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# ON SOME PROPERTIES OF CONJUGATE RELATION AND SUBDIFFERENTIALS IN SET OPTIMIZATION PROBLEM

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Abstract. In this paper, we first give new definitions of set-valued conjugate relation based on comparison of sets introduced by Jahn-Ha in 2011. Then we give some types of weak duality theorems. Next, by using nonlinear scalarizing technique for set, we present a strong duality theorem. We also give some continuity properties of conjugate relation for set-valued map. Lastly, we give some definitions of subdifferentials for set-valued map and investigate its properties.

## 1. Introduction

Let $Y$ be a topological vector space ordered by a closed convex cone $C \subset Y$. Let $X$ be a nonempty set and $F : X \to 2^Y$ a set-valued map with domain $X$ ($F(x) \neq \emptyset$ for each $x \in X$). The set-valued optimization problem is formalized as follows:

$$(\mathrm{P}) \begin{cases} \text{Optimize} \quad F(x) \\ \text{Subject to} \quad x \in X \end{cases}$$

The above problem is based on comparison among values of $F$, that is, whole images $F(x)$ (for details see Kuroiwa-Tanaka-Ha [30] and Jahn-Ha [21]) and seems to be more natural for set-valued optimization problem.

For a given vector optimization problem, there are several approaches to construct a dual problem. One of the difficulties is in the fact that the minimal point in vector optimization problem is not necessarily a singleton. However, in general, it becomes a subset of the image space. There are at least four main ideas to overcome the above difficulties (see [5, 33] and their references therein). The first one is the usage of scalarization in the formulation of the dual problem. The second one is based on the observation that a dual optimization problem is set-valued (see [24, 25, 39, 41]). The third one is based on solution concepts

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with respect to the supremum and infimum in the sense of vector lattice (see [32, 33]). The fourth one gravitates around Wolfe and Mond-Weir duality.

This paper is a continuous research of [3]. We introduce a new approach to obtain duality theory in set optimization problem. We adopt the second type approaches to derive weak duality theorems in the framework of set optimization problem. To derive strong duality theorems, we employ the first type approach which is a nonlinear scalarizing technique for sets. We also investigate some properties of conjugate relation and subdifferentials in set optimization problem.

The organization of this paper is as follows. In section 2, we give some preliminaries of vector optimization problem and set optimization problem. In section 3, we introduce some types of nonlinear scalarizing technique for sets [1, 3] which are generalization of Gerstewitz's scalarizing function for the vector-valued case [9, 11, 12]. Section 4 is the main results. First, we give new definitions of set-valued conjugate relation based on comparison of sets ($l\&u$ type [21]). The new definitions, which are inspired by Chapter 7 of [5], are natural extension of vector-valued conjugate function. Then we give some types of weak duality theorems with respect to $l\&u$ type set relation. We also give some continuity properties of conjugate relation for set-valued map. Next, by using nonlinear scalarizing technique for set, we present a strong duality theorem with respect to $l\&u$ type set relation. Lastly, we give some definitions of subdifferentials for set-valued map and investigate its properties.

## 2. Mathematical preliminaries

Let $\mathbb{R}^n$ be a Euclidean space and $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x_1 \geq 0, x_2 \geq 0, ..., x_n \geq 0\}$ a nonnegative orthant of $\mathbb{R}^n$. Throughout of this paper, let $X$ be a Hilbert space, $Y$ a topological vector space and $0_Y$ the origin of $Y$. For a set $A \subset Y$, $\text{int}A$ and $\text{cl}A$ denote the topological interior and the topological closure of $A$, respectively. We denote the set of linear continuous mappings from $X$ to $Y$ by $\mathcal{L}(X, Y)$. We denote $\mathcal{V}$ by the family of nonempty subsets of $Y$. The sum of two sets $V_1, V_2 \in \mathcal{V}$ and the product of $\alpha \in \mathbb{R}$ and $V \in \mathcal{V}$ are defined by

$$V_1 + V_2 := \{v_1 + v_2 \mid v_1 \in V_1, v_2 \in V_2\} \qquad \alpha V := \{\alpha v \mid v \in V\}.$$

In this paper, we assume that $C \subset Y$ is a closed convex cone, that is, $\text{cl}C = C$, $C + C \subset C$ and $t \cdot C \subset C$ for all $t \in [0, \infty)$.

### 2.1. Preliminaries of vector optimization.

A cone $C$ is called pointed if $C \cap (-C) = \{0_Y\}$ and solid if $\text{int}C \neq \emptyset$.

**Definition 2.1.** For $a, b \in Y$ and a solid convex cone $C \subset Y$, we define

$$a \leq_C b \quad \text{by} \quad b - a \in C.$$

**Proposition 2.2.** *For $x \in Y$ and $y \in Y$, the following statements hold:*

(i) $x \leq_C y$ *implies that* $x + z \leq_C y + z$ *for all* $z \in Y$,
(ii) $x \leq_C y$ *implies that* $\alpha x \leq_C \alpha y$ *for all* $\alpha \geq 0$,
(iii) $\leq_C$ *is reflexive and transitive. Moreover, if* $C$ *is pointed,* $\leq_C$ *is antisymmetric and hence a partial order.*

We say that a point $a \in A \subset Y$ is a maximal [resp. weak maximal] point of $A$ if there is no $\hat{a} \in A \setminus \{a\}$ such that $a \leq_C \hat{a}$ [resp. $a \leq_{\text{int}C} \hat{a}$]. The above definition is equivalent to

$$A \cap (a + C) = \{a\} \quad [\text{resp. } A \cap (a + \text{int}C) = \emptyset].$$

We denote by $\text{Max}(A; C)$[resp. $\text{wMax}(A; \text{int}C)$] the set of maximal [resp. weak maximal] points of $A$ with respect to $C$ [resp. $\text{int}C$], respectively. We can easily see that

$$\text{Max}(A; C) \subset \text{wMax}(A; \text{int}C) \subset A.$$

2.2. **Preliminaries of set optimization.** We consider several types of binary relationships on $\mathcal{V}$ by using a solid convex cone $C \subset Y$.

**Definition 2.3** ([21, 30])**.** For $A$, $B \in \mathcal{V}$ and a solid convex cone $C \subset Y$, we define

$$(\textbf{lower type}) \quad A \leq_C^l B \quad \text{by} \quad B \subset A + C,$$

$$(\textbf{upper type}) \quad A \leq_C^u B \quad \text{by} \quad A \subset B - C,$$

$$(\textbf{lower and upper type}) \quad A \leq_C^{l\&u} B \quad \text{by} \quad B \subset A + C \quad \text{and} \quad A \subset B - C.$$

**Proposition 2.4** (Further investigation of [1, 31])**.** *For $A$, $B$, $D \in \mathcal{V}$, $a, b \in Y$ and $\alpha \geq 0$, the following statements hold.*

(i) $A \leq_C^l B$ *implies* $A+D \leq_C^l B+D$ *and* $A \leq_C^u B$ *implies* $A+D \leq_C^u B+D$.
(ii) $A \leq_C^l B$ *implies* $\alpha A \leq_C^l \alpha B$ *and* $A \leq_C^u B$ *implies* $\alpha A \leq_C^u \alpha B$.
(iii) $\leq_C^l$ *and* $\leq_C^u$ *are reflexive and transitive.*
(iv) $A \leq_C^{l\&u} B$ *implies* $A \leq_C^l B$ *and* $A \leq_C^{l\&u} B$ *implies* $A \leq_C^u B$.
(v) $A \leq_C^l B$ *and* $A \leq_C^u B$ *are not comparable, that is,* $A \leq_C^l B$ *does not imply* $A \leq_C^u B$ *and* $A \leq_C^u B$ *does not imply* $A \leq_C^l B$.
(vi) $A \leq_C^u b$ *implies* $A \leq_C^l b$ *and* $a \leq_C^l B$ *implies* $a \leq_C^u B$.

**Definition 2.5** ([34])**.** It is said that $A \in \mathcal{V}$ is

(i) $C$-closed $\big[(-C)$-closed$\big]$ if $A + C$ $[A - C]$ is a closed set,
(ii) $C$-bounded $\big[(-C)$-bounded$\big]$ if for each neighborhood $U$ of zero in $Z$ there is some positive number $t > 0$ such that

$$A \subset tU + C \quad [A \subset tU - C],$$

(iii) $C$-compact $\big[(-C)$-compact$\big]$ if any cover of $A$ the form $\{U_\alpha + C |\ U_\alpha$ are open$\}$ $[\{U_\alpha - C |\ U_\alpha$ are open$\}]$ admits a finite subcover.

Every $C$-compact set is $C$-closed and $C$-bounded. We denote $\mathrm{cl}(\mathcal{V})_C$ by the family of $C$-closed subsets of $Y$ and $\mathrm{cl}(\mathcal{V})_{-C}$ the family of $(-C)$-closed subsets of $Y$, respectively.

**Definition 2.6** ([18])**.** It is said that $A \in \mathcal{V}$ is $C$-proper $\big[(-C)$-proper$\big]$ if

$$A + C \neq Y \qquad [A - C \neq Y].$$

We denote $\mathcal{V}_C$ by the family of $C$-proper subsets of $Y$ and $\mathcal{V}_{-C}$ the family of $(-C)$-proper subsets of $Y$, respectively.

Introducing the equivalence relations

$$A \sim_l B \Longleftrightarrow A \leq_C^l B \quad \text{and} \quad B \leq_C^l A,$$

$$A \sim_u B \Longleftrightarrow A \leq_C^u B \quad \text{and} \quad B \leq_C^u A,$$

$$A \sim_{l\&u} B \Longleftrightarrow A \leq_C^{l\&u} B \quad \text{and} \quad B \leq_C^{l\&u} A,$$

we can generate a partial ordering on the set of equivalence classes which are denoted by $[\cdot]^l$, $[\cdot]^u$ and $[\cdot]^{l\&u}$, respectively. We can easily see that

$$A \in [B]^l \Longleftrightarrow A + C = B + C,$$

$$A \in [B]^u \Longleftrightarrow A - C = B - C,$$

$$A \in [B]^{l\&u} \Longleftrightarrow A + C = B + C \quad \text{and} \quad A - C = B - C.$$

**Definition 2.7** ($l[u, l\&u]$-minimal and $l[u, l\&u]$-maximal element)**.** Let $\mathcal{S} \subset \mathcal{V}$. We say that $\bar{A} \in \mathcal{S}$ is a $l[u, l\&u]$-minimal element if for any $A \in \mathcal{S}$,

$$A \leq_C^{l[u,l\&u]} \bar{A} \quad \text{implies} \quad \bar{A} \leq_C^{l[u,l\&u]} A.$$

Moreover, we say that $\bar{A} \in \mathcal{S}$ is a $l[u, l\&u]$-maximal element if for any $A \in \mathcal{S}$,

$$\bar{A} \leq_C^{l[u,l\&u]} A \quad \text{implies} \quad A \leq_C^{l[u,l\&u]} \bar{A}.$$

We denote the family of $l[u, l\&u]$-minimal elements of $\mathcal{S}$ by $l[u, l\&u]$-$\mathrm{Min}(\mathcal{S}, C)$ and the family of $l[u, l\&u]$-maximal elements of $\mathcal{S}$ by $l[u, l\&u]$-$\mathrm{Max}(\mathcal{S}, C)$.

## 3. Nonlinear scalarization

In this subsection, we assume that $k^0 \in C \setminus (-C)$. In 1980s, Gerstewitz [9] introduced a nonlinear scalarizing function in vector optimization problem. The nonlinear scalarizing function is known as the Gerstewitz's function. Agreeing $\inf \emptyset = \infty$, we define $\varphi_{C,k^0} : Y \to (-\infty, \infty]$,

$$\varphi_{C,k^0}(y) = \inf\{t \in \mathbb{R} \,|\, y \leq_C tk^0\} = \inf\{t \in \mathbb{R} \,|\, y \in tk^0 - C\}.$$

The above scalarization method, which is also found in a similar form [36], contains the linear scalarization as a special case. After in [10, 11], they derived the essential properties of the Gerstewitz's function in vector optimization problem, for instance, monotonicity properties, sublinear properties. Also, the scalarizing function $\varphi_{C,k^0}$ has a dual form. Agreeing $\sup \emptyset = -\infty$, we define $\psi_{C,k^0} : Y \to [-\infty, \infty)$

$$\psi_{C,k^0}(y) = \sup\{t \in \mathbb{R} \,|\, tk^0 \leq_C y\} = \sup\{t \in \mathbb{R} \,|\, y \in tk^0 + C\}$$
$$\big(\varphi_{C,k^0}(y) = -\psi_{C,k^0}(-y)\big).$$

These functions have wide applications in vector optimization (see also Luc [34], Göpfert-Riahi-Tammer-Zălinescu [12]).

The investigation of scalarizing functions for sets begun at around 2000. In the 2000s decade there were four important papers (see [7, 8, 16, 18]). In the last decade, many authors have been investigated sublinear scalarizing technique for set optimization problem ([1, 2, 3, 13, 14, 17, 23, 27, 28, 29, 35, 37, 42] and their references therein).

In this section, we investigate detailed properties of the following nonlinear scalarizing functions for set, which are natural extension of $\varphi_{C,k^0}$ and $\psi_{C,k^0}$. Agreeing $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$, we define $h^l_{\inf}, h^u_{\inf}, h^{l\&u}_{\inf} : \mathcal{V} \to [-\infty, \infty]$

$$h^l_{\inf}(V) = \inf\{t \in \mathbb{R} \,|\, V \leq^l_C \{tk^0\}\} = \inf\{t \in \mathbb{R} \,|\, tk^0 \in V + C\},$$

$$h^u_{\inf}(V) = \inf\{t \in \mathbb{R} \,|\, V \leq^u_C \{tk^0\}\} = \inf\{t \in \mathbb{R} \,|\, V \subset tk^0 - C\},$$

$$\begin{aligned} h^{l\&u}_{\inf}(V) &= \inf\{t \in \mathbb{R} \,|\, V \leq^{l\&u}_C \{tk^0\}\} \\ &= \inf\{t \in \mathbb{R} \,|\, tk^0 \in V + C \text{ and } V \subset tk^0 - C\}, \end{aligned}$$

and $h^l_{\sup}, h^u_{\sup}, h^{l\&u}_{\sup} : \mathcal{V} \to [-\infty, \infty]$

$$h^l_{\sup}(V) = \sup\{t \in \mathbb{R} \,|\, \{tk^0\} \leq^l_C V\} = \sup\{t \in \mathbb{R} \,|\, V \subset tk^0 + C\},$$

$$h^u_{\sup}(V) = \sup\{t \in \mathbb{R} \,|\, \{tk^0\} \leq^u_C V\} = \sup\{t \in \mathbb{R} \,|\, tk^0 \in V - C\},$$

$$\begin{aligned} h^{l\&u}_{\sup}(V) &= \sup\{t \in \mathbb{R} \,\big|\, \{tk^0\} \leq^{l\&u}_C V\} \\ &= \sup\{t \in \mathbb{R} \,\big|\, V \subset tk^0 + C \text{ and } tk^0 \in V - C\}. \end{aligned}$$

The functions $h^l_{\inf}, h^u_{\inf}, h^{l\&u}_{\inf}, h^l_{\sup}, h^u_{\sup}, h^{l\&u}_{\sup}$ play the role of utility functions. By the definitions of the above scalarizing functions for sets, we obtain the following relationships.

**Proposition 3.1** (see also [1, 3])**.** *The following statements hold;*

(i) $h^l_{\sup}(V) = -h^u_{\inf}(-V)$;
(ii) $h^u_{\sup}(V) = -h^l_{\inf}(-V)$;
(iii) $h^l_{\inf}(V) \leq h^u_{\inf}(V) = h^{l\&u}_{\inf}(V)$;
(iv) $h^{l\&u}_{\sup}(V) = h^l_{\sup}(V) \leq h^u_{\sup}(V)$;
(v) $h^{l\&u}_{\sup}(V) = -h^{l\&u}_{\inf}(-V)$.

*Proof.* (iii) (vi) Using the following inclusions

$$\{V \in \mathcal{V} \,\big|\, V \subset tk^0 - C\} \subset \{V \in \mathcal{V} \,\big|\, tk^0 \in V + C\} \quad \text{and}$$
$$\{V \in \mathcal{V} \,\big|\, V \subset tk^0 + C\} \subset \{V \in \mathcal{V} \,\big|\, tk^0 \in V - C\},$$

we obtain $h^u_{\inf}(V) = h^{l\&u}_{\inf}(V)$ and $h^{l\&u}_{\sup}(V) = h^l_{\sup}(V)$. The inequality parts are from [3].

(v) Combining conclusion (i), (iii) and (iv), we obtain the conclusion. □

**Definition 3.2.** We say that the function $f : \mathcal{V} \to [-\infty, \infty]$ is $\leq^l_C$-increasing if $V_1 \leq^l_C V_2$ implies $f(V_1) \leq f(V_2)$. The definitions of $\leq^u_C$-increasing and $\leq^{l\&u}_C$-increasing are similar to the above one.

3.1. **Infimum type.** In this subsection, we give several properties of infimum type scalarizing functions for sets, which is the revised version of [1]. The reader can check that Lemma 3.3 and 3.4 are almost the same as Corollary 3.3 and 3.5 in [1], respectively. However, the above Corollaries in [1] have some mistakes:

- (ii) of Corollary 3.3 in [1] [$h^l_{\inf}(V) \leq t \iff tk^0 \in V + C$] is wrong,
- (vii) and (viii) of Corollary 3.5 in [1] [if $k^0 \in \mathrm{int}C$ then (vii) $V \subset tk^0 - \mathrm{int}C \iff h^u_{\inf}(V) < t$, (viii) $h^u_{\inf}$ is strictly $\leq^u_{\mathrm{int}C}$-increasing] are wrong.

**Lemma 3.3** ($l$-infimum type [1, 3])**.** *Let $k^0 \in \mathrm{int}C$. The function $h^l_{\inf} : \mathcal{V}_C \to (-\infty, \infty]$ has the following properties:*

(i) $h^l_{\inf}(V) \leq t \iff tk^0 \in \mathrm{cl}(V + C)$;
(ii) $h^l_{\inf}$ *is* $\leq^l_C$*-increasing;*
(iii) $h^l_{\inf}(V + \lambda k^0) = h^l_{\inf}(V) + \lambda$ *for every* $\lambda \in \mathbb{R}$;

(iv) $\hat{V} \in [V]^l \Longrightarrow h^l_{\inf}(\hat{V}) = h^l_{\inf}(V)$;
(v) $h^l_{\inf}$ *is sublinear* (*that is, for* $V_1, V_2 \in \mathcal{V}$ *and* $\alpha \geq 0$, $h^l_{\inf}(V_1 + V_2) \leq h^l_{\inf}(V_1) + h^l_{\inf}(V_2)$ *and* $h^l_{\inf}(\alpha V_1) = \alpha h^l_{\inf}(V_1)$);
(vi) $h^l_{\inf}$ *achieves a real value;*
(vii) $h^l_{\inf}(V) < t \iff tk^0 \in V + \mathrm{int}C$;
(viii) $h^l_{\inf}$ *is strictly* $\leq^l_{\mathrm{int}C}$*-increasing.*

**Lemma 3.4** ($u$ and $l\&u$-infimum type [1, 3])**.** *Let* $k^0 \in \mathrm{int}C$. *The function* $h^u_{\inf}(= h^{l\&u}_{\inf}) : \mathcal{V} \to (-\infty, \infty]$ *has the following properties:*
(i) $h^u_{\inf}(V) \leq t \iff V \subset tk^0 - C$;
(ii) $h^u_{\inf}$ *is* $\leq^u_C$*-increasing* [$h^u_{\inf}$ is $\leq^{l\&u}_C$-increasing];
(iii) $h^u_{\inf}(V + \lambda k^0) = h^u_{\inf}(V) + \lambda$ *for every* $\lambda \in \mathbb{R}$;
(iv) $\hat{V} \in [V]^u \Longrightarrow h^u_{\inf}(\hat{V}) = h^u_{\inf}(V)$ [$\hat{V} \in [V]^{l\&u} \Longrightarrow h^u_{\inf}(\hat{V}) = h^u_{\inf}(V)$];
(v) $h^u_{\inf}$ *is sublinear;*
(vi) $h^u_{\inf}(V) < t \Longrightarrow V \subset tk^0 - \mathrm{int}C$.

*Moreover, if* $k^0 \in \mathrm{int}C$ *and* $V$ *is* $(-C)$*-bounded then* $h^u_{\inf}$ *has the following property:*
(vii) $h^u_{\inf}$ *achieves a real value.*

*Furthermore, if* $k^0 \in \mathrm{int}C$ *and* $V$ *is* $(-C)$*-compact then* $h^u_{\inf}$ *has the following properties:*
(viii) $V \subset tk^0 - \mathrm{int}C \Longrightarrow h^u_{\inf}(V) < t$;
(ix) $h^u_{\inf}$ *is strictly* $\leq^u_{\mathrm{int}C}$*-increasing* [$h^u_{\inf}$ is strictly $\leq^{l\&u}_{\mathrm{int}C}$-increasing].

3.2. **Supremum type.** Using Proposition 3.1, we obtain the following Lemmas in a similar way as Lemma 3.3 and Lemma 3.4.

**Lemma 3.5** ($l$-supremum type)**.** *Let* $k^0 \in \mathrm{int}C$. *The function* $h^l_{\sup} : \mathcal{V} \to [-\infty, \infty)$ *has the following properties:*
(i) $h^l_{\sup}(V) \geq t \iff V \subset tk^0 + C$;
(ii) $h^l_{\sup}$ *is* $\leq^l_C$*-increasing;*
(iii) $h^l_{\sup}(V + \lambda k^0) = h^l_{\sup}(V) + \lambda$ *for every* $\lambda \in \mathbb{R}$;
(iv) $\hat{V} \in [V]^l \Longrightarrow h^l_{\sup}(\hat{V}) = h^l_{\sup}(V)$;
(v) $h^l_{\sup}$ *is super-additive and positively homogeneous* (*that is, for* $V_1, V_2 \in \mathcal{V}$ *and* $\alpha \geq 0$, $h^l_{\sup}(V_1 + V_2) \geq h^l_{\sup}(V_1) + h^l_{\sup}(V_2)$ *and* $h^l_{\sup}(\alpha V_1) = \alpha h^l_{\sup}(V_1)$);
(vi) $h^l_{\sup}(V) > t \Longrightarrow V \subset tk^0 + \mathrm{int}C$.

*Moreover, if* $k^0 \in \mathrm{int}C$ *and* $V$ *is* $C$*-bounded then* $h^l_{\sup}$ *has the following property:*
(vii) $h^l_{\sup}$ *achieves a real value.*

*Furthermore, if $k^0 \in \text{int}C$ and $V$ is $C$-compact then $h^l_{\sup}$ has the following properties:*

(viii) $V \subset tk^0 + \text{int}C \Longrightarrow h^l_{\sup}(V) > t$;
(ix) $h^l_{\sup}$ *is strictly* $\leq^l_{\text{int}C}$*-increasing.*

**Lemma 3.6** ($u$-supremum type)**.** *Let $k^0 \in \text{int}C$. The function $h^u_{\sup} : \mathcal{V}_{-C} \to [-\infty, \infty)$ has the following properties:*

(i) $h^u_{\sup}(V) \geq t \iff tk^0 \in \text{cl}(V - C)$;
(ii) $h^u_{\sup}$ *is* $\leq^u_C$*-increasing;*
(iii) $h^u_{\sup}(V + \lambda k^0) = h^u_{\sup}(V) + \lambda$ *for every* $\lambda \in \mathbb{R}$;
(iv) $\hat{V} \in [V]^u \Longrightarrow h^u_{\sup}(\hat{V}) = h^u_{\sup}(V)$;
(v) $h^u_{\sup}$ *is super-additive and positively homogeneous;*
(vi) $h^u_{\sup}$ *achieves a real value;*
(vii) $h^u_{\sup}(V) > t \iff tk^0 \in V - \text{int}C$;
(viii) $h^u_{\sup}$ *is strictly* $\leq^u_{\text{int}C}$*-increasing.*

3.3. **Inherited properties of continuity and convexity for set-valued map.**

**Definition 3.7** ($l\&u$-$C$-convexity)**.** Let $K$ be a convex set in a real vector space $X$. A set-valued map $F : X \to \mathcal{V}$ is said to be $l\&u$-$C$-convex on $K$ if for each $x_1, x_2 \in K$ and $\lambda \in [0, 1]$, we have

$$F(\lambda x_1 + (1 - \lambda)x_2) \leq^{l\&u}_C \lambda F(x_1) + (1 - \lambda)F(x_2).$$

**Definition 3.8** ($C$-lower semi-continuity)**.** Let $X$ be a topological space. A set-valued map $F : X \to \mathcal{V}$ is said to be $l[u, l\&u]$-$C$-lower semi-continuous at $X$ if the set

$$\{x \in X | F(x) \leq^{l[u,l\&u]}_C V\}$$

is closed for all $V \in \mathcal{V}$.

**Definition 3.9** (strong $C$-lower semi-continuity)**.** Let $X$ be a topological space. A set-valued map $F : X \to \mathcal{V}$ is said to be

(i) strong $l$-$C$-lower semi-continuous at $X$ if the set

$$\{x \in X | F(x) \leq^l_C V\}$$

is closed for all $F(x) \in \text{cl}(\mathcal{V})_{-C}$ and $V \in \mathcal{V}$,

(ii) strong $u$-$C$-lower semi-continuous at $X$ if the set

$$\{x \in X | F(x) \leq^u_C V\}$$

is closed for all $V \in \text{cl}(\mathcal{V})_{-C}$,

(iii) strong $l\&u$-$C$-lower semi-continuous at $X$ if the set

$$\{x \in X | F(x) \leq_C^{l\&u} V\}$$

is closed for all $F(x) \in \mathrm{cl}(\mathcal{V})_{-C}$ and $V \in \mathrm{cl}(\mathcal{V})_{-C}$.

By using (ii) of Lemma 3.4, we obtain the following properties.

**Lemma 3.10.** *Let $K$ be a convex set in a real vector space $X$ and $k^0 \in \mathrm{int}C$. If a set-valued map $F : X \to \mathcal{V}$ is $l\&u$-$C$-convex, then $h_{\inf}^{l\&u}(F(\cdot))$ is convex on $K$.*

**Lemma 3.11.** *Let $X$ be a topological space and $k^0 \in \mathrm{int}C$. If a set-valued map $F : X \to \mathcal{V}$ is $l\&u$-$C$-lower semi-continuous, then $h_{\inf}^{l\&u}(F(\cdot))$ is lower semi-continuous.*

## 4. Main results

Let $H$ be a Hilbert space and let $f : H \to (-\infty, \infty]$ be a proper function. Then we define the conjugate and biconjugate function of $f$ as follows:

$$f^*(x^*) := \sup_{x \in H}\{\langle x, x^* \rangle - f(x)\},$$

$$f^{**}(x) := \sup_{x^* \in H}\{\langle x, x^* \rangle - f^*(x^*)\}.$$

The following result is the one of the most fundamental theorem in duality theory.

**Theorem 4.1** ([38])**.** *Let $H$ be a Hilbert space and let $f : H \to (-\infty, \infty]$ be a proper convex lower semicontinuous function. Then $f^{**} = f$.*

In this section, let $C$ be a solid closed convex cone. We consider the conjugate and biconjugate of set-valued map $F : X \to \mathcal{V}$ and investigate its properties. Moreover, based on the proof of Theorem 4.1 in [38], we extend Theorem 4.1 to set-valued map by using classical Hahn-Banach theorem and nonlinear scalarizing technique mentioned in Section 3.

### 4.1. Conjugate relations and weak duality.

First, we look back on a theory of conjugate duality in vector optimization. We denote the set of $m \times n$ matrix by $\mathbb{R}^{m \times n}$.

**Definition 4.2** (Tanino-Sawaragi [39, 40, 41])**.** Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function. Then the conjugate function of $f$, $f^* : \mathbb{R}^{m \times n} \to \mathcal{V}$, is defined by the following form

$$f^*(A) := \mathrm{Max}\left(\bigcup_{x \in \mathbb{R}^n}\{Ax - f(x)\}; \mathbb{R}_+^m\right),$$

By reiterating the operation $f \to f^*$ on $f^*$, we define the biconjugate of $f$, $f^{**} : \mathbb{R}^n \to \mathcal{V}$, by the following form

$$f^{**}(x) := \mathrm{Max}\left( \bigcup_{A \in \mathbb{R}^{m\times n}} \{Ax - f^*(A)\}; \mathbb{R}^m_+ \right).$$

However, generally speaking, $f^*(A)$ is a set-valued mapping. To overcome the difficulty, Kawasaki [24, 25] introduced set relation on $\mathcal{V}$. Based on his results, we presented new definitions of the biconjugate of $f$.

**Definition 4.3** ([4])**.** For $f^*(A) \neq \emptyset$, we define $f_l^{**}, f_u^{**} : \mathbb{R}^n \to \mathcal{V}$ by

$$f_l^{**}(x) := l\text{-Max}\left( \bigcup_{A \in \mathbb{R}^{m\times n}} [Ax - f^*(A)], \mathbb{R}^m_+ \right),$$

$$f_u^{**}(x) := u\text{-Max}\left( \bigcup_{A \in \mathbb{R}^{m\times n}} [Ax - f^*(A)], \mathbb{R}^m_+ \right).$$

In a similar way as the above, we also gave new definitions of set-valued conjugate maps in infinite dimensional space as a natural extension of [4, 39, 40, 41].

**Definition 4.4** (Araya [3])**.** Let $F : X \to \mathcal{V}$ be a set-valued map. Then the conjugate function of $F$, $F_l^*, F_u^* : \mathcal{L}(X, Y) \to \mathcal{V}$, are defined by the following form

$$F_l^*(T) := l\text{-Max}\left( \bigcup_{x \in X} [Tx - F(x)], C \right),$$

$$F_u^*(T) := u\text{-Max}\left( \bigcup_{x \in X} [Tx - F(x)], C \right).$$

**Definition 4.5** (Araya [3])**.** For $F_l^*(T) \neq \emptyset$ and $F_u^*(T) \neq \emptyset$, we define $F_{ll}^{**}, F_{lu}^{**}, F_{ul}^{**}, F_{uu}^{**} : X \to \mathcal{V}$ by

$$F_{ll}^{**}(x) := l\text{-Max}\left( \bigcup_{T \in \mathcal{L}(X,Y)} [Tx - F_l^*(T)], C \right),$$

$$F_{lu}^{**}(x) := l\text{-Max}\left( \bigcup_{T \in \mathcal{L}(X,Y)} [Tx - F_u^*(T)], C \right),$$

$$F_{ul}^{**}(x) := u\text{-Max}\left( \bigcup_{T \in \mathcal{L}(X,Y)} [Tx - F_l^*(T)], C \right),$$

$$F_{uu}^{**}(x) := u\text{-Max}\left( \bigcup_{T \in \mathcal{L}(X,Y)} [Tx - F_u^*(T)], C \right).$$

In a similar way as the above, we give new definitions of set-valued conjugate maps with respect to $l\&u$ set relation.

**Definition 4.6.** Let $F : X \to \mathcal{V}$ be a set-valued map. Then the conjugate function of $F$, $F^*_{l\&u} : \mathcal{L}(X,Y) \to \mathcal{V}$, is defined by the following form

$$F^*_{l\&u}(T) := l\&u\text{-Max}\left(\bigcup_{x\in X}[Tx - F(x)], C\right).$$

Moreove, for $F^*_{l\&u}(T) \neq \emptyset$, we define $F^{**}_{l\&u} : X \to \mathcal{V}$ by

$$F^{**}_{l\&u}(x) := l\&u\text{-Max}\left(\bigcup_{T\in\mathcal{L}(X,Y)}[Tx - F^*_{l\&u}(T)], C\right).$$

In a similar way as [3], we obtain the following weak duality theorem.

**Theorem 4.7** (Weak duality)**.** *Let $F : X \to \mathcal{V}$ be a set-valued map. Then the biconjugate of $F$ has the following property:*

$$F^{**}_{l\&u}(x) \leq^{l\&u}_C F(x).$$

*Proof.* By the definition of $F^*_{l\&u}$, we have

$$Tx - F(x) \leq^l_C F^*_{l\&u}(T) \text{ and } Tx - F(x) \leq^u_C F^*_{l\&u}(T)\ \forall x \in X,\ \forall T \in \mathcal{L}(X,Y),$$

$(F^*_{l\&u}(T) \subset Tx - F(x) + C$ and $Tx - F(x) \subset F^*_{l\&u}(T) - C)$. Then we obtain the following inclusions:

$$Tx - F^*_{l\&u}(T) \subset F(x) - C \qquad \text{and} \qquad F(x) \subset Tx - F^*_{l\&u}(T) + C,$$

that is, $Tx - F^*_{l\&u}(T) \leq^u_C F(x)$ and $Tx - F^*_{l\&u}(T) \leq^l_C F(x)$. By the definition of $F^{**}_{l\&u}$, we obtain the conclusion. □

Inspired by [5], we give new definitions of conjugate and biconjugate for set-valued map with respect to an element $k^0 \in \text{int}C$.

**Definition 4.8.** Let $F : X \to \mathcal{V}$ be a set-valued map and $k^0 \in \text{int}C$. Then the conjugate function of $F$, $F^*_{k^0,l\&u} : X \to \mathcal{V}$, is defined by the following form

$$F^*_{k^0,l\&u}(x^*) := l\&u\text{-Max}\left(\bigcup_{x\in X}[\langle x, x^*\rangle k^0 - F(x)], C\right)$$

for all $x^* \in X$. Moreover, for $F^*_{k^0,l\&u}(x^*) \neq \emptyset$, we define $F^{**}_{k^0,l\&u} : X \to \mathcal{V}$ by

$$F^{**}_{k^0,l\&u}(x) := l\&u\text{-Max}\left(\bigcup_{x^*\in X^*}[\langle x, x^*\rangle k^0 - F^*_{k^0,l\&u}(x^*)], C\right).$$

By the definition of $F^*_{k^0,l\&u}$ and using (ii), (iii) of Lemma 3.4, we obtain the following properties.

**Lemma 4.9.** *The functions $F^*_{k^0,l\&u}$ have the following properties:*

- $h^u_{\inf}(F^*_{k^0,l\&u}(x^*)) = \sup_{x\in X}\{\langle x, x^*\rangle + h^u_{\inf}(-F(x))\}$,
- $h^u_{\inf}(-F^*_{k^0,l\&u}(x^*)) = \inf_{x\in X}\{-\langle x, x^*\rangle + h^u_{\inf}(F(x))\}$.

In a similar way as Theorem 4.7, we obtain the following weak duality theorem.

**Theorem 4.10** ($k^0$-weak duality)**.** *Let $F : X \to \mathcal{V}$ be a set-valued map. Then the biconjugate of $F$ has the following property:*

$$F^{**}_{k^0,l\&u}(x) \leq^{l\&u}_C F(x).$$

4.2. **Some properties of conjugate relations.**

**Theorem 4.11.** *The following statements hold.*

(i) *$F^*_l$ is strong $l$-$C$-lower semi-continuous,*
(ii) *$F^*_u$ is strong $u$-$C$-lower semi-continuous,*
(iii) *$F^*_{l\&u}$ is strong $l\&u$-$C$-lower semi-continuous.*

*Proof.* We set

$$L^l := \{T \in L(X,Y) \,\big|\, F^*_l(T) \leq^l_C V\}$$

and let $\{T_n\} \subset L^l$ with $T_n \to \hat{T}$ $(n \to \infty)$. Then we have that $T_n x - F(x) \leq^l_C V$, that is,

$$V \subset T_n x - F(x) + C$$

and hence

$$-V \subset -T_n x + F(x) - C.$$

Since $F(x)$ is $(-C)$-closed valued map, we obtain

$$V \subset \hat{T}x - F(x) + C \quad (\hat{T}x - F(x) \leq^l_C V)$$

and hence $\hat{T} \in L^l$. We can show the strong continuity properties of $F^*_u$ and $F^*_{l\&u}$ in a similar way as the above. □

**Example 1.** Assumption of $(-C)$-closedness on $F$ is needed to show that continuity of conjugate relation $F^*_l$. We set

$$X = [1,2], \qquad Y = \mathbb{R}^2, \qquad C = \mathbb{R}^2_+, \qquad k^0 = (1,1),$$

$$V = [0,1] \times [0,1], \qquad F(x) = (-1,0) \times (-3x, 3x), \qquad T_n x = -\frac{x}{2n} k^0.$$

We can check that $T_n$ is linear. Moreover, we can also check that $F$ is not $(-C)$-closed valued map. In this situation, we have that $V \subset T_n x - F(x) + C$ for all $x \in X$ with $T_n x \to \hat{T}x$ $(n \to \infty)$ and

$$V \not\subset \hat{T}x - F(x) + C$$

since $\hat{T}x = 0_Y$.

**Example 2.** Assumption of $(-C)$-closedness on $V$ is needed to show that continuity of conjugate relation $F_u^*$. We set

$$X = [1,2], \qquad Y = \mathbb{R}^2, \qquad C = \mathbb{R}^2_+, \qquad k^0 = (1,1),$$

$$V = \{(x,y)\,|\,xy \leq -1 \quad x < 0\}, \qquad F(x) = [0,x] \times [0,x], \qquad T_n x = -\frac{x}{2n}k^0.$$

We can check that $T_n$ is linear. Moreover, we can also check that $V$ is not $(-C)$-closed. In this situation, we have that $T_n x - F(x) \subset V - C$ for all $x \in X$ with $T_n x \to \tilde{T}x$ $(n \to \infty)$ and

$$\tilde{T}x - F(x) \not\subset V - C$$

since $\tilde{T}x = 0_Y$.

4.3. **Strong duality.**

**Theorem 4.12** ($F^{**}_{k^0,l\&u}$-type)**.** *Let $F : X \to \mathcal{V}$ be a $(-C)$-bounded valued map and $k^0 \in \mathrm{int}C$. We assume the following conditions:*

(i) *$F$ is l&u-$C$-lower semi-continuous,*
(ii) *$F$ is l&u-$C$-convex,*
(iii) *there exists $\hat{s} \in \mathbb{R}$ such that $F(x) \in [\hat{s}k^0]^{l\&u}$,*
(iv) *there exists $\hat{t} \in \mathbb{R}$ such that $F^*_{k^0,l\&u}(x) \in [\hat{t}k^0]^u$.*

*Then we have $h^u_{\inf}(F^{**}_{k^0,l\&u}) = h^u_{\inf}(F)$.*

*Proof.* By Theorem 4.10 and (ii) of Lemma 3.4, we obtain

($\diamond$) $h^u_{\inf}(F^{**}_{k^0,l\&u}(x)) \leq h^u_{\inf}(F(x))$ for all $x \in X$.

By the assumption and (vii) of Lemma 3.4, we have $h^u_{\inf}(F(x)) \in \mathbb{R}$ for all $x \in X$. By the assumption $k^0 \in \mathrm{int}C$ and using property ($\diamond$), we have that for all $x \in X$

$$h^u_{\inf}(F^{**}_{k^0,l\&u}(x)) \leq h^u_{\inf}(F(x)) < \infty.$$

Moreover, since by the assumption, (vii) of Lemma 3.4 and Lemma 4.9, we have

$$h^u_{\inf}(-F^*_{k^0,l\&u}(x^*)) = \inf_{x \in X}\{-\langle x, x^*\rangle + h^u_{\inf}(F(x))\} \in \mathbb{R}$$

and hence

$$h^u_{\inf}(F^{**}_{k^0,l\&u}(x)) = \sup_{x^* \in X^*}\{\langle x, x^*\rangle + h^u_{\inf}(-F^*_{k^0,l\&u}(x^*))\} > -\infty,$$

that is, $h^u_{\inf}(F^{**}_{k^0,l\&u})$ is proper.

By assumption (iii) and (iv) of Proposition 2.4, we have that $F(x) \leq^u_C \hat{s}k^0 \leq^l_C F(x)$. Using Proposition 3.1 and (ii) of Lemma 3.4 and 3.5, we have

$$(\star) \quad h^u_{\inf}(F(x)) \leq \hat{s} \leq h^l_{\sup}(F(x)) = -h^u_{\inf}(-F(x)).$$

By assumption (iv) and (vi) of Proposition 2.4, we obtain $F^*_{k^0,l\&u}(x^*) \leq^l_C \hat{t}k^0 \leq^u_C F^*_{k^0,l\&u}(x^*)$. Using Proposition 3.1 and (ii) of Lemma 3.4 and 3.5, we have

$$(\star\star) \quad -h^u_{\inf}(-F^*_{k^0,l\&u}(x^*)) = h^l_{\sup}(F^*_{k^0,l\&u}(x^*)) \leq \hat{t} \leq h^u_{\inf}(F^*_{k^0,l\&u}(x^*)).$$

We suppose contrary that there exists $z \in X$ such that $h^u_{\inf}(F^{**}_{k^0,l\&u}(z)) < h^u_{\inf}(F(z))$. We set

$$\mathrm{Dom}(h^u_{\inf} \circ F) := \{x \in X \,|\, h^u_{\inf}(F(x)) < \infty\},$$

$$\mathrm{Epi}(h^u_{\inf} \circ F) := \{(x,t) \in X \times \mathbb{R} \,|\, h^u_{\inf}(F(x)) \leq t\}.$$

Then we have by the assumption and Lemma 3.10, 3.11 that $\mathrm{Epi}(h^u_{\inf} \circ F)$ is closed and convex. Moreover, we have

$$(z, h^u_{\inf} \circ F^{**}_{k^0,l\&u}(z)) \notin \mathrm{Epi}(h^u_{\inf} \circ F).$$

From classical Hahn-Banach theorem there exists $(z^*, \alpha) \in X \times \mathbb{R}$ such that $(z^*, \alpha) \neq (0,0)$ and

$$(*) \quad \langle z, z^* \rangle + \alpha \cdot h^u_{\inf} \circ F^{**}_{k^0,l\&u}(z) > \sup\{\langle x, z^* \rangle + \alpha t \,|\, (x,t) \in \mathrm{Epi}(h^u_{\inf} \circ F)\}.$$

It is clear that $\alpha \leq 0$. By using scalarizing function $h^u_{\inf}$ and following the same line as Theorem 4.12 in [3], we obtain $\alpha < 0$.

Dividing $(*)$ by $(-\alpha)$ and using $(\star)$ and Lemma 4.9, we have

$$\begin{aligned}
\frac{\langle z, z^* \rangle}{-\alpha} - h^u_{\inf} \circ F^{**}_{k^0,l\&u}(z) &> \sup\left\{ \frac{\langle x, z^* \rangle}{-\alpha} - t \;\middle|\; (x,t) \in \mathrm{Epi}(h^u_{\inf} \circ F) \right\} \\
&= \sup\left\{ \frac{\langle x, z^* \rangle}{-\alpha} - h^u_{\inf}(F(x)) \;\middle|\; x \in \mathrm{Dom}(h^u_{\inf} \circ F) \right\} \\
&\geq \sup\left\{ \frac{\langle x, z^* \rangle}{-\alpha} + h^u_{\inf}(-F(x)) \;\middle|\; x \in \mathrm{Dom}(h^u_{\inf} \circ F) \right\} \\
&= h^u_{\inf} \circ F^*_{k^0,l\&u}\left(\frac{z^*}{-\alpha}\right)
\end{aligned}$$

On the other hand, by the definition of $F^{**}_{k^0,l\&u}$, $(\star\star)$ and (ii), (iii) of Lemma 3.4, we have

$$\begin{aligned}
\left\langle z, \frac{z^*}{-\alpha} \right\rangle - h^u_{\inf} \circ F^*_{k^0,l\&u}\left(\frac{z^*}{-\alpha}\right) &\leq \left\langle z, \frac{z^*}{-\alpha} \right\rangle + h^u_{\inf}\left( - F^*_{k^0,l\&u}\left(\frac{z^*}{-\alpha}\right)\right) \\
&\leq h^u_{\inf} \circ F^{**}_{k^0,l\&u}\left(\frac{z^*}{-\alpha}\right),
\end{aligned}$$

which is a contradiction. Therefore, we have $h^u_{\inf}(F^{**}_{k^0,l\&u}) = h^u_{\inf}(F)$. □

**Remark 1.** In [3], we have found that by (iii) of Proposition 3.1 it is difficult to obtain $F^{**}_{k^0,ul}$-type duality theorem in this manner. Moreover, we have to assume some conditions on $F$ and $F^*_{k^0,u}$ to obtain $ll$-type (Theorem 4.13 in [3]) and $uu$-type (Theorem 4.14 in [3]) strong duality theorems.

- $F$ satisfies the condition $F(x) - F(x) \subset C$ for all $x \in X$. [$ll$-type]
- $F^*_{k^0,u}(x^*)$ satisfies the condition $F^*_{k^0,u}(x^*) - F^*_{k^0,u}(x^*) \subset -C$ for all $x^* \in X^*$. [$uu$-type]

4.4. **Subdifferentials for set-valued map.**

**Definition 4.13.** Let $F : X \to \mathcal{V}$ be a set-valued map. Then the subdifferential of $F$, $\partial^l F(x), \partial^u F(x)$, are defined by the following set:

$$\partial^l F(x) := \{T \in \mathcal{L}(X,Y) \mid Tv - Tx + F(x) \leq^l_C F(v) \quad \forall v \in X\},$$
$$\partial^u F(x) := \{T \in \mathcal{L}(X,Y) \mid Tv - Tx + F(x) \leq^u_C F(v) \quad \forall v \in X\}.$$

**Proposition 4.14.** *Let $F : X \to \mathcal{V}$ be a set-valued map and $z \in X$. Then we have the following relationship:*

(a) $0_Y \in \partial^l F(z) \Longrightarrow F(z) \in l\text{-Min}\left(\bigcup_{x\in X}[F(x)], C\right)$,

(b) $0_Y \in \partial^u F(z) \Longrightarrow F(z) \in u\text{-Min}\left(\bigcup_{x\in X}[F(x)], C\right)$.

*Proof.* By the definition of $\partial^l F(z)$, we have that

$$0_Y \in \partial^l F(z) \iff 0x - 0z + F(z) \leq^l_C F(x) \quad (\forall x \in X)$$
$$\iff F(z) \leq^l_C F(x) \quad (\forall x \in X) \implies F(z) \in l - \text{Min}\left(\bigcup_{x\in X}[F(x)], C\right)$$

and hence conclusion (a) holds. In a similar way, we obtain conclusion (b). □

**Theorem 4.15.** *Let $F : X \to \mathcal{V}$ be a set-valued map and $z \in X$. Then the following statement is true:*

$$T \in \partial^l F(z) \implies F(z) + F^*_l(T) \leq^l_C Tz.$$

*Proof.* By the definition of $\partial^l F(z)$, we have that

$$T \in \partial^l F(z) \iff Tx - Tz + F(z) \leq^l_C F(x) \quad (\forall x \in X)$$
$$\iff F(x) \subset Tx - Tz + F(z) + C \quad (\forall x \in X)$$
$$\iff Tz + F(x) \subset Tx + F(z) + C \quad (\forall x \in X)$$
$$\Longrightarrow Tz \in Tz + F(x) - F(x) \subset Tx - F(x) + F(z) + C \quad (\forall x \in X)$$
$$\iff F(z) + Tx - F(x) \leq^l_C Tz \quad (\forall x \in X).$$

By the definition of $F^*_l(T)$, we obtain $F(z) + F^*_l(T) \leq^l_C Tz$. □

**Theorem 4.16.** *Let $F : X \to \mathcal{V}$ be a set-valued map and $z \in X$. Then the following statement is true:*

$$F(z) + F_u^*(T) \leq_C^u Tz \implies T \in \partial^u F(z).$$

*Proof.* Let $F(z) + F_u^*(T) \leq_C^u Tz$. By the definition of $F_u^*(T)$, we have

$$\begin{aligned}
&\iff F(z) + Tx - F(x) \leq_C^u Tz \quad (\forall x \in X) \\
&\iff F(z) + Tx - F(x) \subset Tz - C \quad (\forall x \in X) \\
&\iff Tx - Tz + F(z) - F(x) \subset -C \quad (\forall x \in X) \\
&\implies Tx - Tz + F(z) \subset Tx - Tz + F(z) + F(x) - F(x) \subset F(x) - C \quad (\forall x \in X) \\
&\iff Tx - Tz + F(z) \leq_C^u F(x) \quad (\forall x \in X),
\end{aligned}$$

that is, $T \in \partial^u F(z)$. □

## 5. Conclusions

In this paper, first we gave new definitions of conjugate of set-valued map in the framework of set optimization problem. Then we presented weak duality theorems with respect to the set relations $\leq_C^{l\&u}$. Moreover, we presented strong duality theorems which depend on the direction $k^0 \in \mathrm{int}C$ and nonlinear scalarization technique. We also gave some continuity properties of conjugate relation for set-valued map. Then we have found that the concept of $C$-closedness plays an important role to derive some kind of continuity properties of set-valued map $F$ in set optimization problem. Lastly, we gave new definitions of subdifferentials for set-valued map and investigate its properties. We also gave some relationships between the subdifferentials for set-valued map and the Fenchel's type inequalities for set-valued map.

By Proposition 4.14, the author think that Definition 4.13 is one of the natural extensions of subdifferential for (extended) real-valued function. There are some previous researches for the differentials for set-valued map (for instance, [6, 15, 20]), however, the investigation of subdifferential for set-valued map has only just begun. The investigation of relationships between the differentials for set-valued map and the subdifferentials for set-valued map will be one of the most important subject of set optimization problem.

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**Lemma 5.1.** *For $C \subset Y$ a closed convex cone and $A, B, V \in \mathcal{V}$, the following statements hold:*

(i) $C + C = C$;
(ii) $C + \mathrm{int}C = \mathrm{int}C$;
(iii) $\mathrm{cl}A + \mathrm{cl}B \subset \mathrm{cl}(A + B)$;
(iv) $\mathrm{cl}(V + C) + C = \mathrm{cl}(V + C)$.

## Appendix A: Proof of Lemma 3.3

*Proof.* We define

$$\Lambda^l_-(V) := \{t \in \mathbb{R} \,\big|\, tk^0 \in \mathrm{int}(V + C)\},$$

$$\Lambda^l(V) := \{t \in \mathbb{R} \,\big|\, tk^0 \in V + C\},$$

$$\Lambda^l_+(V) := \{t \in \mathbb{R} \,\big|\, tk^0 \in \mathrm{cl}(V + C)\}.$$

Then we have obviously that $\Lambda^l_-(V) \subset \Lambda^l(V) \subset \Lambda^l_+(V)$ and hence

$$\inf \Lambda^l_+(V) \leq \inf \Lambda^l(V)(= h^l_{\mathrm{inf}}(V)) \leq \inf \Lambda^l_-(V).$$

(i) We assume $h^l_{\mathrm{inf}}(V) \leq t$ and let $t \in \mathbb{R}$ be fixed. Then by the definitions of $h^l_{\mathrm{inf}}$ and $\Lambda^l$ being of epigraphical type (that is, $t \in \Lambda^l$ and $\hat{t} > t$ implies $\hat{t} \in \Lambda^l$, see [1]), we have

$$\left(t + \frac{1}{n}\right)k^0 \in V + C$$

for all $n \in \mathbb{N}$. Taking the limit when $n \to \infty$, we obtain $tk^0 \in \mathrm{cl}(V + C)$.

Conversely, by the definitions of $h^l_{\mathrm{inf}}$, we show

$$\inf \Lambda^l_+(V) = \inf \Lambda^l(V) = \inf \Lambda^l_-(V).$$

We assume contrary that $\inf \Lambda^l_+(V) < \inf \Lambda^l_-(V)$. Then there exists $t_1, t_2 \in \mathbb{R}$ such that $\inf \Lambda^l_+(V) < t_1 < t_2 < \inf \Lambda^l_-(V)$. By $\inf \Lambda^l_+(V) < t_1$ $[t_1k^0 \in \mathrm{cl}(V + C)]$ and using (iv) of Lemma 5.1, we have

$$(*) \quad t_1k^0 + C \subset \mathrm{cl}(V + C) + C = \mathrm{cl}(V + C).$$

On the other hand, we have

$$(**) \quad t_2k^0 \in t_2k^0 + C = t_1k^0 + C + (t_2 - t_1)k^0 \subset t_1k^0 + \mathrm{int}C = \mathrm{int}(t_1k^0 + C).$$

By $(*)$, we have the following inclusion

$$(***) \quad \mathrm{int}(t_1k^0 + C) \subset \mathrm{int}(\mathrm{cl}(V + C)) = \mathrm{int}(V + C).$$

By $(**)$ and $(***)$, we obtain $t_2k^0 \in \mathrm{int}(V + C)$, which contradicts $t_2 < \inf \Lambda^l_-(V)$.

(ii) Let $V_1, V_2 \in \mathcal{V}$ be such that $V_1 \leq_C^l V_2$ $(V_2 \subset V_1 + C)$. Then we have

$$V_2 + C \subset V_1 + C + C = V_1 + C.$$

If $h_{\inf}^l(V_2) = \infty$, we have that condition (ii) clearly holds. Taking $h_{\inf}^l(V_2) \in \mathbb{R}$, we obtain

$$h_{\inf}^l(V_2)k^0 \subset \mathrm{cl}(V_2 + C) \subset \mathrm{cl}(V_1 + C).$$

Using (i) of Lemma 3.3, we have $h_{\inf}^l(V_1) \leq h_{\inf}^l(V_2)$.

(iii) see [31]. (iv) is from the definition of equivalence class and the monotonicity of $h_{\inf}^l$.

(v) We prove sub-additivity. For any $V_1, V_2 \in \mathcal{V}$ by the definition of $h_{\inf}^l$ we have

$$h_{\inf}^l(V_1)k^0 \subset \mathrm{cl}(V_1 + C) \quad \text{and} \quad h_{\inf}^l(V_2)k^0 \subset \mathrm{cl}(V_2 + C).$$

If $h_{\inf}^l(V_1) = \infty$ or $h_{\inf}^l(V_2) = \infty$, we have that condition (v) clearly holds. By adding the above inclusions and using (iii) of Lemma 5.1, we obtain

$$\{h_{\inf}^l(V_1) + h_{\inf}^l(V_2)\}k^0 \subset \mathrm{cl}(V_1 + C) + \mathrm{cl}(V_2 + C) \subset \mathrm{cl}(V_1 + V_2 + C).$$

Using (i) of Lemma 3.3, we obtain the sub-additivity of $h_{\inf}^l$. The positively homogeneity of $h_{\inf}^l$ is easy.

(vi) Firstly, we show

$$V \in \mathcal{V} : C\text{−proper} \iff h_{\inf}^l(V) > -\infty$$

If $V + C = Y$ for $V \in \mathcal{V}$, then we have $tk^0 \subset V + C$ for all $t \in \mathbb{R}$, which is equivalent to $h_{\inf}^l(V) = -\infty$. Conversely, let $tk^0 \subset V + C$ for all $t \in \mathbb{R}$. Then we have

$$tk^0 + C \subset V + C + C = V + C.$$

For $k^0 \in \mathrm{int}C$, it is known that

$$\bigcup_{t \in \mathbb{R}} (tk^0 + C) = Y$$

and hence $V + C = Y$.

Moreover, there exist $s \in \mathbb{R}$ such that $sk^0 \in V + C$, that is, $h_{\inf}^l(V) < s < \infty$. Indeed, suppose that for all $t \in \mathbb{R}$ such that $tk^0 \in V + C$. Taking $t = -n$, we have $-nk^0 \in y + C$ for all $y \in V$ and $n \in \mathbb{N}$. Hence, we have

$$-k^0 \in \frac{y}{n} + C.$$

Taking the limit when $n \to \infty$, we obtain $k^0 \in -C$, which is a contradiction.

(vii) Let $h_{\inf}^l(V) < t$. Then there exists $\hat{t} \in \mathbb{R}$ such that $h_{\inf}^l(V) \leq \hat{t} < t$. By using (i), we have

$$tk^0 = \hat{t}k^0 + (t - \hat{t})k^0 \subset V + C + (t - \hat{t})k^0 \subset V + \mathrm{int}C.$$

Conversely, let $tk^0 \in V + \text{int}C$. For $k^0 \in \text{int}C$, it is known that

$$\text{int}C = \bigcup_{\varepsilon>0}(\varepsilon k^0 + \text{int}C).$$

Therefore, we have

$$tk^0 \in V + \text{int}C = \bigcup_{\varepsilon>0}(V + \varepsilon k^0 + \text{int}C + C)$$

and $\{V + \varepsilon k^0 + \text{int}C + C\}_{\varepsilon>0}$ is an open cover of $\{tk^0\}$. Since $\{tk^0\}$ is compact, we can find $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m > 0$ such that

$$tk^0 \in \bigcup_{i=1}^{m}(V + \varepsilon_i k^0 + \text{int}C + C) = V + \varepsilon_0 k^0 + \text{int}C \subset V + \varepsilon_0 k^0 + C,$$

where $\varepsilon_0 := \min\{\varepsilon_i | i = 1, 2 \cdots m\} > 0$. Then we have $(t - \varepsilon_0)k^0 \in \text{cl}(V + C)$ and therefore $h^l_{\inf}(V) \leq t - \varepsilon_0 < t$.

(viii) In a similar way as (ii) by using (vii) instead of (i), remarking $\text{int}C + C = \text{int}C$, we obtain the conclusion. $\square$

## Appendix B: Proof of Lemma 3.4

*Proof.* (i) We define

$$\Lambda^u(V) := \{t \in \mathbb{R} \,|\, V \subset tk^0 - C\}.$$

We assume $h^u_{\inf}(V) \leq t$ and let $t \in \mathbb{R}$ be fixed. Then by the definitions of $h^u_{\inf}$ and $\Lambda^u$ being of epigraphical type, we have

$$v - \left(t + \frac{1}{n}\right)k^0 \in -C$$

for all $v \in V$ and $n \in \mathbb{N}$. Taking the limit when $n \to \infty$, we obtain

$$v - tk^0 \in -\text{cl}C = -C$$

for all $v \in V$, that is, $V \subset tk^0 - C$. The converse is clear from the definition of $h^u_{\inf}$.

(ii) Let $V_1, V_2 \in \mathcal{V}$ be such that $V_1 \leq^u_C V_2$ $(V_1 \subset V_2 - C)$. If $h^u_{\inf}(V_2) = \infty$, we have that condition (ii) clearly holds. Taking $h^u_{\inf}(V_2) \in \mathbb{R}$, we obtain

$$V_2 \subset h^u_{\inf}(V_2)k^0 - C$$

and hence

$$V_2 - C \subset h^u_{\inf}(V_2)k^0 - C - C = h^u_{\inf}(V_2)k^0 - C.$$

Using the inclusion $V_1 \subset V_2 - C$, we have

$$V_1 \subset V_2 - C \subset h^u_{\inf}(V_2)k^0 - C$$

that is, $h^u_{\inf}(V_1) \leq h^u_{\inf}(V_2)$.

(iii) see [31]. (iv) and (v) are similar as Lemma 3.3.

(vi) Let $h^u_{\inf}(V) < t$. Then there exists $\hat{t} \in \mathbb{R}$ such that $h^u_{\inf}(V) \leq \hat{t} < t$. By using (i), we have

$$V \subset \hat{t}k^0 - C = tk^0 - (t - \hat{t})k^0 - C \subset tk^0 - \mathrm{int}C.$$

(vii) Firstly, we show $h^u_{\inf}(V) > -\infty$ for $V \in \mathcal{V}$. Indeed, let $V \subset tk^0 - C$ for all $t \in \mathbb{R}$. Taking $t = -n$, we have $y \in -nk^0 - C$ for all $y \in V$ and $n \in \mathbb{N}$. Hence, we have

$$\frac{y}{n} + k^0 \in -C.$$

Taking the limit when $n \to \infty$, we obtain $k^0 \in -C$, which is a contradiction.

Since $V \in \mathcal{V}$ is $(-C)$-bounded and $k^0 \in \mathrm{int}C$, for the neighborhood of zero

$$U = k^0 - \mathrm{int}C$$

there exists $s > 0$ such that $V \subset s(k^0 - \mathrm{int}C) - C$ and hence

$$V \subset sk^0 - (\mathrm{int}C + C) \subset sk^0 - C.$$

that is, $h^u_{\inf}(V) \leq s < \infty$.

(viii) Let $V \subset tk^0 - \mathrm{int}C$. For $k^0 \in \mathrm{int}C$, it is known that

$$\mathrm{int}C = \bigcup_{\varepsilon > 0} \big((\varepsilon k^0 + \mathrm{int}C) + C\big).$$

Therefore, we have

$$V \subset tk^0 - \mathrm{int}C = tk^0 - \bigcup_{\varepsilon > 0}(\varepsilon k^0 + \mathrm{int}C + C) = \bigcup_{\varepsilon > 0} \big(\{(t - \varepsilon)k^0 - \mathrm{int}C\} - C\big)$$

and $\{(t-\varepsilon)k^0 - \mathrm{int}C - C\}_{\varepsilon > 0}$ is an open cover of $V$. Since $V \in \mathcal{V}$ is $(-C)$-compact, we can find $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m > 0$ such that

$$V \subset \bigcup_{i=1}^{m} \big((t - \varepsilon_i)k^0 - \mathrm{int}C - C\big) = (t - \varepsilon_0)k^0 - \mathrm{int}C \subset (t - \varepsilon_0)k^0 - C$$

where $\varepsilon_0 := \min\{\varepsilon_i | i = 1, 2 \cdots m\} > 0$. Then we have $V \subset (t - \varepsilon_0)k^0 - C$ and therefore $h^u_{\inf}(V) \leq t - \varepsilon_0 < t$.

(ix) In a similar way as (ii) by using (vi) and (viii) instead of (i), remarking $\mathrm{int}C + C = \mathrm{int}C$, we obtain the conclusion. $\square$

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