# ON A CLASS OF NONSMOOTH FRACTIONAL ROBUST MULTI-OBJECTIVE OPTIMIZATION PROBLEMS. PART II: DUALITY* 

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#### Abstract

In the previous paper (Hong et al. Appl. Set-Valued Anal. Optim. 2 (2020), 109-121), the authors did some works on optimality conditions for a class of nonsmooth fractional robust multi-objective optimization problems. In this paper, we further study duality results for such a class of optimization problems. More precisely, we propose modeltypes of both non-fractional and fractional dual problems; then weak, strong, and converse-like duality relations are investigated, respectively.


## 1. Introduction

Duality for fractional multi-objective optimization problems involving locally Lipschitz functions have received a great number of attention from researchers; see e.g., $[7,12,16,17,23]$ and the references therein. On the other hand, due to lack of information or prediction errors, the data of real-world optimization problems is often uncertain, i.e., they are not known exactly when the problem is solved [2-5]. Recently, robust optimization has emerged as a remarkable deterministic framework for studying mathematical optimization problems with data uncertainty; see $[2-6,8,9,13-15,18,19,22,23]$ for theoretical and applied aspects of this area. In particular, Chuong [6] has studied duality (and optimality conditions) for robust multi-objective optimization problem involving nonsmooth real-valued functions.

In this paper, along with optimality conditions proposed in [13], we further introduce types of non-fractional and fractional dual problems and investigate

[^0]weak, strong, and converse-like duality relations under assumptions of (strictly) generalized convex-concavity.

In what follows, we recall some symbols and the problem model from [13]. Let $K=\{1, \ldots, l\}, J=\{1, \ldots, m\}$ be index sets, and the real-valued functions $p_{k}, q_{k}, k \in K$, be locally Lipschitz on $\mathbb{R}^{n}$, and $g_{j}, j \in J$ be given real-valued functions. Furthermore, denote $f(x):=\left(\frac{p_{1}(x)}{q_{1}(x)}, \ldots, \frac{p_{l}(x)}{q_{l}(x)}\right)$ for simplicity. For the sake of convenience, we further assume that $p_{k}(x) \geq 0, q_{k}(x)>0, k \in K$ for all $x \in \mathbb{R}^{n}$. Here after, we use the notation $f:=\left(f_{1}, \ldots, f_{l}\right)$, where $f_{k}:=\frac{p_{k}}{q_{k}}$, $k \in K$, and $g:=\left(g_{1}, \ldots, g_{m}\right)$.

We consider the fractional multi-objective optimization problem in the face of data uncertainty in the constraints of the form:

$$
\begin{equation*}
\operatorname{Min}_{\mathbb{R}_{+}^{l}}\left\{f(x) \mid g_{j}\left(x, v_{j}\right) \leq 0, j \in J\right\} \tag{UP}
\end{equation*}
$$

where $\operatorname{Min}_{\mathbb{R}_{+}^{l}}$ in the above problem will be understood with respect to the ordering cone $\mathbb{R}_{+}^{l}:=\left\{\left(y_{1}, \ldots, y_{l}\right) \mid y_{i} \geq 0, i=1, \ldots, l\right\} ; x \in \mathbb{R}^{n}$ is the vector of decision variables, $v_{j} \in \mathcal{V}_{j}, j \in J$ are uncertain parameters, $g_{j}: \mathbb{R}^{n} \times \mathcal{V}_{j} \rightarrow \mathbb{R}$, $j \in J$ are continuous real-valued functions.

Following the robust approach, we associate with (UP) its robust counterpart:

$$
\begin{equation*}
\operatorname{Min}_{\mathbb{R}_{+}^{l}}\left\{f(x) \mid g_{j}\left(x, v_{j}\right) \leq 0, \forall v_{j} \in \mathcal{V}_{j}, j \in J\right\} \tag{RP}
\end{equation*}
$$

In addition, let $F$ be the feasible set of problem (RP), which is given by

$$
\begin{equation*}
F:=\left\{x \in \mathbb{R}^{n} \mid g_{j}\left(x, v_{j}\right) \leq 0, \forall v_{j} \in \mathcal{V}_{j}, j \in J\right\} \tag{1.1}
\end{equation*}
$$

Definition 1.1. We say that $\bar{x} \in F$ is a local Pareto solution to problem (RP) if and only if there is no $x \in F$ and there is neighborhood $U$ of $\bar{x}$ such that

$$
\begin{equation*}
f_{k}(x) \leq f_{k}(\bar{x}), \quad \forall x \in F \cap U, k \in K \tag{1.2}
\end{equation*}
$$

with at least one strict inequality. If in addition all the inequalities in (1.2) are strict, then one has the definition for local weakly Pareto solution to problem (RP).

Now, we assume the following two assumptions for the functions $g_{j}, j \in J$, given in (1.1); see $[6,13]$ for more detail. The definitions of $\mathbb{B}(\cdot, \cdot)$ and closedness for multifunction are given in the beginning of Section 2.
(A1) For a fixed $\bar{x} \in \mathbb{R}^{n}$, there exists $\delta_{j}^{\bar{x}}>0$ such that the function $v_{j} \in$ $\mathcal{V}_{j} \mapsto g_{j}\left(x, v_{j}\right) \in \mathbb{R}$ is upper semicontinuous for each $x \in \mathbb{B}\left(\bar{x}, \delta_{j}^{\bar{x}}\right)$, and the functions $g_{j}\left(\cdot, v_{j}\right), v_{j} \in \mathcal{V}_{j}$, are Lipschitz of given rank $L_{j}>0$ on $\mathbb{B}\left(\bar{x}, \delta_{j}^{\bar{x}}\right)$, i.e.,
$\left|g_{j}\left(x_{1}, v_{j}\right)-g_{j}\left(x_{2}, v_{j}\right)\right| \leq L_{j}\left\|x_{1}-x_{2}\right\|, \forall x_{1}, x_{2} \in \mathbb{B}\left(\bar{x}, \delta_{j}^{\bar{x}}\right), \forall v_{j} \in \mathcal{V}_{j}$.
(A2) The multifunction $\left(x, v_{j}\right) \in \mathbb{B}\left(\bar{x}, \delta_{j}^{\bar{x}}\right) \times \mathcal{V}_{j} \rightrightarrows \partial_{x} g_{j}\left(x, v_{j}\right) \subset \mathbb{R}^{n}$ is closed at $\left(\bar{x}, \bar{v}_{j}\right)$ for each $\bar{v}_{j} \in \mathcal{V}_{j}(\bar{x})$, where the symbol $\partial_{x}$ stands for the limiting subdifferential operation with respect to $x$, and the notation $\mathcal{V}_{j}(\bar{x})$ signifies active indices in $\mathcal{V}_{j}$ at $\bar{x}$, i.e.,

$$
\begin{equation*}
\mathcal{V}_{j}(\bar{x}):=\left\{v_{j} \in \mathcal{V}_{j} \mid g_{j}\left(\bar{x}, v_{j}\right)=G_{j}(\bar{x})\right\} \tag{1.3}
\end{equation*}
$$

with $G_{j}(\bar{x}):=\sup _{v_{j} \in \mathcal{V}_{j}} g_{j}\left(\bar{x}, v_{j}\right)$.
Definition 1.2. Let $\bar{x} \in F$. We say that the constraint qualification (CQ) is satisfied at $\bar{x}$ if

$$
0 \notin \operatorname{co}\left\{\cup \partial_{x} g_{j}\left(\bar{x}, v_{j}\right) \mid v_{j} \in \mathcal{V}_{j}(\bar{x}), j \in J\right\} .
$$

Next, by using the parametric approach, we transform the problem (RP) into the nonsmooth non-fractional robust multi-objective optimization problem $(\mathrm{RP})_{\gamma}$ with a parameter $\gamma=\left(\gamma_{1}, \ldots, \gamma_{l}\right) \in \mathbb{R}_{+}^{l}$.
$(\mathrm{RP})_{\gamma} \operatorname{Min}_{\mathbb{R}_{+}^{l}}\left\{\hat{f}(x):=\left(p_{1}(x)-\gamma_{1} q_{1}(x), \ldots, p_{l}(x)-\gamma_{l} q_{l}(x)\right) \mid x \in F\right\}$, where the feasible set $F$ is same as (1.1).

We organize the rest of the paper as follows. Section 2 provides some preliminaries and notations. In Section 3, we recall the results on optimality conditions for problem (RP) studied by Hong et al. [13]. Our main findings on duality are proposed in Section 4. Finally, conclusions are given in Section 5.

## 2. Preliminaries

Throughout this paper, we will use some notations and preliminary results; see, e.g., $[20,21]$. Let $\mathbb{R}^{n}$ denote the Euclidean space equipped with the usual Euclidean norm $\|\cdot\|$. The nonnegative orthant of $\mathbb{R}^{n}$ is denoted by $\mathbb{R}_{+}^{n}:=$ $\left\{\left(x_{1}, \ldots, x_{n}\right) \mid x_{i} \geq 0, i=1, \ldots, n\right\}$. The inner product in $\mathbb{R}^{n}$ is defined by $\langle x, y\rangle:=x^{T} y$ for all $x, y \in \mathbb{R}^{n}$. The symbol $\mathbb{B}(x, \rho)$ stands for the open ball centered at $x \in \mathbb{R}^{n}$ with the radius $\rho>0$. For a given set $\Omega \subset \mathbb{R}^{n}$, we use co $\Omega$ to indicate the convex hull of $\Omega$, and the notation $x \xrightarrow{\Omega} \bar{x}$ means that $x \rightarrow \bar{x}$ with $x \in \Omega$.

A given set-valued mapping $F: \Omega \subset \mathbb{R}^{n} \rightrightarrows \mathbb{R}^{m}$ is said to be closed at $\bar{x} \in \Omega$ if for any sequence $\left\{x_{k}\right\} \subset \Omega, x_{k} \rightarrow \bar{x}$, and any sequence $\left\{y_{k}\right\} \subset \mathbb{R}^{m}, y_{k} \rightarrow \bar{y}$, one has $\bar{y} \in F(\bar{x})$.

Given a multifunction $F: \mathbb{R}^{n} \rightrightarrows \mathbb{R}^{m}$ with values $F(x) \subset \mathbb{R}^{m}$ in the collection of all the subsets of $\mathbb{R}^{m}$. The limiting construction
$\operatorname{Limsup}_{x \rightarrow \bar{x}} F(x):=\left\{y \in \mathbb{R}^{m} \mid \exists x_{k} \rightarrow \bar{x}, y_{k} \rightarrow y\right.$ with $y_{k} \in F\left(x_{k}\right)$ for all $\left.k \in \mathbb{N}\right\}$
is known as the Painlevé-Kuratowski upper/outer limit of the multifunction $F$ at $\bar{x}$, in which $\mathbb{N}:=\{1,2, \ldots\}$.

Given $\Omega \subset \mathbb{R}^{n}$, and $\bar{x} \in \Omega$, define the collection of Fréchet/regular normal cone to $\Omega$ at $\bar{x}$ by

$$
\widehat{N}(\bar{x} ; \Omega)=\widehat{N}_{\Omega}(\bar{x}):=\left\{v \in \mathbb{R}^{n} \left\lvert\, \underset{\substack{\Omega \rightarrow \\ \bar{x}}}{\limsup } \frac{\langle v, x-\bar{x}\rangle}{\|x-\bar{x}\|} \leq 0\right.\right\} .
$$

If $\bar{x} \notin \Omega$, we put $\widehat{N}(\bar{x} ; \Omega):=\emptyset$.
The Mordukhovich/limiting normal cone $N(\bar{x} ; \Omega)$ to $\Omega$ at $\bar{x} \in \Omega \subset \mathbb{R}^{n}$ is obtained from regular normal cones by taking the sequential Painlevé-Kuratowski upper limits as

$$
N(\bar{x} ; \Omega):=\underset{x \xrightarrow{\Omega} \bar{x}}{\operatorname{Limsup}} \widehat{N}(x ; \Omega) .
$$

If $\bar{x} \notin \Omega$, we put $N(\bar{x} ; \Omega):=\emptyset$.
For an extended real-valued function $\phi: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}:=[-\infty,+\infty]$ its domain and epigraph are defined by

$$
\operatorname{dom} \phi:=\left\{x \in \mathbb{R}^{n} \mid \phi(x)<\infty\right\} \text { and epi } \phi:=\left\{(x, \mu) \in \mathbb{R}^{n} \times \mathbb{R} \mid \phi(x) \leq \mu\right\}
$$ respectively.

Let $\phi: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$ be finite at $\bar{x} \in \operatorname{dom} \phi$, then the collection of basic subgradients, or the (basic/Mordukhovich/limiting) subdifferential, of $\phi$ at $\bar{x}$ is defined by

$$
\partial \phi(\bar{x}):=\left\{v \in \mathbb{R}^{n} \mid(v,-1) \in N((\bar{x}, \phi(\bar{x})) ; \text { epi } \phi)\right\} .
$$

Recall that a function $\phi: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$ is known as locally Lipschitz at $\bar{x} \in \mathbb{R}^{n}$ with rank $L>0$, i.e., there exists $\rho>0$ such that

$$
\left\|\phi\left(x_{1}\right)-\phi\left(x_{2}\right)\right\| \leq L\left\|x_{1}-x_{2}\right\|, \forall x_{1}, x_{2} \in \mathbb{B}(\bar{x}, \rho)
$$

The following concepts of (strictly) generalized convex-concavity at a given point for locally Lipschitz functions is inspired by [6, Definition 3.9], [12, Definition 3.7] and [11, Definition 3.11]; see also [13, Definition 3.2].

Definition 2.1. (i) We say that $(p, g ; q)$ is generalized convex-concave on $\mathbb{R}^{n}$ at $\bar{x} \in \mathbb{R}^{n}$ if for any $x \in \mathbb{R}^{n}, \xi_{k} \in \partial p_{k}(\bar{x}), \zeta_{k} \in \partial q_{k}(\bar{x}), k \in K$, and $\eta_{v} \in \partial_{x} g_{j}(\bar{x}, v), v \in \mathcal{V}_{j}(\bar{x}), j \in J$, there exists $h \in \mathbb{R}^{n}$ such that

$$
\begin{aligned}
p_{k}(x)-p_{k}(\bar{x}) & \geq\left\langle\xi_{k}, h\right\rangle, & & k \in K, \\
q_{k}(x)-q_{k}(\bar{x}) & \leq\left\langle\zeta_{k}, h\right\rangle, & & k \in K, \\
g_{j}(x, v)-g_{j}(\bar{x}, v) & \geq\left\langle\eta_{v}, h\right\rangle, & & v \in \mathcal{V}_{j}(\bar{x}), j \in J,
\end{aligned}
$$

where $\mathcal{V}_{j}(\bar{x}), j \in J$, are defined as in (1.3).
(ii) We say that $(p, g ; q)$ is strictly generalized convex-concave on $\mathbb{R}^{n}$ at $\bar{x} \in \mathbb{R}^{n}$ if for any $x \in \mathbb{R}^{n} \backslash\{\bar{x}\}, \xi_{k} \in \partial p_{k}(\bar{x}), \zeta_{k} \in \partial q_{k}(\bar{x}), k \in K$, and $\eta_{v} \in \partial_{x} g_{j}(\bar{x}, v), v \in \mathcal{V}_{j}(\bar{x}), j \in J$, there exists $h \in \mathbb{R}^{n}$ such that

$$
\begin{array}{rlrl}
p_{k}(x)-p_{k}(\bar{x}) & >\left\langle\xi_{k}, h\right\rangle, & & k \in K, \\
q_{k}(x)-q_{k}(\bar{x}) & \leq\left\langle\zeta_{k}, h\right\rangle, & k \in K, \\
g_{j}(x, v)-g_{j}(\bar{x}, v) & \geq\left\langle\eta_{v}, h\right\rangle, & v \in \mathcal{V}_{j}(\bar{x}), j \in J,
\end{array}
$$

where $\mathcal{V}_{j}(\bar{x}), j \in J$, are defined as in (1.3).
Remark 2.1. (c.f. [13, Remark 3.2]) We see that if $p_{k}, k \in K$, and $g_{j}, j \in J$ are convex (resp., strictly convex), and $q_{k}, k \in K$ are concave (strictly concave), then $(p, g ; q)$ is generalized convex-concave (resp., strictly generalized convexconcave) on $\mathbb{R}^{n}$ at any $\bar{x} \in \mathbb{R}^{n}$ with $h:=x-\bar{x}$ for each $x \in \mathbb{R}^{n}$.

## 3. Previous results on optimality conditions

In this section, we recall some results on optimality conditions for problem (RP); see [13] for the proof in detail.

First, we recall the Fritz-John type necessary optimality condition for a local weakly Pareto solution to problem (RP) ${ }_{\gamma}$.

Theorem 3.1 (c.f. [13, Theorem 3.1]). Let $\gamma_{k}=\frac{p_{k}(\bar{x})}{q_{k}(\bar{x})}, k \in K$. If $\bar{x}$ is a local weakly Pareto solution to problem (RP), then there exist $\beta_{k} \geq 0, k \in K$ and $\mu_{j} \geq 0, j \in J$ with $\sum_{k \in K} \beta_{k}+\sum_{j \in J} \mu_{j}=1$, such that

$$
\begin{align*}
& 0 \in \sum_{k \in K} \beta_{k} \partial p_{k}(\bar{x})-\sum_{k \in K} \beta_{k} \gamma_{k} \partial q_{k}(\bar{x})+\sum_{j \in J} \mu_{j} \operatorname{co}\left\{\cup \partial_{x} g_{j}\left(\bar{x}, v_{j}\right) \mid v_{j} \in \mathcal{V}_{j}(\bar{x})\right\},  \tag{3.1}\\
& \mu_{j} \sup _{v_{j} \in \mathcal{V}_{j}} g_{j}\left(\bar{x}, v_{j}\right)=0, j \in J .
\end{align*}
$$

Remark 3.1. If the (CQ) given in Definition 1.2 holds, then $\beta_{k}$ in (3.1) can be chosen not all zero, and hence, a point $\bar{x} \in F$ is satisfy (KKT) condition (3.1). Indeed, if $\beta_{k}=0$, for all $k \in K$, then $0 \in \operatorname{co}\left\{\cup \partial_{x} g_{j}\left(\bar{x}, v_{j}\right) \mid v_{j} \in \mathcal{V}_{j}(\bar{x}), j \in J\right\}$, which reaches to a contradiction to our assumption that the (CQ) holds; see the next theorem, which is the Karash-Kuhn-Tucker type necessary optimality condition for a local weakly Pareto solution to the nonsmooth fractional robust multi-objective optimization problem (RP).

Theorem 3.2 (c.f. [13, Theorem 3.2]). Let $\bar{x}$ be a local weakly Pareto solution to problem (RP), then there exist $\beta_{k} \geq 0, k \in K$ and $\mu_{j} \geq 0, j \in J$ with
$\sum_{k \in K} \beta_{k}+\sum_{j \in J} \mu_{j}=1$, such that

$$
\begin{align*}
& 0 \in \sum_{k \in K} \alpha_{k}\left(\partial p_{k}(\bar{x})-\frac{p_{k}(\bar{x})}{q_{k}(\bar{x})} \partial q_{k}(\bar{x})\right)+\sum_{j \in J} \mu_{j} \operatorname{co}\left\{\cup \partial_{x} g_{j}\left(\bar{x}, v_{j}\right) \mid v_{j} \in \mathcal{V}_{j}(\bar{x})\right\}  \tag{3.2}\\
& \mu_{j} \sup _{v_{j} \in \mathcal{V}_{j}} g_{j}\left(\bar{x}, v_{j}\right)=0, j \in J
\end{align*}
$$

where $\alpha_{k}=\frac{\beta_{k}}{q_{k}(\bar{x})}$. If we further assume that the (CQ) holds at $\bar{x}$, then $\sum_{k \in K} \alpha_{k} \neq 0$.

Theorem 3.3 (c.f. [13, Theorem 3.3]). Assume that $\bar{x} \in F$ satisfies the condition (3.1) with $\sum_{k \in K} \beta_{k} \neq 0$.
(i) If $(p, g ; q)$ is generalized convex-concave at $\bar{x}$, then $\bar{x}$ is a weakly Pareto solution to problem (RP).
(ii) If $(p, g ; q)$ is strictly generalized convex-concave at $\bar{x}$, then $\bar{x}$ is a Pareto solution to problem (RP).

Theorem 3.4. (c.f. [13, Theorem 3.4]) Assume that $\bar{x} \in F$ satisfies the condition (3.2) with $\sum_{k \in K} \alpha_{k} \neq 0$.
(i) If $(p, g ; q)$ is generalized convex-concave at $\bar{x}$, then $\bar{x}$ is a weakly Pareto solution to problem (RP).
(ii) If $(p, g ; q)$ is strictly generalized convex-concave at $\bar{x}$, then $\bar{x}$ is a Pareto solution to problem (RP).

## 4. Main Results: duality Relations

In this section, we propose model-types of both non-fractional and fractional problems for problem (RP). Then weak, strong, and converse robust duality relations between them are examined, respectively. In what follows, we use the following notation for convenience.

$$
\begin{aligned}
& u \prec v \Leftrightarrow u-v \in-\operatorname{int} \mathbb{R}_{+}^{l}, \quad u \nprec v \text { is the negation of } u \prec v, \\
& u \preceq v \Leftrightarrow u-v \in-\mathbb{R}_{+}^{l} \backslash\{0\}, \quad u \npreceq v \text { is the negation of } u \preceq v .
\end{aligned}
$$

Moreover, we also define $\mathbb{R}_{+}^{\mathbb{N}}:=\left\{\mu:=\left(\mu_{j}, \mu_{j i}\right), j \in J=\{1, \ldots, m\}, i \in I_{j}=\right.$ $\left.\left\{1, \ldots, i_{j}\right\} \mid i_{j} \in \mathbb{N}, \mu_{j} \geq 0, \mu_{j i} \geq 0, \sum_{i \in I_{j}} \mu_{j i}=1\right\}$ for simplicity.
4.1. Non-fractional dual model. For $z \in \mathbb{R}^{n}, \beta:=\left(\beta_{1}, \ldots, \beta_{l}\right) \in \mathbb{R}_{+}^{l} \backslash\{0\}$, $\gamma:=\left(\gamma_{1}, \ldots, \gamma_{l}\right) \in \mathbb{R}_{+}^{l}$, and $\mu \in \mathbb{R}_{+}^{\mathbb{N}}$, in connection with the fractional multiobjective optimization problem (RP), we consider its non-fractional robust multiobjective dual problem of the form:
$(\mathrm{RD})_{\gamma} \quad \operatorname{Max}_{\mathbb{R}_{+}^{l}}\left\{\breve{f}(z, \beta, \mu):=\left(\gamma_{1}, \ldots, \gamma_{l}\right) \mid(z, \beta, \mu) \in F_{D_{1}}\right\}$,
where the feasible set $F_{D_{1}}$ is defined by

$$
\begin{aligned}
F_{D_{1}} & :=\left\{(z, \beta, \mu) \in \mathbb{R}^{n} \times\left(\mathbb{R}_{+}^{l} \backslash\{0\}\right) \times \mathbb{R}_{+}^{\mathbb{N}} \mid 0 \in \sum_{k \in K} \beta_{k} \partial p_{k}(z)-\sum_{k \in K} \beta_{k} \gamma_{k} \partial q_{k}(z)\right. \\
& \left.+\sum_{j \in J} \mu_{j}\left(\sum_{i \in I_{j}} \mu_{j i} \eta_{j i}\right), \sum_{j \in J} \mu_{j} g_{j}\left(z, v_{j}\right) \geq 0, \eta_{j i} \in\left\{\cup \partial_{x} g_{j}\left(z, v_{j i}\right) \mid v_{j i} \in \mathcal{V}_{j}(z)\right\}\right\},
\end{aligned}
$$

where $v_{j} \in \mathcal{V}_{j}(z), \mathcal{V}_{j}(z)$ is defined as in (1.3) by replacing $\bar{x}$ with $z$, and $j \in J$, $i \in I_{j}=\left\{1, \ldots, i_{j}\right\}$.

Definition 4.1. A feasible point $(\bar{z}, \bar{\beta}, \bar{\mu}) \in F_{D_{1}}$ is said to be
(i) a local Pareto solution to problem (RD) $)_{\gamma}$, if and only if there is neighborhood $U$ of $(\bar{z}, \bar{\beta}, \bar{\mu})$ such that

$$
\breve{f}(z, \beta, \mu)-\breve{f}(\bar{z}, \bar{\beta}, \bar{\mu}) \notin \mathbb{R}_{+}^{l} \backslash\{0\}, \quad \forall(z, \beta, \mu) \in F_{D_{1}} \cap U .
$$

(ii) a local weakly Pareto solution to problem $(\mathrm{RD})_{\gamma}$, if and only if there is neighborhood $U$ of $(\bar{z}, \bar{\beta}, \bar{\mu})$ such that

$$
\breve{f}(z, \beta, \mu)-\breve{f}(\bar{z}, \bar{\beta}, \bar{\mu}) \notin \operatorname{int} \mathbb{R}_{+}^{l}, \quad \forall(z, \beta, \mu) \in F_{D_{1}} \cap U .
$$

The following theorem describes weak duality relations between the primal problem (RP) and the dual problem (RD) ${ }_{\gamma}$.

Theorem 4.1 (Weak duality). Let $x \in F$ and $(z, \beta, \mu) \in F_{D_{1}}$ be given.
(i) If $(p, g ; q)$ is generalized convex-concave on $\mathbb{R}^{n}$ at $z$, then

$$
f(x) \nprec \breve{f}(z, \beta, \mu) .
$$

(ii) If $(p, g ; q)$ is strictly generalized convex-concave on $\mathbb{R}^{n}$ at $z$, then

$$
f(x) \npreceq \breve{f}(z, \beta, \mu) .
$$

Proof. Since $(z, \beta, \mu) \in F_{D_{1}}$, there exist $\beta_{k} \geq 0, \xi_{k} \in \partial p_{k}(z), \zeta_{k} \in \partial q_{k}(z), \gamma_{k}=$ $\frac{p_{k}(z)}{q_{k}(z)}, k \in K$ with $\sum_{k \in K} \beta_{k} \neq 0$, and $\mu_{j} \geq 0, j \in J, \mu_{j i} \geq 0, \eta_{j i} \in \partial_{x} g_{j}\left(z, v_{j i}\right)$,
$v_{j i} \in \mathcal{V}_{j}(z), i \in I_{j}=\left\{1, \ldots, i_{j}\right\}, i_{j} \in \mathbb{N}, \sum_{i \in I_{j}} \mu_{j i}=1$, such that

$$
0=\sum_{k \in K} \beta_{k} \xi_{k}-\sum_{k \in K} \beta_{k} \gamma_{k} \zeta_{k}+\sum_{j \in J} \mu_{j}\left(\sum_{i \in I_{j}} \mu_{j i} \eta_{j i}\right),
$$

$$
\begin{equation*}
\sum_{j \in J} \mu_{j} g_{j}\left(z, v_{j}\right) \geq 0 \tag{4.1}
\end{equation*}
$$

We first justify (i). Assume to the contrary that

$$
f(x) \prec \breve{f}(z, \beta, \mu) .
$$

In other words,

$$
\begin{equation*}
f(x)-\breve{f}(z, \beta, \mu) \in-\operatorname{int} \mathbb{R}_{+}^{l} . \tag{4.2}
\end{equation*}
$$

Along with the generalized convex-concavity of $(p, g ; q)$ on $\mathbb{R}^{n}$ at $z$, for such $x$, there exists $h \in \mathbb{R}^{n}$ such that

$$
\begin{aligned}
0= & \sum_{k \in K} \beta_{k}\left\langle\xi_{k}, h\right\rangle-\sum_{k \in K} \beta_{k} \gamma_{k}\left\langle\zeta_{k}, h\right\rangle+\sum_{j \in J} \mu_{j}\left(\sum_{i \in I_{j}} \mu_{j i}\left\langle\eta_{j i}, h\right\rangle\right) \\
\leq & \sum_{k \in K} \beta_{k}\left(p_{k}(x)-p_{k}(z)\right)-\sum_{k \in K} \beta_{k} \gamma_{k}\left(q_{k}(x)-q_{k}(z)\right) \\
& +\sum_{j \in J} \mu_{j}\left(\sum_{i \in I_{j}} \mu_{j i}\left[g_{j}\left(x, v_{j i}\right)-g_{j}\left(z, v_{j i}\right)\right]\right)
\end{aligned}
$$

$$
\begin{equation*}
=\sum_{k \in K} \beta_{k} p_{k}(x)-\sum_{k \in K} \beta_{k} \gamma_{k} q_{k}(x)+\sum_{j \in J} \mu_{j}\left(\sum_{i \in I_{j}} \mu_{j i}\left[g_{j}\left(x, v_{j i}\right)-g_{j}\left(z, v_{j i}\right)\right]\right) . \tag{4.3}
\end{equation*}
$$

Due to the fact that $x \in F$, one has

$$
\sum_{j \in J} \mu_{j}\left(\sum_{i \in I_{j}} \mu_{j i} g_{j}\left(x, v_{j i}\right)\right) \leq 0
$$

Thus, it follows from (4.3) that

$$
\begin{aligned}
0 & \leq \sum_{k \in K} \beta_{k} p_{k}(x)-\sum_{k \in K} \beta_{k} \gamma_{k} q_{k}(x)-\sum_{j \in J} \mu_{j}\left(\sum_{i \in I_{j}} \mu_{j i} g_{j}\left(z, v_{j i}\right)\right) \\
& \leq \sum_{k \in K} \beta_{k} p_{k}(x)-\sum_{k \in K} \beta_{k} \gamma_{k} q_{k}(x)
\end{aligned}
$$

where the last inequality holds true due to (4.1). Since $\beta \in \mathbb{R}_{+}^{l} \backslash\{0\}$, we have there is $k_{0} \in K$ such that

$$
\begin{equation*}
0 \leq p_{k_{0}}(x)-\gamma_{k_{0}} q_{k_{0}}(x) . \tag{4.4}
\end{equation*}
$$

Observe that the inequality (4.4) is lead to the $k_{0}$-th component of $\breve{f}(z, \beta, \mu)=$ $\gamma_{k_{0}} \leq \frac{p_{k_{0}}(x)}{q_{k_{0}}(x)}=$ the $k_{0}$-th component of $f(x)$, which contradicts (4.2).

Now, let us prove (ii). Assume to the contrary that

$$
f(x) \preceq \breve{f}(z, \beta, \mu),
$$

which means

$$
\begin{equation*}
f(x)-\breve{f}(z, \beta, \mu) \in-\mathbb{R}_{+}^{l} \backslash\{0\} . \tag{4.5}
\end{equation*}
$$

By the strictly generalized convex-concavity of $(p, g ; q)$ on $\mathbb{R}^{n}$ at $z$, for such $x$, there exists $h \in \mathbb{R}^{n}$ such that

$$
\begin{aligned}
0= & \sum_{k \in K} \beta_{k}\left\langle\xi_{k}, h\right\rangle-\sum_{k \in K} \beta_{k} \gamma_{k}\left\langle\zeta_{k}, h\right\rangle+\sum_{j \in J} \mu_{j}\left(\sum_{i \in I_{j}} \mu_{j i}\left\langle\eta_{j i}, h\right\rangle\right) \\
< & \sum_{k \in K} \beta_{k}\left(p_{k}(x)-p_{k}(z)\right)-\sum_{k \in K} \beta_{k} \gamma_{k}\left(q_{k}(x)-q_{k}(z)\right) \\
& +\sum_{j \in J} \mu_{j}\left(\sum_{i \in I_{j}} \mu_{j i}\left[g_{j}\left(x, v_{j i}\right)-g_{j}\left(z, v_{j i}\right)\right]\right) \\
= & \sum_{k \in K} \beta_{k} p_{k}(x)-\sum_{k \in K} \beta_{k} \gamma_{k} q_{k}(x)+\sum_{j \in J} \mu_{j}\left(\sum_{i \in I_{j}} \mu_{j i}\left[g_{j}\left(x, v_{j i}\right)-g_{j}\left(z, v_{j i}\right)\right]\right) .
\end{aligned}
$$

Similarly as in the proof of (i), we arrive at

$$
0<\sum_{k \in K} \beta_{k} p_{k}(x)-\sum_{k \in K} \beta_{k} \gamma_{k} q_{k}(x) .
$$

This entails that there is $k_{0} \in K$ such that

$$
0<p_{k_{0}}(x)-\gamma_{k_{0}} q_{k_{0}}(x)
$$

Equivalently, the $k_{0}$-th component of $\breve{f}(z, \beta, \mu)=\gamma_{k_{0}}<\frac{p_{k_{0}}(x)}{q_{k_{0}}(x)}=$ the $k_{0}$-th component of $f(x)$, which contradicts (4.5). Thus, the proof is complete.

The next theorem provides strong duality relations between the primal problem (RP) and the dual problem (RD) $\gamma$.
Theorem 4.2 (Strong duality). Let $\bar{x}$ be a local weakly Pareto solution to problem ( $\mathrm{RP)} \mathrm{such} \mathrm{that} \mathrm{the} \mathrm{(CQ)} \mathrm{is} \mathrm{satisfied} \mathrm{at} \mathrm{this} \mathrm{point}$. $(\bar{\beta}, \bar{\mu}) \in\left(\mathbb{R}_{+}^{l} \backslash\{0\}\right) \times \mathbb{R}_{+}^{\mathbb{N}}$ such that $(\bar{x}, \bar{\beta}, \bar{\mu}) \in F_{D_{1}}$ and $f(\bar{x})=\breve{f}(\bar{x}, \bar{\beta}, \bar{\mu})$. Furthermore,
(i) If $(p, g ; q)$ is generalized convex-concave on $\mathbb{R}^{n}$ at any $z \in \mathbb{R}^{n}$, then $(\bar{x}, \bar{\beta}, \bar{\mu})$ is a weakly Pareto solution to problem (RD) ${ }_{\gamma}$.
(ii) If $(p, g ; q)$ is strictly generalized convex-concave on $\mathbb{R}^{n}$ at any $z \in \mathbb{R}^{n}$, then $(\bar{x}, \bar{\beta}, \bar{\mu})$ is a Pareto solution to problem $(\mathrm{RD})_{\gamma}$.

Proof. By Theorem 3.1, we find $\beta_{k} \geq 0, \xi_{k} \in \partial p_{k}(\bar{x}), \zeta_{k} \in \partial q_{k}(\bar{x}), \gamma_{k}=\frac{p_{k}(\bar{x})}{q_{k}(\bar{x})}$, $k \in K$ with $\sum_{k \in K} \beta_{k} \neq 0$, and $\mu_{j} \geq 0, j \in J, \mu_{j i} \geq 0, \eta_{j i} \in \partial_{x} g_{j}\left(\bar{x}, v_{j i}\right)$, $v_{j i} \in \mathcal{V}_{j}(\bar{x}), i \in I_{j}=\left\{1, \ldots, i_{j}\right\}, i_{j} \in \mathbb{N}, \sum_{i \in I_{j}} \mu_{j i}=1$ such that

$$
\begin{align*}
& 0=\sum_{k=1}^{l} \beta_{k} \xi_{k}-\sum_{k=1}^{l} \beta_{k} \gamma_{k} \zeta_{k}+\sum_{j \in J} \mu_{j}\left(\sum_{i \in I_{j}} \mu_{j i} \eta_{j i}\right),  \tag{4.6}\\
& \mu_{j} \sup _{v_{j} \in \mathcal{V}_{j}} g_{j}\left(\bar{x}, v_{j}\right)=0, j \in J . \tag{4.7}
\end{align*}
$$

Since $v_{j i} \in \mathcal{V}_{j}(\bar{x})$, we have $g_{j}\left(\bar{x}, v_{j i}\right)=\sup _{v_{j} \in \mathcal{V}_{j}} g_{j}\left(\bar{x}, v_{j}\right)$ for $j \in J$, and $i \in I_{j}=\left\{1, \ldots, i_{j}\right\}$. Thus, it follows from (4.7) that $\mu_{j} g_{j}\left(\bar{x}, v_{j i}\right)=0$ for $j \in J$ and $i \in I_{j}$. This entails that

$$
\begin{equation*}
\sum_{j \in J} \mu_{j}\left(\sum_{i \in I_{j}} \mu_{j i} g_{j}\left(\bar{x}, v_{j i}\right)\right)=\sum_{j \in J}\left(\sum_{i \in I_{j}} \mu_{j i} \mu_{j} g_{j}\left(\bar{x}, v_{j i}\right)\right)=0 \tag{4.8}
\end{equation*}
$$

Letting $\bar{\beta}:=\left(\beta_{1}, \ldots, \beta_{l}\right) \in \mathbb{R}_{+}^{l} \backslash\{0\}$ and $\bar{\mu}:=\left(\mu_{j}, \mu_{j i}\right)$. Along with (4.6) and (4.8), we conclude that $(\bar{x}, \bar{\beta}, \bar{\mu}) \in F_{D_{1}}$. Thus, $f(\bar{x})=\breve{f}(\bar{x}, \bar{\beta}, \bar{\mu})$.
(i) As $(p, g ; q)$ is generalized convex-concave at any $z \in \mathbb{R}^{n}$, by invoking Theorem 4.1 (i), we obtain

$$
\breve{f}(\bar{x}, \bar{\beta}, \bar{\mu})=f(\bar{x}) \nprec \breve{f}(z, \beta, \mu)
$$

for any $(z, \beta, \mu) \in F_{D_{1}}$. This means that $(\bar{x}, \bar{\beta}, \bar{\mu})$ is a weakly Pareto solution to problem (RD) ${ }_{\gamma}$.
(ii) Since $(p, g ; q)$ is strictly generalized convex-concave at any $z \in \mathbb{R}^{n}$, by invoking Theorem 4.1 (ii), we assert that

$$
\breve{f}(\bar{x}, \bar{\beta}, \bar{\mu}) \npreceq \breve{f}(z, \beta, \mu)
$$

for any $(z, \beta, \mu) \in F_{D_{1}}$. This means that $(\bar{x}, \bar{\beta}, \bar{\mu})$ is a Pareto solution to problem (RD) ${ }_{\gamma}$.

Now, we present converse-like duality relations between the primal problem (RP) and the dual problem (RD) ${ }_{\gamma}$.
Theorem 4.3 (Converse-like duality). $\operatorname{Let}(\bar{x}, \bar{\beta}, \bar{\mu}) \in F_{D_{1}}$.
(i) If $\bar{x} \in F$ and $(p, g ; q)$ is generalized convex-concave on $\mathbb{R}^{n}$ at $\bar{x}$, then $\bar{x}$ is a weakly Pareto solution to problem (RP).
(ii) If $\bar{x} \in F$ and $(p, g ; q)$ is strictly generalized convex-concave on $\mathbb{R}^{n}$ at $\bar{x}$, then $\bar{x}$ is a Pareto solution to problem (RP).

Proof. Since $(\bar{x}, \bar{\beta}, \bar{\mu}) \in F_{D_{1}}$, there exist $\bar{\beta}:=\left(\beta_{1}, \ldots, \beta_{l}\right) \in \mathbb{R}_{+}^{l} \backslash\{0\}, \xi_{k} \in$ $\partial p_{k}(\bar{x}), \zeta_{k} \in \partial q_{k}(\bar{x}), \gamma_{k}=\frac{p_{k}(\bar{x})}{q_{k}(\bar{x})}, k \in K$, and $\bar{\mu}:=\left(\mu_{j}, \mu_{j i}\right), \mu_{j} \geq 0, j \in J, \mu_{j i} \geq$ $0, \eta_{j i} \in \partial_{x} g_{j}\left(\bar{x}, v_{j i}\right), v_{j i} \in \mathcal{V}_{j}(\bar{x}), i \in I_{j}=\left\{1, \ldots, i_{j}\right\}, i_{j} \in \mathbb{N}, \sum_{i \in I_{j}} \mu_{j i}=1$, such that

$$
\begin{align*}
& 0=\sum_{k=1}^{l} \beta_{k} \xi_{k}-\sum_{k=1}^{l} \beta_{k} \gamma_{k} \zeta_{k}+\sum_{j \in J} \mu_{j}\left(\sum_{i \in I_{j}} \mu_{j i} \eta_{j i}\right),  \tag{4.9}\\
& \sum_{j \in J} \mu_{j} g_{j}\left(\bar{x}, v_{j}\right) \geq 0 \tag{4.10}
\end{align*}
$$

Let $\bar{x} \in F$. Then $g_{j}\left(\bar{x}, v_{j}\right) \leq 0$, for all $v_{j} \in \mathcal{V}_{j}, j \in J$, and thus, $\mu_{j} g_{j}\left(\bar{x}, v_{j}\right) \leq 0$. This, together with (4.10), yields

$$
\mu_{j} g_{j}\left(\bar{x}, v_{j}\right)=0, j \in J
$$

i.e.,

$$
\mu_{j} \sup _{v_{j} \in \mathcal{V}_{j}} g_{j}\left(\bar{x}, v_{j}\right)=0, j \in J
$$

So, we assert by virtue of (4.9) that $\bar{x}$ satisfies condition (3.1). To finish the proof, it remains to apply Theorem 3.3.
4.2. Fractional dual model. For $z \in \mathbb{R}^{n}, \alpha:=\left(\alpha_{1}, \ldots, \alpha_{l}\right) \in \mathbb{R}_{+}^{l} \backslash\{0\}$, and $\mu \in \mathbb{R}_{+}^{\mathbb{N}}$, in connection with the fractional multiobjective optimization problem (RP), we consider its fractional robust multiobjective dual problem of the form:

$$
\begin{equation*}
\operatorname{Max}_{\mathbb{R}_{+}^{l}}\left\{\tilde{f}(z, \alpha, \mu): \left.=\left(\frac{p_{1}(z)}{q_{1}(z)}, \ldots, \frac{p_{l}(z)}{q_{l}(z)}\right) \right\rvert\,(z, \alpha, \mu) \in F_{D_{2}}\right\} \tag{RD}
\end{equation*}
$$

where the feasible set $F_{D_{2}}$ is defined by

$$
\begin{aligned}
F_{D_{2}} & :=\left\{(z, \alpha, \mu) \in \mathbb{R}^{n} \times\left(\mathbb{R}_{+}^{l} \backslash\{0\}\right) \times \mathbb{R}_{+}^{\mathbb{N}} \left\lvert\, 0 \in \sum_{k \in K} \alpha_{k}\left(\partial p_{k}(z)-\frac{p_{k}(z)}{q_{k}(z)} \partial q_{k}(z)\right)\right.\right. \\
& \left.+\sum_{j \in J} \mu_{j}\left(\sum_{i \in I_{j}} \mu_{j i} \eta_{j i}\right), \sum_{j \in J} \mu_{j} g_{j}\left(z, v_{j}\right) \geq 0, \eta_{j i} \in\left\{\cup \partial_{x} g_{j}\left(z, v_{j i}\right) \mid v_{j i} \in \mathcal{V}_{j}(z)\right\}\right\},
\end{aligned}
$$

where $v_{j} \in \mathcal{V}_{j}(z), \mathcal{V}_{j}(z)$ is defined as in (1.3) by replacing $\bar{x}$ with $z$, and $j \in J$, $i \in I_{j}=\left\{1, \ldots, i_{j}\right\}$.
Definition 4.2. A feasible point $(\bar{z}, \bar{\alpha}, \bar{\mu}) \in F_{D_{2}}$ is said to be
(i) a local Pareto solution to problem (RD), if and only if there is neighborhood $U$ of $(\bar{z}, \bar{\alpha}, \bar{\mu})$ such that

$$
\tilde{f}(z, \alpha, \mu)-\tilde{f}(\bar{z}, \bar{\alpha}, \bar{\mu}) \notin \mathbb{R}_{+}^{l} \backslash\{0\}, \quad \forall x \in F_{D_{2}} \cap U
$$

(ii) a local weakly Pareto solution to problem (RD), if and only if there is neighborhood $U$ of $(\bar{z}, \bar{\alpha}, \bar{\mu})$ such that

$$
\tilde{f}(z, \alpha, \mu)-\tilde{f}(\bar{z}, \bar{\alpha}, \bar{\mu}) \notin \operatorname{int} \mathbb{R}_{+}^{l}, \quad \forall x \in F_{D_{2}} \cap U
$$

We first show the weak duality relations between the primal problem (RP) and the dual problem (RD).

Theorem 4.4 (Weak duality). Let $x \in F$ and $(z, \alpha, \mu) \in F_{D_{2}}$.
(i) If $(p, g ; q)$ is generalized convex-concave on $\mathbb{R}^{n}$ at $z$, then

$$
f(x) \nprec \tilde{f}(z, \alpha, \mu) .
$$

(ii) If $(p, g ; q)$ is strictly generalized convex-concave on $\mathbb{R}^{n}$ at $z$, then

$$
f(x) \npreceq \tilde{f}(z, \alpha, \mu) .
$$

Proof. Since $(z, \alpha, \mu) \in F_{D_{2}}$, there exist $\alpha_{k} \geq 0, \xi_{k} \in \partial p_{k}(z), \zeta_{k} \in \partial q_{k}(z)$, $k \in K$ with $\sum_{k \in K} \alpha_{k} \neq 0$, and $\mu_{j} \geq 0, j \in J, \mu_{j i} \geq 0, \eta_{j i} \in \partial_{x} g_{j}\left(z, v_{j i}\right)$, $v_{j i} \in \mathcal{V}_{j}(z), i \in I_{j}=\left\{1, \ldots, i_{j}\right\}, i_{j} \in \mathbb{N}, \sum_{i \in I_{j}} \mu_{j i}=1$, such that

$$
\begin{align*}
& 0=\sum_{k \in K} \alpha_{k}\left(\xi_{k}-\frac{p_{k}(z)}{q_{k}(z)} \zeta_{k}\right)+\sum_{j \in J} \mu_{j}\left(\sum_{i \in I_{j}} \mu_{j i} \eta_{j i}\right), \\
& \sum_{j \in J} \mu_{j} g_{j}\left(z, v_{j}\right) \geq 0 \tag{4.11}
\end{align*}
$$

We first justify the item (i). Assume to the contrary that

$$
f(x) \prec \tilde{f}(z, \alpha, \mu) .
$$

This means that

$$
\begin{equation*}
f(x)-\tilde{f}(z, \alpha, \mu) \in-\operatorname{int} \mathbb{R}_{+}^{l} \tag{4.12}
\end{equation*}
$$

By the generalized convex-concavity of $(p, g ; q)$ on $\mathbb{R}^{n}$ at $z$, for such $x$, there exists $h \in \mathbb{R}^{n}$ such that

$$
\begin{aligned}
0= & \sum_{k \in K} \alpha_{k}\left(\left\langle\xi_{k}, h\right\rangle-\frac{p_{k}(z)}{q_{k}(z)}\left\langle\zeta_{k}, h\right\rangle\right)+\sum_{j \in J} \mu_{j}\left(\sum_{i \in I_{j}} \mu_{j i}\left\langle\eta_{j i}, h\right\rangle\right) \\
\leq & \sum_{k \in K} \alpha_{k}\left[p_{k}(x)-p_{k}(z)-\frac{p_{k}(z)}{q_{k}(z)}\left(q_{k}(x)-q_{k}(z)\right)\right] \\
& +\sum_{j \in J} \mu_{j}\left(\sum_{i \in I_{j}} \mu_{j i}\left[g_{j}\left(x, v_{j i}\right)-g_{j}\left(z, v_{j i}\right)\right]\right)
\end{aligned}
$$

$$
\begin{equation*}
=\sum_{k \in K} \alpha_{k}\left[p_{k}(x)-\frac{p_{k}(z)}{q_{k}(z)} q_{k}(x)\right]+\sum_{j \in J} \mu_{j}\left(\sum_{i \in I_{j}} \mu_{j i}\left[g_{j}\left(x, v_{j i}\right)-g_{j}\left(z, v_{j i}\right)\right]\right) . \tag{4.13}
\end{equation*}
$$

Since $x \in F$, then $\sum_{j \in J} \mu_{j}\left(\sum_{i \in I_{j}} \mu_{j i} g_{j}\left(x, v_{j i}\right)\right) \leq 0$. Thus, it yields form (4.13) that

$$
0 \leq \sum_{k \in K} \alpha_{k}\left[p_{k}(x)-\frac{p_{k}(z)}{q_{k}(z)} q_{k}(x)\right]-\sum_{j \in J} \mu_{j}\left(\sum_{i \in I_{j}} \mu_{j i} g_{j}\left(z, v_{j i}\right)\right) .
$$

Moreover, since (4.11) is valid, thus $\sum_{j \in J} \mu_{j}\left(\sum_{i \in I_{j}} \mu_{j i} g_{j}\left(z, v_{j i}\right)\right) \geq 0$, and so

$$
0 \leq \sum_{k \in K} \alpha_{k}\left[p_{k}(x)-\frac{p_{k}(z)}{q_{k}(z)} q_{k}(x)\right] .
$$

This entails that there is $k_{0} \in K$ such that

$$
\begin{equation*}
0 \leq p_{k_{0}}(x)-\frac{p_{k_{0}}(z)}{q_{k_{0}}(z)} q_{k_{0}}(x) \tag{4.14}
\end{equation*}
$$

due to $\alpha \in \mathbb{R}_{+}^{l} \backslash\{0\}$. The inequality (4.14) is lead to the $k_{0}$-th component of $\tilde{f}(z, \alpha, \mu)=\frac{p_{k_{0}}(z)}{q_{k_{0}}(z)} \leq \frac{p_{k_{0}}(x)}{q_{k_{0}}(x)}=$ the $k_{0}$-th component of $f(x)$, which contradicts (4.12). Thus, the item (i) is proved.

Now, we show the item (ii). Assume to the contrary that

$$
f(x) \preceq \tilde{f}(z, \alpha, \mu),
$$

i.e.,

$$
\begin{equation*}
f(x)-\tilde{f}(z, \alpha, \mu) \in-\mathbb{R}_{+}^{l} \backslash\{0\} \tag{4.15}
\end{equation*}
$$

By the strictly generalized convex-concavity of $(p, g ; q)$ on $\mathbb{R}^{n}$ at $z$, for such $x$, there exists $h \in \mathbb{R}^{n}$ such that

$$
\begin{aligned}
0= & \sum_{k \in K} \alpha_{k}\left(\left\langle\xi_{k}, h\right\rangle-\frac{p_{k}(z)}{q_{k}(z)}\left\langle\zeta_{k}, h\right\rangle\right)+\sum_{j \in J} \mu_{j}\left(\sum_{i \in I_{j}} \mu_{j i}\left\langle\eta_{j i}, h\right\rangle\right) \\
< & \sum_{k \in K} \alpha_{k}\left[p_{k}(x)-p_{k}(z)-\frac{p_{k}(z)}{q_{k}(z)}\left(q_{k}(x)-q_{k}(z)\right)\right] \\
& +\sum_{j \in J} \mu_{j}\left(\sum_{i \in I_{j}} \mu_{j i}\left[g_{j}\left(x, v_{j i}\right)-g_{j}\left(z, v_{j i}\right)\right]\right) \\
= & \sum_{k \in K} \alpha_{k}\left[p_{k}(x)-\frac{p_{k}(z)}{q_{k}(z)} q_{k}(x)\right]+\sum_{j \in J} \mu_{j}\left(\sum_{i \in I_{j}} \mu_{j i}\left[g_{j}\left(x, v_{j i}\right)-g_{j}\left(z, v_{j i}\right)\right]\right) .
\end{aligned}
$$

A similar argument as in the proof of item (i), we arrive at

$$
0<\sum_{k \in K} \alpha_{k}\left[p_{k}(x)-\frac{p_{k}(z)}{q_{k}(z)} q_{k}(x)\right] .
$$

This entails that there is $k_{0} \in K$ such that

$$
0<p_{k_{0}}(x)-\frac{p_{k_{0}}(z)}{q_{k_{0}}(z)} q_{k_{0}}(x) .
$$

Equivalently, the $k_{0}$-th component of $\tilde{f}(z, \alpha, \mu)=\frac{p_{k_{0}}(z)}{q_{k_{0}}(z)}<\frac{p_{k_{0}}(x)}{q_{k_{0}}(x)}=$ the $k_{0}-$ th component of $f(x)$. which contradicts (4.15). Thereby, the desired result follows.

Next, we proposes strong duality relations between the primal problem (RP) and the dual problem (RD).

Theorem 4.5 (Strong duality). Let $\bar{x}$ be a local weakly Pareto solution to problem $(\mathrm{RP})$ such that the $(\mathrm{CQ})$ holds at this point. Then there exists $(\bar{\alpha}, \bar{\mu}) \in$ $\left(\mathbb{R}_{+}^{l} \backslash\{0\}\right) \times \mathbb{R}_{+}^{\mathbb{N}}$ such that $(\bar{x}, \bar{\alpha}, \bar{\mu}) \in F_{D_{2}}$ and $f(\bar{x})=\tilde{f}(\bar{x}, \bar{\alpha}, \bar{\mu})$. Furthermore,
(i) If $(p, g ; q)$ is generalized convex-concave on $\mathbb{R}^{n}$ at any $z \in \mathbb{R}^{n}$, then ( $\bar{x}, \bar{\alpha}, \bar{\mu}$ ) is weakly Pareto solution to problem (RD).
(ii) If $(p, g ; q)$ is strictly generalized convex-concave on $\mathbb{R}^{n}$ at any $z \in \mathbb{R}^{n}$, then $(\bar{x}, \bar{\alpha}, \bar{\mu})$ is Pareto solution to problem (RD).

Proof. Along with Theorem 3.2, we find $\alpha_{k} \geq 0, \xi_{k} \in \partial p_{k}(\bar{x}), \zeta_{k} \in \partial q_{k}(\bar{x})$, $k \in K$ with $\sum_{k \in K} \alpha_{k} \neq 0$, and $\mu_{j} \geq 0, j \in J, \mu_{j i} \geq 0, \eta_{j i} \in \partial_{x} g_{j}\left(\bar{x}, v_{j i}\right)$,
$v_{j i} \in \mathcal{V}_{j}(\bar{x}), i \in I_{j}=\left\{1, \ldots, i_{j}\right\}, i_{j} \in \mathbb{N}, \sum_{i \in I_{j}} \mu_{j i}=1$, such that

$$
\begin{align*}
& 0=\sum_{k \in K} \alpha_{k}\left(\xi_{k}-\frac{p_{k}(\bar{x})}{q_{k}(\bar{x})} \zeta_{k}\right)+\sum_{j \in J} \mu_{j}\left(\sum_{i \in I_{j}} \mu_{j i} \eta_{j i}\right),  \tag{4.16}\\
& \mu_{j} \sup _{v_{j} \in \mathcal{V}_{j}} g_{j}\left(\bar{x}, v_{j}\right)=0, j \in J \tag{4.17}
\end{align*}
$$

Since $v_{j i} \in \mathcal{V}_{j}(\bar{x})$, then $g_{j}\left(\bar{x}, v_{j i}\right)=\sup _{v_{j} \in \mathcal{V}_{j}} g_{j}\left(\bar{x}, v_{j}\right)$ for $j \in J$ and $i \in I_{j}=$ $\left\{1, \ldots, i_{j}\right\}$. Thus, it stems from (4.17) that $\mu_{j} g_{j}\left(\bar{x}, v_{j i}\right)=0$ for $j \in J$ and $i \in I_{j}$. This entails that

$$
\begin{equation*}
\sum_{j \in J} \mu_{j}\left(\sum_{i \in I_{j}} \mu_{j i} g_{j}\left(\bar{x}, v_{j i}\right)\right)=\sum_{j \in J}\left(\sum_{i \in I_{j}} \mu_{j i} \mu_{j} g_{j}\left(\bar{x}, v_{j i}\right)\right)=0 . \tag{4.18}
\end{equation*}
$$

Letting $\bar{\alpha}:=\left(\alpha_{1}, \ldots, \alpha_{l}\right) \in \mathbb{R}_{+}^{l} \backslash\{0\}$, and $\bar{\mu}:=\left(\mu_{j}, \mu_{j i}\right)$. It follows from (4.16) and (4.18) that $(\bar{x}, \bar{\alpha}, \bar{\mu}) \in F_{D_{2}}$. Observe that $f(\bar{x})=\tilde{f}(\bar{x}, \bar{\alpha}, \bar{\mu})$.
(i) As $(p, g ; q)$ is generalized convex-concave at any $z \in \mathbb{R}^{n}$, by invoking Theorem 4.4 (i), we obtain

$$
\tilde{f}(\bar{x}, \bar{\alpha}, \bar{\mu})=f(\bar{x}) \nprec \tilde{f}(z, \alpha, \mu)
$$

for any $(z, \alpha, \mu) \in F_{D}$, i.e., $(\bar{x}, \bar{\alpha}, \bar{\mu})$ is a weakly Pareto solution to problem (RD).
(ii) Since $(p, g ; q)$ is strictly generalized convex-concave at any $z \in \mathbb{R}^{n}$, by invoking Theorem 4.4 (ii), we assert that

$$
\tilde{f}(\bar{x}, \bar{\alpha}, \bar{\mu}) \npreceq \tilde{f}(z, \alpha, \mu)
$$

for any $(z, \alpha, \mu) \in F_{D_{2}}$, that is, $(\bar{x}, \bar{\alpha}, \bar{\mu})$ is a Pareto solution to problem (RD).

Finally, we present converse-like duality relations between the primal problem (RP) and the dual problem (RD).

Theorem 4.6 (Converse-like duality). $\operatorname{Let}(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in F_{D_{2}}$.
(i) If $\bar{x} \in F$ and $(p, g ; q)$ is generalized convex-concave on $\mathbb{R}^{n}$ at $\bar{x}$, then $\bar{x}$ is a weakly Pareto solution of problem (RP).
(ii) If $\bar{x} \in F$ and $(p, g ; q)$ is strictly generalized convex-concave on $\mathbb{R}^{n}$ at $\bar{x}$, then $\bar{x}$ is a Pareto solution of problem (RP).

Proof. Since $(\bar{x}, \bar{\alpha}, \bar{\mu}) \in F_{D_{2}}$, there exist $\bar{\alpha}:=\left(\alpha_{1}, \ldots, \alpha_{l}\right) \in \mathbb{R}_{+}^{l} \backslash\{0\}, \xi_{k} \in$ $\partial p_{k}(\bar{x}), \zeta_{k} \in \partial q_{k}(\bar{x}), k \in K$, and $\bar{\mu}:=\left(\mu_{j}, \mu_{j i}\right), \mu_{j} \geq 0, j \in J, \mu_{j i} \geq 0$, $\eta_{j i} \in \partial_{x} g_{j}\left(\bar{x}, v_{j i}\right), v_{j i} \in \mathcal{V}_{j}(\bar{x}), i \in I_{j}=\left\{1, \ldots, i_{j}\right\}, i_{j} \in \mathbb{N}, \sum_{i \in I_{j}} \mu_{j i}=1$, such
that

$$
\begin{align*}
& 0=\sum_{k \in K} \alpha_{k}\left(\xi_{k}-\frac{p_{k}(\bar{x})}{q_{k}(\bar{x})} \zeta_{k}\right)+\sum_{j \in J} \mu_{j}\left(\sum_{i \in I_{j}} \mu_{j i} \eta_{j i}\right),  \tag{4.19}\\
& \sum_{j \in J} \mu_{j} g_{j}\left(\bar{x}, v_{j}\right) \geq 0 \tag{4.20}
\end{align*}
$$

Let $\bar{x} \in F$. Then $g_{j}\left(\bar{x}, v_{j}\right) \leq 0$, for all $v_{j} \in \mathcal{V}_{j}, j \in J$, and thus, $\mu_{j} g_{j}\left(\bar{x}, v_{j}\right) \leq 0$. This, together with (4.20), yields that $\mu_{j} g_{j}\left(\bar{x}, v_{j}\right)=0, j \in J$, i.e.,

$$
\mu_{j} \sup _{v_{j} \in \mathcal{V}_{j}} g_{j}\left(\bar{x}, v_{j}\right)=0, j \in J
$$

So, we assert by virtue of (4.19) that $\bar{x}$ satisfies condition (3.2). To finish the proof, it remains to apply Theorem 3.4.

## 5. Conclusions

In this paper, along with optimality conditions proposed in [13], we further introduced types of non-fractional and fractional dual problems and examined weak, strong, and converse-like duality relations under assumptions of (strictly) generalized convex-concavity.

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