Proceedings of the International Conference on
Nonlinear Analysis and Convex Analysis &
International Conference on Optimization:
Techniques and Applications -I-
(Hakodate, Japan, 2019), 183–195

A VICINAL MAPPING ON GEODESIC SPACES

TAKUTO KAJIMURA AND YASUNORI KIMURA

Abstract. Kohsaka proposed the concept of vicinal mapping on geodesic spaces with curvature bounded above. He also showed its fundamental properties and a fixed point theorem. On the other hand, the authors proposed a new notion of spherically vicinal mappings containing the original one. They also showed its fundamental properties. In this paper, we propose a new concept of vicinal mappings on geodesic spaces and show its fundamental properties. We also show a convergence theorem to its fixed point.

1. Introduction

Kohsaka and Takahashi [11] proposed the concept of nonspreading mappings on smooth Banach spaces. In a smooth Banach space E, a mapping T from E into itself is said to be nonspreading if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leqq \phi(Tx, y) + \phi(Ty, x)$$

for all $x, y \in E$, where $\phi(u, v) = \|u\|^2 - 2\langle u, Jv\rangle + \|v\|^2$ for all $u, v \in E$ and J is the duality mapping from E into E^*. On the other hand, Kohsaka [10] proposed the concept of metrically nonspreading mappings and firmly metrically nonspreading mappings on metric spaces. In a metric space X, a mapping T from X into itself is said to be

- metrically nonspreading if
$$2d(Tx, Ty)^2 \leqq d(Tx, y)^2 + d(Ty, x)^2$$
for all $x, y \in X$;
- firmly metrically nonspreading if
$$d(Tx, x)^2 + d(Ty, y)^2 + 2d(Tx, Ty)^2 \leqq d(Tx, y)^2 + d(Ty, x)^2$$
for all $x, y \in X$.

2010 *Mathematics Subject Classification.* 52A41.
Key words and phrases. Convex function, resolvent, vicinal mapping, firmly vicinal mapping.

Every firmly metrically nonspreading mapping is clearly metrically nonspreading. In particular, nonspreadingness and metrical nonspreadingness naturally coincide with each other in Hilbert spaces. It is well known that the classical resolvent which was proposed by Mayer [12] on CAT(0) spaces is firmly metrically nonspreading.

The concepts of vicinal mappings and firmly vicinal mappings were proposed by Kohsaka [9] on metric spaces. In a metric space X, a mapping T from X into itself is said to be

- vicinal if
$$\begin{aligned}&(L_x^2(1+L_y^2)+L_y^2(1+L_x^2))\cos d(Tx,Ty)\\&\geqq L_x^2(1+L_y^2)\cos d(Tx,y)+L_y^2(1+L_x^2)\cos d(x,Ty)\end{aligned}$$
for all $x, y \in X$;
- firmly vicinal if
$$\begin{aligned}&(L_x^2(1+L_y^2)L_y+L_y^2(1+L_x^2)L_x)\cos d(Tx,Ty)\\&\geqq L_x^2(1+L_y^2)\cos d(Tx,y)+L_y^2(1+L_x^2)\cos d(x,Ty)\end{aligned}$$
for all $x, y \in X$,

where $L_z = \cos d(Tz, z)$ for all $z \in X$. If a metric space X satisfies $d(u, v) < \pi/2$ for all $u, v \in X$, then every firmly vicinal mapping is vicinal. It is known that the resolvent which was proposed by Kimura and Kohsaka [5] on CAT(1) spaces is firmly vicinal.

The authors [3] proposed spherically vicinal mappings with η and firmly spherically vicinal mappings with η on admissible CAT(1) spaces. In an admissible CAT(1) space X, a mapping T from X into itself is said to be

- spherically vicinal with η if
$$\begin{aligned}&(\eta'(L_x)+\eta'(L_y))\cos d(Tx,Ty)\\&\leqq \eta'(L_y)\cos d(Tx,y)+\eta'(L_x)\cos d(Ty,x)\end{aligned}$$
for all $x, y \in X$;
- firmly spherically vicinal with η if
$$\begin{aligned}&(\eta'(L_x)L_x+\eta'(L_y)L_y)\cos d(Tx,Ty)\\&\leqq \eta'(L_y)\cos d(Tx,y)+\eta'(L_x)\cos d(Ty,x)\end{aligned}$$
for all $x, y \in X$,

where $L_z = \cos d(Tz, z)$ for all $z \in X$ and η is a function from $]0, 1]$ into $[0, \infty[$ satisfying η is differentiable, η' is continuous at 1, and $\eta'(t) < 0$ for all $t \in]0, 1]$. Since X is an admissible CAT(1) space, every firmly spherically nonspreading mapping with η is obviously spherically nonspreading with η. They showed that every firmly vicinal mapping on admissible CAT(1) spaces is firmly spherically vicinal with $\eta\colon t \mapsto 1/t - t$ and also showed that the resolvent which was proposed by themselves [4] on CAT(1) spaces is firmly spherically vicinal with $\eta\colon t \mapsto -\log t$.

In this paper, we propose vicinal mappings with φ and firmly vicinal mappings with φ on CAT(κ) spaces and show that both firmly metrically nonspreading mappings on CAT(0) spaces and firmly spherically vicinal mappings with ψ are examples of vicinal mappings with φ. We also show fundamental properties and an approximation theorem to a fixed point for those mappings.

2. Preliminaries

Let X be a metric space. We denote the set of all fixed points of a mapping T from X into itself by $\mathcal{F}(T)$ and the set of all minimizers of a function f from X into $]-\infty, \infty]$ by $\operatorname{argmin}_{x\in X} f(x)$, respectively. An asymptotic center of a sequence $\{x_n\}$ of X is defined by the set

$$\left\{ u \in X \;\middle|\; \limsup_{n\to\infty} d(u, x_n) = \inf_{y\in X} \limsup_{n\to\infty} d(y, x_n) \right\},$$

and we denote it by $\mathcal{A}(\{x_n\})$. $\{x_n\}$ is Δ-convergent to $x_0 \in X$ which is denoted by $x_n \overset{\Delta}{\rightharpoonup} x_0$ if $\mathcal{A}(\{x_{n_i}\}) = \{x_0\}$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$, and we refer to x_0 as Δ-limit of $\{x_n\}$. A mapping T form X into itself is said to be

- quasi-nonexpansive if $\mathcal{F}(T)$ is nonempty and $d(Tx, p) \leqq d(x, p)$ for all $x \in X$ and $p \in \mathcal{F}(T)$;
- asymptotically regular if $\lim_{n\to\infty} d(T^{n+1}x, T^n x) = 0$ for all $x \in X$;
- Δ-demiclosed if p belongs to $\mathcal{F}(T)$ whenever a sequence $\{x_n\}$ of X satisfies $x_n \overset{\Delta}{\rightharpoonup} p \in X$ and $\lim_{n\to\infty} d(Tx_n, x_n) = 0$.

For $D \in\,]0, \infty]$, a metric space X is called a D-geodesic space if for each $x, y \in X$ with $d(x, y) < D$, there exists a mapping γ_{xy} from $[0, \ell]$ into X satisfying $\gamma_{xy}(0) = x$, $\gamma_{xy}(\ell) = y$, and $d(\gamma_{xy}(s), \gamma_{xy}(t)) = |s - t|$ for all $s, t \in [0, \ell]$, where $\ell = d(x, y)$. We refer to γ_{xy} as a geodesic joining x and y. We denote the image of γ_{xy} by $\mathrm{Im}\gamma_{xy}$, that is, $\mathrm{Im}\gamma_{xy} = \{\gamma(s) \mid s \in [0, \ell]\}$. In this paper, we assume the uniqueness of $\mathrm{Im}\gamma_{xy}$ for any two points x and y in D-geodesic spaces. Let X be a D-geodesic space. For each $x, y \in X$ and $t \in [0, 1]$, there exists a unique element $z \in X$ such that $d(x, z) = (1 - t)d(x, y)$, and we denote it by $z = tx \oplus (1 - t)y$. Such an element $z \in X$ is called a convex combination between x and y. For each $x, y, z \in X$, the set $\triangle(x, y, z) = \mathrm{Im}\gamma_{xy} \cup \mathrm{Im}\gamma_{yz} \cup \mathrm{Im}\gamma_{zx}$ is called a geodesic triangle. A function f from X into $]-\infty, \infty]$ is said to be convex if $f(tx \oplus (1 - t)y) \leqq tf(x) + (1 - t)f(y)$ for all $x, y \in X$ and $t \in\,]0, 1[$.

Let X be a D-geodesic space. For $\kappa \in \mathbb{R}$, a two-dimensional model space of X with curvature κ is defined by

$$M_\kappa^2 = \begin{cases} \dfrac{1}{\sqrt{\kappa}}\mathbb{S}^2 & (\kappa > 0); \\ \mathbb{R}^2 & (\kappa = 0); \\ \dfrac{1}{\sqrt{-\kappa}}\mathbb{H}^2 & (\kappa < 0), \end{cases}$$

where $\mathbb{S}^2$ is the two-dimensional unit sphere in $\mathbb{R}^3$, and $\mathbb{H}^2$ is the two-dimensional hyperbolic space. We denote the diameter of M_κ^2 by D_κ, that is, $D_\kappa = \pi/\sqrt{\kappa}$ for all $\kappa > 0$ and $D_\kappa = \infty$ for all $\kappa \leqq 0$. It is well known that M_κ^2 is a D_κ-geodesic space. For each $\triangle(x, y, z) \subset X$ satisfying $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$, a comparison triangle of $\triangle(x, y, z)$ is defined by the set $\triangle(\bar{x}, \bar{y}, \bar{z}) \subset M_\kappa^2$ satisfying $d(x, y) = d(\bar{x}, \bar{y})$, $d(y, z) = d(\bar{y}, \bar{z})$, and $d(z, x) = d(\bar{z}, \bar{x})$, respectively. For each $p \in \triangle(x, y, z) \subset X$, we call $\bar{p} \in \triangle(\bar{x}, \bar{y}, \bar{z}) \subset M_\kappa^2$ a comparison point of p whenever at least one of the following conditions hold:

- if $p \in \mathrm{Im}\gamma_{xy}$, then $\bar{p} \in \mathrm{Im}\gamma_{\bar{x}\bar{y}}$ and $d(x, p) = d(\bar{x}, \bar{p})$;
- if $p \in \mathrm{Im}\gamma_{yz}$, then $\bar{p} \in \mathrm{Im}\gamma_{\bar{y}\bar{z}}$ and $d(y, p) = d(\bar{y}, \bar{p})$;
- if $p \in \mathrm{Im}\gamma_{zx}$, then $\bar{p} \in \mathrm{Im}\gamma_{\bar{z}\bar{x}}$ and $d(z, p) = d(\bar{z}, \bar{p})$.

A D-geodesic space X is called a CAT(κ) space if $d(p, q) \leqq d(\bar{p}, \bar{q})$ for all $x, y, z \in X$ and $p, q \in \triangle(x, y, z)$, where $\bar{p}$ and $\bar{q}$ are the comparison points of p and q, respectively. A CAT(κ) space X is said to be admissible if $d(x, y) < D_\kappa/2$ for all $x, y \in X$. In CAT(κ) spaces, the following inequalities are well known. See [2, 7] for example.

Lemma 2.1. *For $\kappa \in \{1, 0, -1\}$, let X be a* CAT(κ) *space and $x, y, z \in X$ satisfying $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$. Then the following inequalities hold:*

- *if $\kappa = 1$, then*
$$\begin{aligned}&\cos d(tx \oplus (1-t)y, z) \sin d(x, y) \\ &\geqq \cos d(x, z) \sin(td(x, y)) + \cos d(y, z) \sin((1-t)d(x, y));\end{aligned}$$
- *if $\kappa = 0$, then*
$$d(tx \oplus (1-t)y, z)^2 \leqq td(x, z)^2 + (1-t)d(y, z)^2 - t(1-t)d(x, y)^2;$$
- *if $\kappa = -1$, then*
$$\begin{aligned}&\cosh d(tx \oplus (1-t)y, z) \sinh d(x, y) \\ &\leqq \cosh d(x, z) \sinh(td(x, y)) + \cosh d(y, z) \sinh((1-t)d(x, y)).\end{aligned}$$

By using the properties of the functions cosine and hyperbolic cosine, it is easy to check that these inequalities are equivalent to the following inequalities.

Corollary 2.2. *For $\kappa \in \{1, 0, -1\}$, let X be a* CAT(κ) *space and $x, y, z \in X$ satisfying $d(x,y)+d(y,z)+d(z,x) < 2D_\kappa$. Then the following inequalities hold:*

- *if $\kappa = 1$, then*

$$\begin{aligned}&(1-\cos d(tx \oplus (1-t)y, z)) \sin d(x,y)\\&\leqq (1-\cos d(x,z)) \sin(td(x,y)) + (1-\cos d(y,z)) \sin((1-t)d(x,y))\\&\quad - 4\sin(d(x,y)/2)\sin(td(x,y)/2)\sin((1-t)d(x,y)/2);\end{aligned}$$

- *if $\kappa = 0$, then*

$$d(tx \oplus (1-t)y, z)^2 \leqq td(x,z)^2 + (1-t)d(y,z)^2 - t(1-t)d(x,y)^2;$$

- *if $\kappa = -1$, then*

$$\begin{aligned}&(\cosh d(tx \oplus (1-t)y, z) - 1) \sinh d(x,y)\\&\leqq (\cosh d(x,z) - 1) \sinh(td(x,y)) + (\cosh d(y,z) - 1) \sinh((1-t)d(x,y))\\&\quad - 4\sinh(d(x,y)/2)\sinh(td(x,y)/2)\sinh((1-t)d(x,y)/2).\end{aligned}$$

Let $\kappa \in \mathbb{R}$ and X a CAT(κ) space. For $D \in]0, \infty]$, a sequence $\{x_n\}$ of X is said to be D-bounded if $\{x_n\}$ satisfies

$$\inf_{y \in X} \limsup_{n\to\infty} d(x_n, y) < D.$$

The following fundamental properties are well known.

Lemma 2.3 ([1, 8])**.** *Let $\kappa \in \mathbb{R}$, X an admissible* CAT(κ) *space and $\{x_n\}$ a $D_\kappa/2$-bounded sequence of X. Then $\mathcal{A}(\{x_n\})$ consists of one point, and $\{x_n\}$ has a Δ-convergent subsequence.*

Lemma 2.4 ([6])**.** *Let $\kappa \in \mathbb{R}$, X an admissible* CAT(κ) *space and $\{x_n\}$ a $D_\kappa/2$-bounded sequence of X. If $\{d(x_n, z)\}$ is convergent for each Δ-limit z of subsequences of $\{x_n\}$, then $\{x_n\}$ is Δ-convergent to an element of X.*

3. A function c_κ

In this section, we introduce a function c_κ which plays an important role in this paper and we show fundamental properties of the function. For $\kappa \in \mathbb{R}$, we define a function c_κ from $\left[0, D_{\kappa/2}\right]$ into $[0,\infty]$ by

$$\begin{aligned}
c_\kappa(t) &= \frac{t^2}{2} - \frac{\kappa t^4}{24} + \frac{\kappa^2 t^6}{720} + \cdots \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\kappa^{n-1}t^{2n}}{(2n)!} \\
&= \begin{cases} \dfrac{1}{\kappa}\left(1-\cos\left(\sqrt{\kappa}t\right)\right) & (\kappa > 0); \\ \dfrac{1}{2}t^2 & (\kappa = 0); \\ \dfrac{1}{-\kappa}\left(\cosh\left(\sqrt{-\kappa}t\right) - 1\right) & (\kappa < 0), \end{cases}
\end{aligned}$$

for $t \in [0, D_\kappa/2[$ and $c_\kappa(D_\kappa/2) = \lim_{t\uparrow D_\kappa/2} c_\kappa(t)$. Then it follows that

$$c_\kappa'(t) = \begin{cases} \dfrac{1}{\sqrt{\kappa}}\sin\left(\sqrt{\kappa}t\right) & (\kappa > 0); \\ t & (\kappa = 0); \\ \dfrac{1}{\sqrt{-\kappa}}\sinh\left(\sqrt{-\kappa}t\right) & (\kappa < 0), \end{cases}$$

and that

$$c_\kappa''(t) = \begin{cases} \cos\left(\sqrt{\kappa}t\right) & (\kappa > 0); \\ 1 & (\kappa = 0); \\ \cosh\left(\sqrt{-\kappa}t\right) & (\kappa < 0) \end{cases}$$

for all $t \in [0, D_\kappa/2[$. We know that

- c_κ is convex and increasing function with $c_\kappa(0) = 0$;
- c_κ' is an increasing function with $c_\kappa'(0) = 0$;
- $c_\kappa''(0) = 1$.

We also know that

$$\kappa c_\kappa(t) + c_\kappa''(t) = 1$$

and

$$c_\kappa''(t)^2 + \kappa c_\kappa'(t)^2 = 1.$$

The sum theorems are formulated as:

$$\begin{aligned} c'_\kappa(s+t) &= c'_\kappa(s)c''_\kappa(t) + c''_\kappa(s)c'_\kappa(t), \\ c'_\kappa(s-t) &= c'_\kappa(s)c''_\kappa(t) - c''_\kappa(s)c'_\kappa(t), \\ c''_\kappa(s+t) &= c''_\kappa(s)c''_\kappa(t) - \kappa c'_\kappa(s)c'_\kappa(t), \\ c''_\kappa(s-t) &= c''_\kappa(s)c''_\kappa(t) + \kappa c'_\kappa(s)c'_\kappa(t). \end{aligned}$$

In particular, we have

$$\begin{aligned} c'_\kappa(2t) &= 2c'_\kappa(t)c''_\kappa(t), \\ c''_\kappa(2t) &= c''_\kappa(t)^2 - \kappa c'_\kappa(t)^2 \\ &= 1 - 2\kappa c'_\kappa(t)^2 \\ &= 2\kappa c''_\kappa(t)^2 - 1. \end{aligned}$$

From the formulas above, we obtain the following:

$$\begin{aligned} c'_\kappa(a) + c'_\kappa(b) &= 2c'_\kappa\left(\frac{a+b}{2}\right)c''_\kappa\left(\frac{a-b}{2}\right), \\ c'_\kappa(a) - c'_\kappa(b) &= 2c''_\kappa\left(\frac{a+b}{2}\right)c'_\kappa\left(\frac{a-b}{2}\right), \\ c''_\kappa(a) + c''_\kappa(b) &= 2c''_\kappa\left(\frac{a+b}{2}\right)c''_\kappa\left(\frac{a-b}{2}\right), \\ c''_\kappa(a) - c''_\kappa(b) &= -2\kappa c'_\kappa\left(\frac{a+b}{2}\right)c'_\kappa\left(\frac{a-b}{2}\right). \end{aligned}$$

We also have

$$\begin{aligned} c'_\kappa(s)c''_\kappa(t) &= \frac{1}{2}(c'_\kappa(s+t) + c'_\kappa(s-t)), \\ c''_\kappa(s)c'_\kappa(t) &= \frac{1}{2}(c'_\kappa(s+t) - c'_\kappa(s-t)), \\ c''_\kappa(s)c''_\kappa(t) &= \frac{1}{2}(c''_\kappa(s+t) + c''_\kappa(s-t)), \\ -\kappa c'_\kappa(s)c'_\kappa(t) &= \frac{1}{2}(c''_\kappa(s+t) - c''_\kappa(s-t)). \end{aligned}$$

By using the function c_κ and Corollary 2.2, we get the following lemma.

Lemma 3.1. *Let $\kappa \in \mathbb{R}$, X a* CAT(κ) *space, $x, y, z \in X$ satisfying $d(x, y) + d(y, z) + d(z, x) < D_\kappa$ and $t \in [0, 1]$. Then*

$$\begin{aligned} & c_\kappa(d(tx \oplus (1-t)y, z))c'_\kappa(d(x, y)) \\ & \leqq c_\kappa(d(x, z))c'_\kappa(td(x, y)) + c_\kappa(d(y, z))c'_\kappa(d(x, y)) \\ & \quad - 4c'_\kappa(d(x, y)/2)c'_\kappa(td(x, y)/2)c'_\kappa((1-t)d(x, y)/2). \end{aligned}$$

4. Vicinal mappings with φ on CAT(κ) spaces

Let X be an admissible CAT(κ) spaces, T a mapping from X into itself and ψ a function from $[0, \infty[$ into $]0, \infty[$, which is continuous at 0. T is said to be

- vicinal with ψ if
$$\begin{aligned} & (\psi(d(Tx, x)) + \psi(d(Ty, y)))\, c_\kappa(d(Tx, Ty)) \\ & \leqq \psi(d(Ty, y))c_\kappa(d(Tx, y)) + \psi(d(Tx, x))c_\kappa(d(x, Ty)) \end{aligned}$$
for all $x, y \in X$;
- firmly vicinal with ψ if
$$\begin{aligned} & (\psi(d(Tx, x))c_\kappa(d(Tx, x)) + \psi(d(Ty, y))c_\kappa(d(Ty, y)))c''_\kappa(d(Tx, Ty)) \\ & + (\psi(d(Tx, x)) + \psi(d(Ty, y)))\, c_\kappa(d(Tx, Ty)) \\ & \leqq \psi(d(Ty, y))c_\kappa(d(Tx, y)) + \psi(d(Tx, x))c_\kappa(d(x, Ty)) \end{aligned}$$
for all $x, y \in X$.

It is clear that every firmly vicinal mapping with ψ is vicinal with ψ. We first show the following important theorem. In what follows, we assume that ψ is a function from $[0, \infty[$ into $]0, \infty[$, which is continuous at 0.

Theorem 4.1. *Let X be an admissible* CAT(κ) *space, f a convex function from X into $]-\infty, \infty]$ and φ an increasing and differentiable function from $[0, c_\kappa(D_\kappa/2)[$ into $[0, \infty[$, and suppose that φ' is continuous. If*

$$Tx = \operatorname*{argmin}_{y \in X}\{f(y) + \varphi(c_\kappa(d(y, x)))\}$$

is a single-valued mapping, then

$$\begin{aligned} & (\varphi'(K_x)K_x + \varphi'(K_y)K_y)c''_\kappa(d(Tx, Ty)) + (\varphi'(K_x) + \varphi'(K_y))\, c_\kappa(d(Tx, Ty)) \\ & \leqq \varphi'(K_y)c_\kappa(d(Tx, y)) + \varphi'(K_x)c_\kappa(d(x, Ty)) \end{aligned}$$

for all $x, y \in X$, where $K_z = c_\kappa(d(Tz, z))$ for all $z \in X$.

Proof. Let $x, y \in X$ satisfying $Tx \neq Ty$, and put $z_t = tTx \oplus (1-t)Ty$ for all $t \in \,]0,1[$. By the definition of T and convexity of f, we have

$$\begin{aligned} f(Ty) + \varphi(K_y) &\leqq f(z_t) + \varphi(c_\kappa(d(z_t, y))) \\ &\leqq tf(Tx) + (1-t)f(Ty) + \varphi(c_\kappa(d(z_t, y))) \end{aligned}$$

and hence

$$t(f(Ty) - f(Tx)) \leqq \varphi(c_\kappa(d(z_t, y))) - \varphi(K_y).$$

Dividing by t and using the Lemma 3.1, we get

$$\begin{aligned} &f(Ty) - f(Tx) \\ &\leqq \frac{1}{t}(\varphi(c_\kappa(d(z_t, y))) - \varphi(K_y)) \\ &\leqq \frac{1}{t}\left(\varphi\left(\frac{c_\kappa(d(Tx,y))c'_\kappa(tD) + K_y c'_\kappa((1-t)D) - 4c'_\kappa\left(\frac{D}{2}\right)c'_\kappa\left(\frac{tD}{2}\right)c'_\kappa\left(\frac{(1-t)D}{2}\right)}{c'_\kappa(D)}\right)\right. \\ &\qquad \left. - \varphi(K_y)\right), \end{aligned}$$

where $D = d(Tx, Ty)$. Taking the limit as t tends to 0 and using l'Hospital's rule, we have

$$\begin{aligned} f(Ty) - f(Tx) &\leqq \varphi'(K_y)\left(c_\kappa(d(Tx,y)) - K_y c''_\kappa(D) - 2c'_\kappa(D/2)^2\right)\frac{D}{c'_\kappa(D)} \\ &= \varphi'(K_y)(c_\kappa(d(Tx,y)) - K_y c''_\kappa(D) - c_\kappa(D))\frac{D}{c'_\kappa(D)}. \end{aligned}$$

Since x and y are arbitrary, we also get

$$f(Tx) - f(Ty) \leqq \varphi'(K_x)(c_\kappa(d(x,Ty)) - K_x c''_\kappa(D) - c_\kappa(D))\frac{D}{c'_\kappa(D)}.$$

Adding each side of these inequalities, we get the conclusion. In the case that $Tx = Ty$, we clearly obtain the conclusion. $\square$

By Theorem 4.1, the resolvent R_f which was proposed by the authors [2] is firmly vicinal mapping with ψ. In fact, we define a function $\varphi\colon [0,\infty[\,\to [0,\infty[$

$$\varphi(t) = t + 1 - \frac{1}{t+1}.$$

Since $\tanh a \sinh a = \cosh a - 1/\cosh a$ for all $a \in \mathbb{R}$, we can express

$$R_f x = \operatorname*{argmin}_{y \in X}\{f(y) + \varphi(\cosh d(y,x))\}$$

for each $x \in X$, and we know that R_f is a single-valued mapping. It is obvious that the function φ is differentiable and

$$\varphi'(t) = 1 + \frac{1}{(t+1)^2} > 0$$

for all $t \in [0, \infty[$. Therefore, it follows from Theorem 4.1 that

$$\begin{aligned}&(\varphi'(C_x))C_x + \varphi'(C_y)C_y)c''_\kappa(d(Tx,Ty)) + (\varphi'(C_x) + \varphi'(C_y))\, c_\kappa(d(Tx,Ty))\\ &\leqq \varphi'(C_y)c_\kappa(d(Tx,y)) + \varphi'(C_x)c_\kappa(d(x,Ty))\end{aligned}$$

for all $x, y \in X$, where $C_z = \cosh d(R_f z, z)$ for all $z \in X$. We also know that

$$\psi(s) = 1 + \frac{1}{(\cosh s + 1)^2}$$

is continuous at 0. Thus R_f is firmly vicinal with ψ.

We remark that every firmly metrically nonspreading mapping on CAT(0) spaces is firmly vicinal with a positive constant function. In fact, in the case that $\kappa = 0$ and $\psi(t) = K$, the inequality of firm vicinality with ψ becomes

$$Kd(Tx,x)^2 + Kd(Ty,y)^2 + 2Kd(Tx,Ty)^2 \leqq Kd(Tx,y)^2 + Kd(x,Ty)^2,$$

where K is a positive constant in $\mathbb{R}$. Thus every firmly metrically nonspreading mapping is vicinal with a positive constant function. Also, every metrically nonspreading, spherically nonspreading mapping of sum-type, and hyperbolically nonspreading mapping is vicinal with a positive constant function. In fact, in the case that $\psi(t) = K$, the inequality of vicinality with ψ becomes

$$\begin{cases} 2K\cos d(Tx,Ty) \geqq K\cos d(Tx,y) + K\cos d(x,Ty) & (\kappa = 1);\\ 2Kd(Tx,Ty)^2 \leqq Kd(Tx,y)^2 + Kd(x,Ty)^2 & (\kappa = 0);\\ 2K\cosh d(Tx,Ty) \leqq K\cosh d(Tx,y) + K\cosh d(x,Ty) & (\kappa = -1),\end{cases}$$

where K is a positive constant in $\mathbb{R}$. Thus every metrically nonspreading, spherically nonspreading mapping of sum-type, and hyperbolically nonspreading mapping is vicinal with a positive constant function. We next show fundamental properties of firmly vicinal mappings with ψ.

Theorem 4.2. *Let X be an admissible* CAT(κ) *space and T a vicinal mapping with ψ from X into itself. If $\mathcal{F}(T)$ is nonempty, then T is quasi-nonexpansive.*

Proof. Let $x \in X$ and $p \in \mathcal{F}(T)$. Then the vicinality with ψ of T implies that

$$\begin{aligned}&(\psi(d(Tx,x)) + \psi(0))c_\kappa(d(Tx,p))\\ &\leqq \psi(0)c_\kappa(d(Tx,p)) + \psi(d(Tx,x))c_\kappa(d(x,p))\end{aligned}$$

and hence

$$\psi(d(Tx,x))c_\kappa(d(Tx,p)) \leqq \psi(d(Tx,x))c_\kappa(d(x,p)).$$

Since $\psi(d(Tx,x)) \in]0,\infty[$, we have

$$c_\kappa(d(Tx,p)) \leqq c_\kappa(d(x,p))$$

and hence

$$d(Tx,p) \leqq d(x,p).$$

Therefore T is quasi-nonexpansive. $\square$

Theorem 4.3. *Let X be an admissible* CAT(κ) *space, T a vicinal mapping with ψ from X into itself and $p \in X$. If a sequence $\{x_n\}$ of X satisfies $\mathcal{A}(\{x_n\}) = \{p\}$ and $\lim_{n\to\infty} d(Tx_n, x_n) = 0$, then p is a fixed point of T.*

Proof. Let $\{x_n\}$ be a sequence of X satisfying $\mathcal{A}(\{x_n\}) = \{p\}$ and $d(Tx_n, x_n) \to 0$ as $n \to \infty$. The vicinality with ψ of T implies that

$$\begin{aligned} &c_\kappa(d(Tx_n,Tp)) \\ &\leqq c_\kappa(d(Tx_n,p)) + \frac{\psi(d(Tx_n,x_n))}{\psi(d(Tp,p))}(c_\kappa(d(Tx_n,Tp)) - c_\kappa(d(x_n,Tp))). \end{aligned}$$

On the other hand, since $\mathcal{A}(\{x_n\}) = \{p\}$, we know that $\{x_n\}$ is bounded. Since $d(Tx_n,x_n) \to 0$, $\{Tx_n\}$ is also bounded, and it follows that $|d(Tx_n,Tp) - d(x_n,Tp)| \to 0$. We also know that the function c_κ is uniformly continuous on a bounded set. Therefore, using the boundedness of those sequences and continuity of φ at 0, we get

$$\limsup_{n\to\infty} c_\kappa(d(Tx_n,Tp)) \leqq \limsup_{n\to\infty} c_\kappa(d(Tx_n,p))$$

and hence

$$\limsup_{n\to\infty} d(Tx_n,Tp) \leqq \limsup_{n\to\infty} d(Tx_n,p).$$

Since $\mathcal{A}(\{x_n\}) = \{p\}$, we get $p \in \mathcal{F}(T)$. $\square$

Corollary 4.4. *Let X be an admissible* CAT(κ) *space. Then, every vicinal mapping with ψ from X into itself is Δ-demiclosed.*

Proof. Let $\{x_n\}$ be a sequence of X satisfying $x_n \overset{\Delta}{\rightharpoonup} p$ and $d(Tx_n,x_n) \to 0$. By the definition of Δ-convergence, we know that $\mathcal{A}(\{x_n\}) = \{p\}$. Therefore, by Theorem 4.3, p is a fixed point of T. Thus we get the conclusion. $\square$

Theorem 4.5. *Let X be an admissible* CAT(κ) *space and T a firmly vicinal mapping with ψ from X into itself. If $\mathcal{F}(T)$ is nonempty, then T is asymptotically regular.*

Proof. Let $x \in X$ and $p \in \mathcal{F}(T)$. The firm vicinalty with ψ of T implies that

$$\begin{aligned}&(\psi(d(T^{n+1}x, T^n x))c_\kappa(d(T^{n+1}x, T^n x)) + \psi(0)c_\kappa(0))c''_\kappa(d(T^{n+1}x, p))\\ &+ (\psi(d(T^{n+1}x, T^n x)) + \psi(0))c_\kappa(d(T^{n+1}x, p))\\ &\leqq \psi(0)c_\kappa(d(T^{n+1}x, p)) + \psi(d(T^{n+1}x, T^n x))c_\kappa(d(T^n x, p))\end{aligned}$$

and hence

$$\begin{aligned}&\psi(d(T^{n+1}x, T^n x))c_\kappa(d(T^{n+1}x, T^n x))c''_\kappa(d(T^{n+1}x, p))\\ &\leqq \psi(d(T^{n+1}x, T^n x))c_\kappa(d(T^n x, p)) - \psi(d(T^{n+1}x, T^n x))c_\kappa(d(T^{n+1}x, p)).\end{aligned}$$

Since $\psi(d(T^{n+1}x, T^n x)) \in]0, \infty[$ and $c''_\kappa(d(T^{n+1}x, p)) \in]0, \infty[$, we get

$$c_\kappa(d(T^{n+1}x, T^n x)) \leqq \frac{c_\kappa(d(T^n x, p)) - c_\kappa(d(T^{n+1}x, p))}{c''_\kappa(d(T^{n+1}x, p))}.$$

On the other hand, it follows from Theorem 4.2 that

$$0 \leqq d(T^n x, p) \leqq d(T^{n-1}x, p) \leqq \cdots \leqq d(x, p) < \frac{D_\kappa}{2}.$$

Thus there exists $a \in [0, D_\kappa/2[$ such that $d(T^n x, p) \to a$. Therefore, it follows that

$$0 \leqq c_\kappa(d(T^{n+1}x, T^n x)) \leqq \frac{c_\kappa(d(T^n x, p)) - c_\kappa(d(T^{n+1}x, p))}{c''_\kappa(d(T^{n+1}x, p))} \to 0$$

and hence $d(T^{n+1}x, T^n x) \to 0$. Thus T is asymptotically regular. □

Using Theorems 4.2, 4.4, and 4.5, we next show the following convergence theorem.

Theorem 4.6. *Let X be an admissible complete* CAT(κ) *space and T a firmly vicinal mapping with ψ from X into itself. If $\mathcal{F}(T)$ is nonempty, then $\{T^n x\}$ is Δ-convergent to an element of $\mathcal{F}(T)$.*

Proof. Let $x \in X$. It follows from Theorem 4.2 that

$$\inf_{y \in X} \limsup_{n \to \infty} d(T^n x, y) \leqq \inf_{y \in \mathcal{F}(T)} \limsup_{n \to \infty} d(T^n x, y) \leqq \inf_{y \in \mathcal{F}(T)} d(x, y) < \frac{\pi}{2}.$$

Therefore, by using Lemma 2.3, $\{T^n x\}$ has a Δ-convergent subsequence. Let $\{T^{n_i} x\}$ be a subsequence of $\{T^n x\}$ satisfying $T^{n_i} x \overset{\Delta}{\rightharpoonup} z$. Using Theorem 4.5, we get

$$\lim_{n \to \infty} d(T(T^n x), T^n x) = \lim_{n \to \infty} d(T^{n+1}x, T^n x) = 0.$$

Thus Corollary 4.4 implies that $z \in \mathcal{F}(T)$. It follows from Theorem 4.2 that

$$0 \leqq d(T^n x, z) \leqq d(T^{n-1}x, z) \leqq \cdots \leqq d(x, y) < \frac{\pi}{2}.$$

Therefore $\{d(T^n x, z)\}$ is convergent. Hence, by Lemma 2.4, $\{T^n x\}$ is Δ-convergent to an element of X. From Corollary 4.4 and Theorem 4.5, the Δ-limit of $\{T^n x\}$ belongs to $\mathcal{F}(T)$. $\square$

References

[1] R. Espínola and A. Fernández-León, CAT(κ)*-spaces, weak convergence and fixed points*, J. Math. Anal. Appl. **65** (2006), 762–772.

[2] T. Kajimura and Y. Kimura, *Resolvents of convex functions in complete geodesic spaces with negative curvature*, J. Fixed Point Theory Appl. **21** (2019).

[3] T. Kajimura and Y. Kimura, *A spherically vicinal mapping on geodesic spaces with curvature bounded above*, Thai J. Math. **18** (2020), 384–393.

[4] T. Kajimura and Y. Kimura, *A new definition of resolvents on complete geodesic spaces*, RIMS Kôkyûroku No. 2112 (2018).

[5] Y. Kimura and F. Kohsaka, *Spherical nonspreadingness of resolvents of convex functions in geodesic spaces*, J. Fixed Point Theory Appl. **18** (2016), 93–115.

[6] Y. Kimura, S. Saejung and P. Yotkaew, *The Mann algorithm in a complete geodesic space with curvature bounded above*, Fixed Point Theory Appl. **2013** (2013), doi: 10.1186/1687-1812-2013-336, 13 pages.

[7] Y. Kimura and K. Satô, *Convergence of subset of a complete geodesic space with curvature bounded above*, Nonlinear Analysis: Theory, Methods & Applications **75** (2012), 5079–5085.

[8] W. A. Kirk and B. Panyanak, *A concept of convergence in geodesic spaces*, Nonlinear Anal. **68** (2008), 3689–3696.

[9] F. Kohsaka, *Existence and approximation of fixed points of vicinal mappings in geodesic spaces*, Pure Appl. Funct. Anal. **3** (2018), 91–106.

[10] F. Kohsaka, *Fixed points of metrically nonspreading mappings in Hadamard spaces*, arXiv:1811.12698v2 [math.FA] (2019).

[11] F. Kohsaka and W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Arch. Math. (Basel) **91** (2008), 166–177.

[12] U. F. Mayer, *Gradient flows on nonpositively curved metric spaces and harmonic maps*, Comm. Anal. Geom. **6** (1998), 199–206.

T. Kajimura

Department of Information Science, Toho University, Miyama, Funabashi, Chiba, 274-8510, Japan

E-mail address: 6518004k@st.toho-u.ac.jp

Y. Kimura

Department of Information Science, Toho University, Miyama, Funabashi, Chiba, 274-8510, Japan

E-mail address: yasunori@is.sci.toho-u.ac.jp