k f^k}x, x\right)\right) \\
&\qquad\qquad\qquad\qquad d\left(R_{\lambda_{n+1}^k f^k}x, R_{\lambda_n^k f^k}x\right) \\
&\quad\le \frac{\left|\lambda_{n+1}^k-\lambda_n^k\right|}{2a}\left(d\left(R_{\lambda_{n+1}^k f^k}x, z\right) + d\left(R_{\lambda_n^k f^k}x, z\right) + 2d(x,z)\right) \\
&\qquad\qquad\qquad\qquad d\left(R_{\lambda_{n+1}^k f^k}x, R_{\lambda_n^k f^k}x\right) \\
&\quad\le \frac{\left|\lambda_{n+1}^k-\lambda_n^k\right|}{2a}\cdot 4d(x,z)d\left(R_{\lambda_{n+1}^k f^k}x, R_{\lambda_n^k f^k}x\right).
\end{aligned}
$$

Then we get

$$
d\left(R_{\lambda_{n+1}^k f^k}x, R_{\lambda_n^k f^k}x\right) \le \frac{2\left|\lambda_{n+1}^k-\lambda_n^k\right|}{a}d(x,z).
$$

By the above inequality, we have

$$
\begin{aligned}
&d(U_{n+1}x, U_nx)^2 \\
&\qquad \le 4d(x,z)^2\sum_{k=1}^N\left(\left|\alpha_{n+1}^k-\alpha_n^k\right| + \frac{(\lambda_{n+1}^k-\lambda_n^k)^2}{a^2} + \frac{2\left|\lambda_{n+1}^k-\lambda_n^k\right|}{a}\right).
\end{aligned}
$$

Since $\sum_{n=1}^\infty\sum_{k=1}^N\left|\alpha_{n+1}^k-\alpha_n^k\right| < \infty$, $\sum_{n=1}^\infty\sum_{k=1}^N\left|\lambda_{n+1}^k-\lambda_n^k\right| < \infty$ and boundedness of $\{x_n\}$, we obtain $\sum_{n=1}^\infty d(U_{n+1}x_{n+1}, U_nx_{n+1}) < \infty$. By Lemma 2.4, we have $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$. Furthermore,

$$
\begin{aligned}
d(U_nx_n, x_n) &\le d(U_nx_n, x_{n+1}) + d(x_{n+1}, x_n) \\
&\le d(U_nx_n, \beta_nu \oplus (1-\beta_n)U_nx_n) + d(x_{n+1}, x_n) \\
&\le \beta_nd(U_nx_n, u) + d(x_{n+1}, x_n).
\end{aligned}
$$

Since $\lim_{n\to\infty}\beta_n = 0$ and $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$, we get $\lim_{n\to\infty} d(U_n x_n, x_n) = 0$. We show $\limsup_{n\to\infty}\left(d(u, P_F u)^2 - (1-\beta_n)d(u, U_n x_n)^2\right) \le 0$. We have

$$\begin{aligned}
&\left|\left(d(u, P_F u)^2 - (1-\beta_n)d(u, U_n x_n)^2\right) - \left(d(u, P_F u)^2 - d(u, x_n)^2\right)\right| \\
&\quad = \left|d(u, x_n)^2 - d(u, U_n x_n)^2 + \beta_n d(u, U_n x_n)^2\right| \\
&\quad \le \left|d(u, x_n)^2 - d(u, U_n x_n)^2\right| + \beta_n d(u, U_n x_n)^2 \\
&\quad = |(d(u, x_n) + d(u, U_n x_n))(d(u, x_n) - d(u, U_n x_n))| + \beta_n d(u, U_n x_n)^2 \\
&\quad = |d(u, x_n) + d(u, U_n x_n)|\,|d(u, x_n) - d(u, U_n x_n)| + \beta_n d(u, U_n x_n)^2 \\
&\quad \le |d(u, x_n) + d(u, U_n x_n)|\, d(U_n x_n, x_n) + \beta_n d(u, U_n x_n)^2.
\end{aligned}$$

Since $\lim_{n\to\infty}\beta_n = 0$, $\lim_{n\to\infty} d(U_n x_n, x_n) = 0$ and boundedness of $\{x_n\}$, $\{U_n x_n\}$ we get $\lim_{n\to\infty}\left|\left(d(u, P_F u)^2 - (1-\beta_n)d(u, U_n x_n)^2\right) - \left(d(u, P_F u)^2 - d(u, x_n)^2\right)\right| = 0$. From boundedness of $\{x_n\}$, we can take a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ Δ-converges to x_0 and $\liminf_{n\to\infty} d(u, x_n) = \lim_{i\to\infty} d(u, x_{n_i})$. Thus we obtain

$$\begin{aligned}
\limsup_{n\to\infty}\left(d(u, P_F u)^2 - (1-\beta_n)d(u, U_n x_n)^2\right) &= \limsup_{n\to\infty}\left(d(u, P_F u)^2 - d(u, x_n)^2\right) \\
&= d(u, P_F u)^2 - \liminf_{i\to\infty} d(u, x_{n_i})^2 \\
&\le d(u, P_F u)^2 - d(u, x_0)^2.
\end{aligned}$$

Since $\sum_{n=1}^{\infty}\sum_{k=1}^{N}\left|\alpha_{n+1}^k - \alpha_n^k\right| < \infty$ and $\sum_{n=1}^{\infty}\sum_{k=1}^{N}\left|\lambda_{n+1}^k - \lambda_n^k\right| < \infty$, we obtain $\left\{\alpha_n^k\right\}$ converges to $\alpha^k \in [a, 1-a]$ and $\left\{\lambda_n^k\right\}$ converges to $\lambda^k \in [a, \infty[$ for every $k = 1, 2, \ldots, N$. Let $Ux = \operatorname{argmin}_{y\in X}\sum_{k=1}^{N}\alpha^k d\left(R_{\lambda^k f^k} x, y\right)^2$. Then we have

$$\begin{aligned}
&\lim_{n\to\infty} d(U_n x_n, U x_n) \\
&\quad \le 2\lim_{n\to\infty} d(x_n, z)\sqrt{\sum_{k=1}^{N}\left(|\alpha_n^k - \alpha^k| + \frac{(\lambda_n^k - \lambda^k)^2}{a^2} + \frac{2\,|\lambda_n^k - \lambda^k|}{a}\right)} \\
&\quad = 0.
\end{aligned}$$

Therefore, we obtain $\lim_{n\to\infty} d(U x_n, x_n) = 0$. Since $F = \bigcap_{k=1}^{N}\operatorname{argmin}_X f^k$ and Lemma 2.2, we have $x_0 \in F$. Therefore, we get $d(u, P_F u) \le d(u, x_0)$. We also obtain $\limsup_{n\to\infty}(d(u, P_F u)^2 - (1-\beta_n)d(u, U_n x_n)^2) \le 0$. From this inequality, we have

$$\begin{aligned} d(x_{n+1}, P_F u)^2 &\leq d(\beta_n u \oplus (1-\beta_n) U_n x_n, P_F u)^2 \\ &\leq \beta_n d(u, P_F u)^2 + (1-\beta_n) d(U_n x_n, P_F u)^2 \\ &\quad - \beta_n (1-\beta_n) d(u, U_n x_n)^2 \\ &\leq (1-\beta_n) d(x_n, P_F u)^2 + \beta_n (d(u, P_F u)^2 \\ &\quad - (1-\beta_n) d(u, U_n x_n)^2). \end{aligned}$$

By Lemma 2.4, we have $\lim_{n\to\infty} d(x_n, P_F u) = 0$. □

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